NIL RINGS SATISFYING CERTAIN CHAIN CONDITIONS

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In general, given the fact that every element in a ring is nilpotent, one cannot conclude that the ring itself is nilpotent. However, there are theorems which do assert that, in the presence of certain side conditions, nil implies nilpotent. We shall prove some theorems of this nature here; among them they contain or subsume many of the earlier known theorems of this sort.

Definition. The left ideal λ of R is a *left annihilator* if λ is the set of *all* the elements in R which annihilate a certain subset of R from the left.

Of course, we can also define right annihilator in an analogous fashion. We shall use the notation

$$l(S) = \{x \in R | xS = (0)\} \quad \text{and} \quad r(S) = \{x \in R | Sx = (0)\}.$$

We say that R satisfies the *ascending chain condition* on left annihilators if every ascending chain of left annihilators becomes stationary at some point. This is equivalent to the fact that every non-empty set of left annihilators has a maximal element. If a ring R satisfies the ascending chain condition on left annihilators, so does every subring of R, for the left annihilator in a subring is the intersection of the subring with the left annihilator in R.

Another well-known fact, but one worthy of pointing out again, is that the assumption of an *ascending* chain condition on *left* annihilators is equivalent to the assumption of a *descending* chain condition on *right* annihilators.

We begin with

LEMMA 1. *Let R be a ring satisfying the ascending chain condition on left annihilators. Then, if R is nil, every non-zero homomorphic image of R contains a non-zero nilpotent ideal.*

Proof. Let $\bar{R}$ be a non-zero homomorphic image of R; thus, $\bar{R}$ is isomorphic to R/M where $M \neq R$ is a two-sided ideal of R.

Let $\mathfrak{M} = \{l(x) | x \in R, x \notin M\}$. By hypothesis $\mathfrak{M}$ has a maximal element; that is, there is an element a in R and not in M such that $l(a)$ is maximal in $\mathfrak{M}$.

For any x in R, $l(ax) \supset l(a)$; if, in addition, ax is not in M, then by the maximality of $l(a)$ we are forced to conclude that $l(ax) = l(a)$. Suppose that $ar \notin M$. Since ar is nilpotent there is an integer k such that $(ar)^k \in M$ but

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$(ar)^{k-1} \notin M$. However, ar is nilpotent. Thus, there is a least positive integer n such that $(ar)^n = 0$. Since $(ar)^{k-1}$ is of the form ax and is not in M, $l((ar)^{k-1}) = l(a)$. Because $(ar)^{n-k+1}$ is in $l((ar)^{k-1})$, we are led to $(ar)^{n-k+1}a = 0$, and so $(ar)^{n-k+2} = (ar)^{n-k+1}ar = 0$. By our choice of n we must conclude that $n - k + 2 \geqslant n$ and so $k \leqslant 2$. Consequently, $(ax)^2$ is in M for every x in R.

For any x in R, since $x + 1$ is formally invertible, we have that $ax + a$ is not in M. From this it results that $l(ax + a) = l(a)$ and the argument used above yields $(ax + a)^2$ is in M.

In $\bar{R} = R/M$ these translate into: $\bar{a} \neq 0$, $(\bar{a}\bar{x})^2 = 0$, $(\bar{a}\,\bar{x} + \bar{a})^2 = 0$ for all $\bar{x}$, which immediately yield $\bar{a}\bar{x}\bar{a} = 0$ for all $\bar{x}$ in $\bar{R}$. Thus, $\bar{a}\bar{R}$ is a nilpotent right ideal of $\bar{R}$. If $\bar{a}\bar{R} = (0)$, then the left annihilator of $\bar{R}$ would be a non-zero nilpotent ideal. If $\bar{a}\bar{R} \neq (0)$ it generates a nilpotent two-sided ideal. In this way the lemma has been established.

As an immediate corollary we have the basic result:

COROLLARY. *If R is a nil ring satisfying the ascending chain condition on left annihilators, then R is locally nilpotent.*

Proof. As is well known (cf. **3**), R contains a maximal locally nilpotent ideal $L(R)$, the Levitzki radical of R, which contains all locally nilpotent ideals of R. Moreover, $L(R/L(R)) = (0)$; hence $R/L(R)$ certainly contains no nilpotent ideals.

If $L(R) \neq R$, by the lemma, $R/L(R)$ would have a non-zero nilpotent ideal, which it cannot. Thus $R/L(R) = (0)$, and hence $R = L(R)$. Therefore, R is locally nilpotent.

We continue the investigation of our class of nil rings with

LEMMA 2. *Let R be a nil ring satisfying the ascending chain condition on left annihilators. Then there exists an element $x_0 \neq 0$ in R such that $Rx_0 = (0)$.*

Proof. Let $\mathfrak{M}$ be the set of finitely generated subrings in R and let $\mathfrak{N} = \{r(S) | S \in \mathfrak{M}\}$. By the remark previously made, there is an S_0 in $\mathfrak{M}$ such that $r(S_0)$ is minimal in $\mathfrak{N}$. Let $x \in R$; then S_1, the subring generated by S_0 and x, is finitely generated. Since $S_0 \subset S_1$, $r(S_0) \supset r(S_1)$, whence, by the minimality of $r(S_0)$ we conclude that $r(S_0) = r(S_1)$. However, $r(S_1) = r(S_0) \cap r(x)$. Together with the above equality we then conclude that $r(S_0) = r(S_0) \cap r(x)$, that is $r(S_0) \subset r(x)$ for *every* x in R. But then $Rr(S_0) = (0)$.

To finish the proof, we note that since S_0 is finitely generated, by the corollary to Lemma 1, it must be nilpotent. Hence for some integer k, $S_0{}^k = (0)$, $S_0{}^{k-1} \neq (0)$. Since $(0) \neq S_0{}^{k-1} \subset r(S_0)$, $r(S_0) \neq (0)$, and we have proved the lemma.

Before proceeding with our study we digress for a moment to record a result whose proof is completely trivial but which has some independent interest. We state it without proof.

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SUBLEMMA. *If R is any ring and if A is a two-sided ideal of R which happens to be a left annihilator, then the inverse image in R of any left annihilator in R/A is itself a left annihilator in R.*

Trivial as the sublemma is it allows us to assert that in certain factor rings the ascending chain conditions persist.

LEMMA 3. *If R satisfies the ascending chain condition on left (on right) annihilators and if A is a two-sided ideal of R which happens to be a left annihilator in R, then R/A satisfies the ascending chain condition on left (on right) annihilators.*

Proof. As we have previously observed, the ascending chain condition on right annihilators translates into the descending chain condition on left annihilators. In view of the one-to-one correspondence set up by the sublemma between the left annihilators in R/A and certain left annihilators in R the lemma now becomes clear.

We are now in a position to prove the first of the theorems of this paper.

THEOREM 1. *If the ring R satisfies the ascending chain conditions on left and right annihilators, then any nil subring of R is nilpotent.*

Proof. Since the ascending chain conditions on left and right annihilators are inherited from the ring by its subrings, we may, without loss of generality, assume that R is nil. We wish to show that R is nilpotent.

Let

$$T_k = \{x \in R | xR^k = (0)\}.$$

Since $T_1 \subset T_2 \subset \ldots \subset T_k \subset \ldots$ and since they are left annihilators, by our hypothesis there is an n such that $T_n = T_{n+1} = \ldots$. If $T_n = R$, then $R^{n+1} = (0)$ and the proof is complete. Suppose that $T_n \neq R$; then $\bar{R} = R/T_n \neq (0)$. By Lemma 3, $\bar{R}$ satisfies the ascending chain condition on right annihilators; so by Lemma 2 (in its reflected form), there exists an $\bar{x} \neq 0$ in $\bar{R}$ such that $\bar{x}\bar{R} = (0)$. Therefore, $xR \subset T_n$ (where x is an inverse image of $\bar{x}$ in R) and so $xRR^n = (0)$. Thus $xR^{n+1} = (0)$. From our choice of n, this forces $xR^n = (0)$ and so $x \in T_n$. But then $\bar{x} = 0$, contrary to hypothesis. This contradiction proves that $\bar{R} = (0)$ and so $R = T_n$ and $R^{n+1} = (0)$.

If one examines the proof given for Theorem 1, one sees that a more general result has actually been proved, that only a part of the ascending chain condition on the right has been used. The exact theorem proved has been:

Let R be a nil ring satisfying the ascending chain condition on left annihilators and such that the ascending chain of right annihilators $T_n = \{x \in R | R^n x = (0)\}$ becomes stationary. Then R is nilpotent.

Indeed it seems reasonable that imposing any additional chain condition on the right is superfluous, that one should be able to draw the same conclusion with only the chain condition on the left. This leads us to make the

CONJECTURE. *If R satisfies the ascending chain condition on left annihilators, then any nil subring of R is nilpotent.*

We turn to a consideration of nil rings that satisfy polynomial identities. Using a very simple argument due to Posner (**7**, p. 181, first paragraph), we have that any prime ring satisfying a polynomial identity over its centroid satisfies the ascending chain condition on left and right annihilators. Thus, any nil subring of such a ring, by our Theorem 1, is nilpotent. We summarize this in

LEMMA 4. *Let R be a prime ring satisfying a polynomial identity over its centroid. Then any nil subring of R is nilpotent.*

The above result can also be derived as a consequence of the characterization given by Posner for prime rings satisfying polynomial identities.

Let R be any nil ring satisfying a polynomial identity with properly conditioned coefficients (e.g., R may be an algebra over a field satisfying a polynomial identity with coefficients in the field, R may be any ring, the identity having as coefficients operators, some of which are invertible, R/P may be supposed to satisfy some polynomial identity, depending on P, for each prime ideal P of R, etc.). A theorem by Kaplansky (**4**) asserts that such a ring is locally nilpotent. We give a new, and very elementary, proof of this in the

COROLLARY 1. *A nil ring satisfying a polynomial identity is locally nilpotent.*

Proof. Let S be a finitely generated subring of R which we suppose is not nilpotent. Since S^k is also finitely generated, by Zorn's Lemma there exists an ideal, P, of R maximal with respect to not containing any S^k. However, P is a prime ideal of R; for if $AB \subset P$ with A, B ideals properly larger than P, then $A \supset S^{k_1}$, $B \supset S^{k_2}$, whence $P \supset AB \supset S^{k_1+k_2}$, a contradiction. However, $\bar{R} = R/P$ is a prime ring satisfying a polynomial identity; since, in addition, it is nil, by the lemma it is nilpotent. This is inconsistent with $\bar{R}$ being prime. Thus no such P exists and we conclude the $S^k = (0)$ for some k.

As another immediate corollary to our results we have a result of Levitzki (**6**):

COROLLARY 2. *A nil ring satisfying a polynomial identity has a non-zero nilpotent ideal.*

Proof. If R has no non-zero nilpotent ideals, then by a known result (**3**, p. 196, Theorem 2) R has prime ideals. If P is a prime ideal of R, then R/P is a nil prime ring satisfying a polynomial identity. Hence, by the lemma, it is nilpotent.

Recently Kegel (**5**) has shown that if $R = A + B$ where A and B are nilpotent subrings of R, then R is nilpotent. In as-yet unpublished work he has proved that if $R = A + B$ where A is nilpotent and where B is locally nilpotent, then R is locally nilpotent. It is conjectured that if $R = A + B$ where A and B are locally nilpotent, then R is locally nilpotent. In the presence of a polynomial identity we settle this in

THEOREM 2. *Let $R = A + B$ be a ring satisfying a polynomial identity where A and B are nil subrings of R. Then R itself is locally nilpotent.*

Proof. We want to show that $R = L(R)$, the Levitzki radical of R. By going to $R/L(R)$ we may suppose that $L(R) = (0)$. We must then show that $R = (0)$.

So we suppose that $L(R) = (0)$, $R \neq (0)$. But then R has a prime ideal P (**3**, Chapter 8). Now $\bar{R} = R/P$ is a prime ring satisfying a polynomial identity and $\bar{R} = \bar{A} + \bar{B}$ where $\bar{A}$, $\bar{B}$ are nil. By Lemma 4 each of $\bar{A}$ and $\bar{B}$ is nilpotent; hence by Kegel's result, $\bar{R}$ must be nilpotent. Since $\bar{R}$ is a prime ring, this is impossible. Thus we get $R = (0)$. This establishes the theorem.

Were the conjecture we made earlier true, we would have no need of the next theorem; but since we do not know it as yet, we state and prove the theorem. The kind of conditions used are those that arise so naturally in the recent, very important work of Goldie (**1, 2**)

THEOREM 3. *Let R be a nil ring satisfying the ascending chain condition on left annihilators and on direct sums of left ideals. Then R is nilpotent.*

Proof. Let $A = \{x \in R | Rx = (0)\}$ and let T be the torsion part of A. Of course, T is a two-sided ideal of R. We can write T as a direct sum of its primary components, T_p, where if $t \in T_p$, then $p^{m(t)}t = 0$ where p is a prime.

Let $V_0 = (0)$ and $V_1 = \{x \in T_p | px = 0\}$. V_1 is an ideal of R and is a vector space over the field, P, of p elements. Since there are no infinite direct sums of left ideals in R and since every subgroup of V_1, which is an algebra over P, is a left ideal of R, we must conclude that V_1 is finite-dimensional over P. Let $V_2 = \{x \in T_p | p^2 x = 0\}$ and let $W_2 = V_2/V_1$. If $u_1, \ldots, u_k, \ldots$ are in V_2, then $pu_1, \ldots, pu_k, \ldots$ are in V_1, which is finite-dimensional over P; hence $\sum \alpha_i pu_i = 0$ for some non-zero α_i in P; hence $p \sum \alpha_i u_i = 0$ and so $\sum \alpha_i u_i \in V_1$. But then W_2 is finite-dimensional over P. Similarly, if $V_i = \{x \in T_p | p^i x = 0\}$, then $W_i = V_i/V_{i-1}$ is a finite-dimensional vector space over P.

Each V_i is a right ideal of R; hence V_i/V_{i-1} is an R-module which is a finite-dimensional vector space over P. Since R induces a nil finite ring of linear transformations on these, this ring of linear transformations must be nilpotent. Therefore $V_i R^{n_i} \subset V_{i-1}$ for some integer n_i. From this we get that $V_i R^{m_i} = (0)$, where $m_i = n_1 + n_2 + \ldots + n_i$.

However, by the ascending chain condition on left annihilators, there exists an integer k_0 such that if $xR^m = (0)$, then $xR^{k_0} = (0)$. Given any element a in T_p, it is in some V_i; hence $aR^{m_i} = (0)$ and so $aR^{k_0} = (0)$. In other words $T_p R^{k_0} = (0)$.

Since T is a direct sum of the T_p's, we get that $TR^{k_0} = (0)$.

Consider $\bar{R} = R/T$ and let $\bar{A} = A/T$; $\bar{A}$ is torsion free. Let $\bar{V} = \bar{A} \otimes_Z Q$, where Z is the ring of integers and Q is the field of rational numbers. If $\bar{u}_1, \ldots, \bar{u}_n$ in $\bar{A}$ are linearly independent (as elements in $\bar{V}$) over Q, they are *a priori* linearly independent over Z; hence their inverse images $u_1, \ldots, u_n$ in A are linearly independent over Z. Since these then generate a direct sum of left-ideals of R (since any subgroup of A is a left-ideal of R), we get, from our hypothesis, that $\bar{V}$ is finite-dimensional over Q. Thus $\bar{R}$ induces a ring of linear transformations on $\bar{V}$ by $(\bar{a} \otimes q)t_{\bar{r}} = \bar{a}\bar{r} \otimes q$. The homomorphism $\bar{r} \rightarrow t_{\bar{r}}$ has kernel exactly $r(\bar{A})$; thus $\bar{R}/r(\bar{A})$ is a nil ring of $n \times n$ matrices over the rationals, where $n = \dim_Q \bar{V}$. By a well-known result (or by invoking our Theorem 1), $\bar{R}/r(\bar{A})$ is nilpotent; hence $(\bar{R}/r(\bar{A}))^m = (0)$; so $\bar{R}^m \subset r(\bar{A})$ and so $\bar{A}\bar{R}^m = (0)$, whence $AR^m \subset T$. Since $TR^{k_0} = (0)$, we get $AR^mR^{k_0} = (0)$ and so $AR^{m+k_0} = (0)$. By our choice of k_0, it results that $AR^{k_0} = (0)$.

Let $U = \{x \in R | xR^{k_0} = (0)\}$. In $R' = R/U$ we know, by Lemma 3, that the ascending chain condition on left annihilators holds. If $R' \neq (0)$, since it is nil, we would have by Lemma 2 that there is an ideal $B' \neq (0)$ so that $R'B' = (0)$. Thus, if B denotes the inverse image in R of B', $RB \subset U$, whence $RBR^{k_0} = (0)$. However, this places BR^{k_0} in A, and the discussion above allows us to conclude that $(BR^{k_0})R^{k_0} = (0)$. Since $BR^{2k_0} = (0)$, by the choice of k_0 we are led to $BR^{k_0} = (0)$ and so $B \subset U$ and $B' = (0)$. This last conclusion is in contradiction with $B' \neq (0)$. Thus we are forced to conclude that $R' = (0)$; hence that $R = U$. From the definition of U, we obtain $R^{k_0+1} = (0)$; that is, R is nilpotent. This proves the theorem.

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