# ON HARMONIC CONTINUATION

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**1.** Let $D$ be a bounded, closed, simply-connected domain whose boundary $C$ consists of a finite number of analytic Jordan curves. Let $\gamma$ be any analytic arc of $C$. Then we shall prove the following theorem.

THEOREM 1. *Let $u(x, y)$ be harmonic in the interior of $D$ and continuous on $\gamma$, and let $\partial u(x, y)/\partial n = g(s)$ when $(x, y)$ is on $\gamma$, where $g(s)$ is an analytic function of arc-length $s$ along $\gamma$. Then $u(x, y)$ can be harmonically continued across $\gamma$.*

Here, $\partial/\partial n$ denotes differentiation along the inner normal. A similar result for the case in which $u(x, y)$ is analytic on $\gamma$ is known [3, pp. 220–3]. The proof of Theorem 1 is given in §§ 2–4, and an extension to the case in which $D$ is a more general region is given in § 5.

**2.** First let $v(r, \theta)$ be any function which is harmonic in $r < R$, $0 < \theta < \pi$, and continuous in the closure of this region, $r, \theta$ being polar coordinates. Also, let $\partial v(r, \theta)/\partial y$ tend to zero as $y \to 0$, when $-R < x < R$. Define $v(\theta)$ by $v(\theta) = v(R, \theta)$ if $0 \leqq \theta \leqq \pi$, and $v(\theta) = v(R, 2\pi - \theta)$ if $\pi \leqq \theta \leqq 2\pi$. Then it is proved in [4] (with a different notation) that the function

$$V(r, \theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{v(\phi)}{R^2 + r^2 - 2Rr\cos(\theta - \phi)}\, d\phi,$$

which is harmonic in $r < R$ and continuous in $r \leqq R$, is equal to $v(r, \theta)$ in $r \leqq R$, $0 \leqq \theta \leqq \pi$. Thus $V(r, \theta)$ is a harmonic continuation of $v(r, \theta)$ into the lower half of the region $r \leqq R$.

**3.** Next, let $w(x, y)$ be harmonic in $x^2 + y^2 < 1$, $y > 0$ and continuous in $x^2 + y^2 < 1$, $y \geqq 0$, and let $\partial w(x, y)/\partial y$ tend to $g(x)$ as $y \to 0$ when $-1 < x < 1$, where $g(x)$ is analytic in $-1 < x < 1$.

If $-1 < a < 1$, we have

$$g(x) = \sum_{n=0}^{\infty} c_n (x - a)^n,$$

the expansion being valid in some neighbourhood of $a$. Then the series $\sum c_n (z - a)^n$, where $z = x + iy$, represents an analytic function of $z$, $g(z)$ say, in some region $|z - a| \leqq R$, and we can take $R$ to satisfy $-1 < a - R < a + R < 1$.

Let
$$G(z) = \sum_{n=0}^{\infty} \frac{c_n (z - a)^{n+1}}{n+1},$$

and let $W(x, y) = \operatorname{im} G(z)$. Then $G(z)$ is analytic in $|z - a| \leqq R$, and so $W(x, y)$ is harmonic there. Also,

$$\frac{\partial W(x, y)}{\partial y} = \frac{\partial}{\partial x}\{\text{re } G(z)\},$$

and the right-hand side is $g(x)$ when $y=0$ and $a-R \leqq x \leqq a+R$. Hence, by § 2, $w(x, y)-W(x, y)$ can be harmonically continued across the segment $y=0$, $a-R \leqq x \leqq a+R$. But $W(x, y)$ is harmonic in the whole region $|z-a| \leqq R$ and so, by varying $a$, we obtain a harmonic continuation of $w(x, y)$ across the $x$-axis, $-1 \leqq x \leqq 1$.

**4.** We are now in a position to establish Theorem 1. We can transform the interior of $D$ conformally into the region $x_1^2+y_1^2 < 1$, $y_1 > 0$ in the $z_1$-plane, where $z_1 = x_1+iy_1$, in such a way that $\gamma$ becomes the diameter $y_1 = 0$, $-1 \leqq x_1 \leqq 1$, the transformation being $z = f(z_1)$, say. Then the transformation is conformal in a domain which extends outside $D$ across the arc $\gamma$ [**2**, p. 186]. Furthermore, $g(s)$ is transformed into an analytic function of $x_1$.

Let $u(x, y)$ be transformed into the function $u_1(x_1, y_1)$. Then we have the relation between the normal derivatives

$$\left[\frac{\partial u_1(x_1, y_1)}{\partial y_1}\right]_{y_1=0} = |f'(x_1)| \frac{\partial u(x, y)}{\partial n} \tag{1}$$

for $(x, y)$ on $\gamma$, where $|f'(x_1)|$ denotes $[|f'(z_1)|]_{y_1=0}$, which can be shown to be an analytic function of $x_1$. Since now the right-hand side of (1) is an analytic function of $x_1$, it follows from § 3 that $u_1(x_1, y_1)$ can be harmonically continued across the segment $y_1 = 0$, $-1 \leqq x_1 \leqq 1$. On transforming back to the $(x, y)$-plane, we obtain the result that $u(x, y)$ can be harmonically continued across the arc $\gamma$.

**5.** Suppose for the moment that $\gamma$ is any analytic Jordan arc. Then there is a region $D'$ containing $\gamma$ the points of which can be said to be on one side or the other of $\gamma$ [**1**, pp. 192–3]. Let $D'_+$ denote the subregion consisting of points on one side of $\gamma$ and $D'_-$ the subregion consisting of points on the other side of $\gamma$, the points of $\gamma$ belonging to both $D'_+$ and $D'_-$. We can now obtain the extension of Theorem 1.

THEOREM 2. *Let $D$ be any region for which there is an analytic Jordan arc $\gamma$ with the property that if $P$ is any point of $\gamma$ then there is a neighbourhood $N(P)$ of $P$ such that $N(P) \cap D = N(P) \cap D'_+$. Let $u(x, y)$ satisfy the same conditions as in Theorem* 1 *with respect to $D$ and $\gamma$. Then $u(x, y)$ can be harmonically continued across $\gamma$.*

It is clearly possible to construct a subregion of $D$, $D_1$ say, which is bounded, closed, simply-connected, and whose boundary consists of $\gamma$ (or any finite part of $\gamma$ if $\gamma$ extends to infinity) and a finite number of analytic Jordan arcs. Indeed,

$$D_1 \subset \bigcup_{P \in \gamma} \{N(P) \cap D\}.$$

Also, $u(x, y)$ is harmonic in the interior of $D_1$. It follows from Theorem 1 that $u(x, y)$ can be harmonically continued across $\gamma$, and this is the required result.

## REFERENCES

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