A VOLUME ESTIMATE FOR STRONG SUBHARMONICITY AND MAXIMUM PRINCIPLE ON COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. A generalized maximum principle on a complete Riemannian manifold (M, g) is shown under a certain volume growth condition of (M, g) and its geometric applications are given.

Introduction

Let (M, g) be a complete and connected Riemannian manifold of dimension $m \geq 1$. In this article we study the following problem:

PROBLEM. *For any $\varepsilon > 0$ and a smooth function f bounded from above on M does there exist $x_\varepsilon \in M$ such that* (i) $\sup_M f - \varepsilon < f(x_\varepsilon)$ (ii) $|\operatorname{grad} f|(x_\varepsilon) < \varepsilon$ *and* (iii) $\Delta f(x_\varepsilon) < \varepsilon$?

Omori first formulated such a maximum principle on a complete Riemannian manifold with sectional curvature bounded from below (cf. [Om]). Since then, the maximum principle has been generalized by several authors and applied to the study of value distribution of maps between manifolds as first observed by Omori, [Om]. Especially Yau generalized Omori's result to a complete Riemannian manifold with Ricci curvature bounded from below (cf. [C-Y], [Y-1]). Recently Chen & Xin [C-X] and Ratto, Rigoli & Setti [R-R-S] have extended the principle to a complete Riemannian manifold whose Ricci curvature decays slower than a certain decreasing function tending to minus infinity. Their curvature condition is optimal in a sense, and the idea of the proof heavily depends on an upper estimate for the Laplacian of the distance function by the curvature condition.

On the other hand it is known that the maximum principle no longer holds on a general complete non-compact Riemannian manifold and the problem is deeply related to the volume growth property of (M, g). Here for any $x \in M$ and $r > 0$ we define a function h_x by $h_x(r) := \log V_x(r)/r^2$,

where $V_x(r)$ is the volume of the geodesic ball $B_x(r)$ centered at $x \in M$ of radius $r > 0$. Every complete Riemannian manifold (M, g) in the articles [Om], [C-Y], [C-X], [R-R-S], [Y-1] admits a point $x \in M$ with $\int_1^{+\infty} dr/rh_x(r) = +\infty$ in view of their curvature condition. Furthermore for a given smooth monotone increasing function $h(r)$ satisfying $\int_1^{+\infty} dr/rh(r) < +\infty$, we can construct a two dimensional complete Riemannian manifold (N, g_h) admitting a bounded strongly subharmonic function $u \geq 0$; i.e., $\Delta u \geq c > 0$ on N and $\sup_N u < +\infty$, and a point x such that $h_x(r) \sim h(r)$ for sufficiently large r (cf. §2, Remark 2.4). From this observation we would like to propose the following conjecture:

CONJECTURE. *Suppose a complete Riemannian manifold (M, g) of dimension $m \geq 1$ admits a point $x \in M$ such that $\int_1^{+\infty} dr/rh_x(r) = +\infty$. Then the maximum principle holds on (M, g).*

In this article we show the following theorem which is weaker than the above conjecture.

THEOREM. *Let (M, g) be a complete Riemannian manifold of dimension $m \geq 1$. Suppose M admits a point x such that $\liminf_{r \to +\infty} h_x(r) < +\infty$. Then the maximum principle holds on (M, g).*

As far as we know, this result is the first one for the maximum principle on complete Riemannian manifolds without any curvature condition and it generalizes the maximum principle obtained in [C-Y], [Om], [Kr], [Y-1]. Our method is based on a volume estimate for strong subharmonicity on complete Riemannian manifolds. Furthermore there exists a two dimensional complete Riemannian manifold with finite volume whose Gaussian curvature decays faster than the curvature condition assumed by [C-X] or [R-R-S] (cf. [L-S], §3, Example 1).

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§1. A volume estimate for strong subharmonicity on complete Riemannian manifolds

Let (M, g) be a complete non-compact Riemannian manifold of dimension $m \geq 1$ and let $V_x(r)$ denote the volume of the geodesic ball $B_x(r)$

centered at $x \in M$ of radius $r > 0$. Let u (resp. K) be a smooth (resp. non-negative continuous) function on M. We define subsets of M as follows: for given constants $\alpha \geq -1$, $\beta > 0$, we set

$$\Omega(u, \alpha, \beta) := \{x \in M \; ; \; u(x) \geq 0 \text{ and } \Delta u(x) \geq \beta K(x) u(x)^{\alpha+1}\},$$

where $\Delta u := \mathrm{Trace}_g \nabla^2 u$,   and for a given constant $\gamma > 0$, we set

$$M(u, \gamma) := \{x \in M \; ; \; u(x) > \gamma\}.$$

The following theorem plays a crucial role in this article.

THEOREM 1.1.   *Suppose $M(u, \gamma)$ is a non-empty subset of $\Omega(u, \alpha, \beta)$ for some $\alpha \geq 1$.*

(i) If $K(x) \equiv 1$ on M, then for any point $x \in M$ there exist a positive constant $r(x, \gamma)$ such that

$$\frac{\log \mathrm{Vol}(B_x(r) \cap M(u, \gamma))}{r^2} \geq \left(\frac{\beta \log 2}{2^6 c_*} \right) \gamma^\alpha \quad for \ any \ \ r \geq r(x, \gamma),$$

where c_* is a positive constant independent of α, β, γ and $x \in M$. In particular, it holds

$$\liminf_{r \to +\infty} \frac{\log V_x(r)}{r^2} = +\infty \quad for \ any \ \ x \in M.$$

(ii) If there exist a point $x_1 \in M$, positive constants c_1 and r_* such that $K(x) \geq c_1/d(x_1, x)$ for any $x \in M \setminus B_{x_1}(r_*)$ and $\gamma > \sup\limits_{y \in B_{x_1}(r_*)} u(y)$, then there exists a positive constant $r(x_1, \gamma)$ such that

$$\frac{\log \mathrm{Vol}(B_{x_1}(r) \cap M(u, \gamma))}{r} \geq \left(\frac{\beta c_1 \log 2}{2^7 c_*} \right) \gamma^\alpha \quad for \ any \ \ r \geq r(x_1, \gamma).$$

In particular, it holds

$$\liminf_{r \to +\infty} \frac{\log V_{x_1}(r)}{r} = +\infty.$$

(iii) If there exist a point $x_2 \in M$, positive constants c_2 and r_{**} such that $K(x) \geq c_2/d(x_2, x)^2$ for any $x \in M \setminus B_{x_2}(r_{**})$ and $\gamma > \sup\limits_{y \in B_{x_2}(r_{**})} u(y)$, then there exists a positive constant $r(x_2, \gamma)$ such that

$$\frac{\log \mathrm{Vol}(B_{x_2}(r) \cap M(u, \gamma))}{\log r} \geq \left(\frac{\beta c_2}{2^8 c_*} \right) \gamma^\alpha \quad for \ any \ \ r \geq r(x_2, \gamma).$$

In particular, it holds

$$\liminf_{r \to +\infty} \frac{\log V_{x_2}(r)}{\log r} = +\infty.$$

Remark 1.2. In Theorem 1.1 if $u^* := \sup_M u < +\infty$, then we have only to consider an unbounded function $v := 1/(u^* - u)$ such that $M(v, 1/(u^* - \gamma))$ is a non empty subset of $\Omega(v, 1, \beta\gamma^{\alpha+1})$. The above volume growth properties follow immediately from this observation.

To show Theorem 1.1 we need the following lemma, which can be proved by a direct calculation.

LEMMA 1.3.   *Let v (resp. ω) be a smooth (resp. Lipschitz continuous) function on M and let λ be a smooth function on a real line $\mathbf{R}$. Then the following holds*

$$\begin{aligned}
&\mathrm{div}(\omega^2 \mathrm{grad}\, \lambda(v^2)) \\
&= \omega^2 \{2\lambda'(v^2)(v\Delta v + |dv|^2) + \lambda''(v^2)|dv^2|^2\} + 4\omega v \lambda'(v^2)\langle d\omega, dv\rangle
\end{aligned}$$

where $\Delta v = \mathrm{div}(\mathrm{grad}\, v)$.

Proof of Theorem 1.1. Since $M(u, \gamma) = M(u/\gamma, 1)$ and $\Omega(u, \alpha, \beta) = \Omega(u/\gamma, \alpha, \beta\gamma^\alpha)$, we replace u by u/γ and set $\delta := \beta\gamma^\alpha$. Hence we have $M(u, 1)(\neq \phi) \subset \Omega(u, \alpha, \delta)$. For any positive number $p > 1$, setting $v^p = u^{p/2}$ we obtain the following on the subset $M(u, 1)$:

$$(1.4) \qquad v^p(\Delta v^p) + |dv^p|^2 \geq \left(\frac{\delta p K}{2}\right)u^{p+\alpha} + \frac{2(p-1)}{p}|dv^p|^2.$$

For a fixed positive number $\rho > 1$ with $M(u, \rho) \neq \phi$, let λ be a smooth function defined on real line such that $\lambda(t) \equiv 0$ if $t \leq 1$, $\lambda(t) > 0, \lambda'(t) > 0, \lambda''(t) \geq 0$ if $t > 1$ and $\lambda(t) \equiv t$ if $t \geq \rho > 1$. Since the Riemannian manifold (M, g) is complete, for any fixed point $x \in M$ and $r > 0$ there exists a Lipschitz continuous function ω_r with $0 \leq \omega_r \leq 1$ on M such that $\omega_r \equiv 1$ on $B_x(r)$, supp $\omega_r \subset \overline{B_x(2r)}$ and $|d\omega_r|^2 \leq c_*/r^2$, where $c_* > 0$ does not depend on x and r. We apply Lemma 1.3 to $\omega = \omega_r^{p+\alpha}$ and $v = v^p$. Integrating the left hand side of Lemma 1.3 and using (1.4), we obtain for any $\varepsilon > 0$ and $B_x(2r, r) := B_x(2r) \setminus B_x(r)$ the following:

$$\frac{4(p-1)}{p}\int \omega_r^{2(p+\alpha)}\lambda'(u^p)|dv^p|^2 + \delta p \int K\omega_r^{2(p+\alpha)}\lambda'(u^p)u^{p+\alpha}$$

$$\leq -4(p+\alpha)\int \omega_r^{2(p+\alpha)-1}\lambda'(u^p)v^p\langle d\omega_r, dv^p\rangle$$

$$\leq 2\varepsilon \int \omega_r^{2(p+\alpha)}\lambda'(u^p)|dv^p|^2 \; + \; \frac{2(p+\alpha)^2}{\varepsilon}\int_{B_x(2r,r)}\omega_r^{2(p+\alpha-1)}\lambda'(u^p)u^p|d\omega_r|^2.$$

Taking $\varepsilon = (p-1)/p$ we obtain the following:

$$\int K\omega_r^{2(p+\alpha)}\lambda'(u^p)u^{p+\alpha} \; \leq \; \frac{2c_*(p+\alpha)^2}{\delta(p-1)r^2}\int_{B_x(2r,r)}\omega_r^{2(p+\alpha-1)}\lambda'(u^p)u^p.$$

On the other hand since $\alpha \geq 1$ and $\lambda'(u^p) > 0$ if and only if $u > 1$, we obtain

$$\int_{B_x(2r,r)}\omega_r^{2(p+\alpha-1)}\lambda'(u^p)u^p$$

$$\leq \left(\int K\omega_r^{2(p+\alpha)}\lambda'(u^p)u^{p(p+\alpha)/(p+\alpha-1)}\right)^{(p+\alpha-1)/(p+\alpha)}$$

$$\times\left(\int_{B_x(2r,r)}K^{-(p+\alpha-1)}\lambda'(u^p)\right)^{1/(p+\alpha)}$$

$$\leq \left(\int K\omega_r^{2(p+\alpha)}\lambda'(u^p)u^{p+\alpha}\right)^{(p+\alpha-1)/(p+\alpha)}$$

$$\times\left(\int_{B_x(2r,r)}K^{-(p+\alpha-1)}\lambda'(u^p)\right)^{1/(p+\alpha)}.$$

Hence we obtain the following:

$$(1.5) \qquad \int K\omega_r^{2(p+\alpha)}\lambda'(u^p)u^{p+\alpha}$$

$$\leq \left(\frac{2c_*(p+\alpha)^2}{\delta(p-1)r^2}\right)^{p+\alpha}\int_{B_x(2r,r)}K^{-(p+\alpha-1)}\lambda'(u^p)$$

for any $p > 1$ and $r \geq r_0 = r_0(x,\gamma)$ with $B_x(r_0)\cap M(u,1) \neq \phi$. If $\inf_{y\in B_x(2r,r)} K(y) \geq c_\sigma/(2r)^\sigma$ for any $r > r_0$ and $c_\sigma > 0$ with $\sigma \in \{0, 1, 2\}$, then setting $F(p,r) := \int_{B_x(r)}\lambda'(u^p)$ and taking $p \geq \max\{2,\alpha\}$ we obtain the following from (1.5) :

$$(1.6) \qquad F(p,r) \; \leq \; \left(\frac{2^{\sigma+4}c_*p}{\delta c_\sigma r^{2-\sigma}}\right)^{p+\alpha}F(p,2r)$$

for any $r \geq r_0$.

　　We first show (i). For $\sigma = 0$, put $c_0 = 1$ and set $p(r) = \delta r^2 / 2^5 c_* \geq \max\{2, \alpha\}$ with $\delta = \beta \gamma^\alpha$. We define $F(r) := F(p(r), r)$. Then putting $\sigma = 0$ in (1.6) , we have the following:

$$(1.7) \qquad F(r) \leq \left(\frac{1}{2} \right)^{p(r) + \alpha} F(2r)$$

for any $r \geq r_0$. We fix r such that $r \geq 2r_0$. Since there exists $k \geq 1$ such that $2^{-(k+1)} < r_0/r \leq 2^{-k}$, by putting $r_i = 2^i r_0$ and by (1.7) we obtain

$$F(r_0) \leq \left(\frac{1}{2} \right)^{\sum_{i=0}^{k-1} p(r_i) + k\alpha} F(r_k) \leq \left(\frac{1}{2} \right)^{\delta r^2 / 2^6 c_*} \left(\frac{r_1}{r} \right)^\alpha F(r)$$

for any $r \geq r_1$. Therefore there exists $r(x, \gamma) > 0$ such that

$$(1.8) \qquad \frac{\log F(r)}{r^2} \geq \left(\frac{\beta \log 2}{2^6 c_*} \right) \gamma^\alpha$$

for any $r \geq r(x, \gamma)$. Since u was replaced by u/γ in the beginning and it may be assumed that $\sup_{\mathbf{R}} \lambda'(t) = 1$, it holds that $F(r) \leq \mathrm{Vol}(B_x(r) \cap M(u, \gamma))$ for any $r \gg 0$. Therefore we can obtain the desired estimate in (i) from (1.8). To show (ii) (resp. (iii)) we have only to put $x = x_1$ (resp. $x = x_2$), $\sigma = 1$ (resp. $\sigma = 2$) in (1.6) and set $p(r) = \delta c_1 r / 2^6 c_*$ (resp. $p(r) = \delta c_2 / 2^7 c_*$) in (1.7). The desired estimate follows from the same argument as above respectively. This completes the proof of Theorem 1.1.

§2.　Maximum principle on a complete Riemannian manifold satisfying a certain volume growth condition

　　Let u be a smooth function on a Riemannian manifold M. For given constants α and β, we set

$$\Gamma(u, \alpha, \beta) := \{ \ x \in M \ ; \ u(x) > 0 \ \ and \ \ \Delta u(x) < \beta u(x)^{\alpha+1} \ \}.$$

First we show the following maximum principle:

　　THEOREM 2.1. *Let (M, g) be a complete Riemannian manifold of dimension $m \geq 1$. Suppose M admits a point x such that the following volume growth condition holds*

$$(*) \qquad \liminf_{r \to +\infty} \frac{\log V_x(r)}{r^2} < \ +\infty.$$

For a given smooth function u on M such that $0 < \sup_M u \leq +\infty$, and for $\alpha > 0, \beta > 0$, the following holds by setting $u^* = \sup_M u$:

(i) *$\Gamma(u, \alpha, \beta)$ is non-empty. In particular, M admits no non-zero smooth solution $w \geq 0$ (resp. $v \geq 0$) satisfying $\Delta w \geq \beta w^{\alpha+1}$ (resp. $\Delta \log v \geq \beta v^\alpha$) on M (cf.[Os]).*

(ii) *If $u^* < +\infty$ and $\sigma \geq -1$, then $\Gamma(u, \sigma, \beta)$ is non-empty. In particular, M admits no non-zero smooth solution $v \geq 0$ with $\sup_M v < +\infty$ satisfying $\Delta v \geq \beta v^{\sigma+1}$ on M.*

(iii) *$u(x) \leq u^*(\alpha, \beta) := \sup_{y \in \Gamma(u, \alpha, \beta)} u(y)$ for any $x \in M$. Especially if $u^*(\alpha, \beta)$ is finite for a certain pair (α, β), then $u^*(\alpha, \beta)$ is independent of α and β and hence $u^* = u^*(\alpha, \beta)$.*

(iv) *If $u^* < +\infty$ and $\Gamma(u) := \{x \in M; u(x) > 0 \text{ and } \Delta u(x) < 0\}$ is non-empty, then $u^* = \sup_{y \in \Gamma(u)} u$.*

Proof for the case $\alpha \geq 1$. If M is compact, then the above assertions are obvious. We may assume that M is non-compact. First remark that the hypothesis $(*)$ implies (i) by Theorem 1.1, (i). (ii) follows from Remark 1.2 and (i). To see (iii) if $u^*(\alpha, \beta) = +\infty$, then the assertion is trivial. If $u^*(\alpha, \beta)$ is finite and there exists a point $z \in M$ with $u^*(\alpha, \beta) < u(z)$, then it is clear that $M(u, u^*(\alpha, \beta))$ is a non-empty subset of $\Omega(u, \alpha, \beta)$, which contradicts to the assumed growth condition $(*)$ by Theorem 1.1, (i). By the same argument, we can see the latter half of (iii). (iv) follows from (iii) immediately. $\square$

PROPOSITION 2.2. *Let (X, g_X) be a complete Riemannian manifold of dimension $n \geq 1$ and let f be a smooth function bounded from above on X. For any $\varepsilon > 0$ take a point $y_\varepsilon \in X$ with $\sup_X f - \varepsilon^2 < f(y_\varepsilon)$. Then there exists a point $x_\varepsilon \in X$ such that* (i) $f(y_\varepsilon) \leq f(x_\varepsilon)$, (ii) $d_X(x_\varepsilon, y_\varepsilon) \leq \varepsilon$ *and* (iii) $|\text{grad } f|(x_\varepsilon) \leq \varepsilon$, *where d_X is the distance function relative to g_X.*

Proof. This was proved in [H] in the case $X = \mathbf{R}^n$. In this general case the proof is essentially the same and therefore omitted. $\square$

Now we are in a position to show the following our main result.

THEOREM 2.3. *Under the volume growth condition $(*)$ of Theorem 2.1 let f be a smooth function bounded from above on M. Then for any given point $x \in M$ and $\varepsilon > 0$ there exists a point $x_\varepsilon \in M$ depending on x such that* (i) $f(x) \leq f(x_\varepsilon)$, (ii) $|\text{grad } f|(x_\varepsilon) < \varepsilon$ *and* (iii) $\Delta f(x_\varepsilon) < \varepsilon$.

Proof. Since the assertion is trivial, if f attains $f^* := \sup_M f$, we suppose that f does not attain f^* on M. We put $\varepsilon_* := \min\{\varepsilon, f^* - f(x)\}/(1 + \min\{\varepsilon, f^* - f(x)\}) > 0$. Set $w := 1/(1 + f^* - f) > 0$ and $M_p := M(w^p, 1 - \varepsilon_*^2)$ for any positive integer p. Then clearly $M_p \subset M_q$ and $\partial M_p \cap \partial M_q = \phi$ if $p > q \geq 1$. On the other hand $\Gamma_p := \Gamma(w^p, 1, \varepsilon_*)$ is non-empty by Theorem 2.1, (i). By the fact $(p-1)\Delta w^p \geq pw\Delta w^{p-1}$ for any $p \geq 2$ and $0 < w < 1$ on M, we obtain $\Gamma_p \subset \Gamma_q$ and $\partial \Gamma_p \cap \partial \Gamma_q = \phi$ if $p > q \geq 1$. Setting $\Sigma_p := \Gamma_p \cap M_p$, Σ_p is also non-empty and $\sup_{\Sigma_p} w^p = 1$ by Theorem 2.1, (iii). In particular $\Sigma_p \subset \Sigma_q$ and $\partial\Sigma_p \cap \partial\Sigma_q = \phi$ if $p > q \geq 1$. Suppose Σ_p converges to a non-empty subset $\Sigma_\infty \subset M$ containing a point x_∞ as p tends to infinity. Then w should attain 1 at x_∞. This is a contradiction. Hence $M \setminus \Sigma_p$ converges to the whole space M as p tends to infinity. This implies that $d_M(x_*, \Sigma_p)$ is unbounded for a fixed point $x_* \in M$. Set $\lambda_p := \sup_{y \in \partial\Sigma_p} d_M(y, \partial\Sigma_1)$ for any $p > 1$. Then $\lambda_p \in (0, +\infty]$ and λ_p is non-decreasing in p. We claim that $\lim_{p \to +\infty} \lambda_p = +\infty$. Otherwise there exists a constant $c > 0$ with $\lim_{p \to +\infty} \lambda_p \leq c < +\infty$. Then we obtain $\partial\Sigma_p \subset \{y \in M; d_M(y, \partial\Sigma_1) \leq c\}$ for any $p \geq 1$. This implies $d_M(x_*, \Sigma_p) \leq d_M(x_*, \partial\Sigma_1) + c < +\infty$, which is a contradiction. Therefore there exists a large positive integer p_* such that $\varepsilon_* < \lambda_p \leq +\infty$ for any integer p with $p \geq p_*$. For a fixed $p \geq p_*$, there exists a point $y_* \in \partial\Sigma_p$ with $d_M(y_*, \partial\Sigma_1) > \varepsilon_*$. Clearly such a point admits a small positive constant δ_* such that $\overline{B_z(\varepsilon_*)} \subset \Sigma_1$ if $z \in B_{y_*}(\delta_*) \cap \Sigma_p$. Now we take a point $z_\varepsilon \in B_{y_*}(\delta_*) \cap \Sigma_p$. By Proposition 2.2, there exists a point $x_\varepsilon \in \overline{B_{z_\varepsilon}(\varepsilon_*)} \cap M_p \subset \Sigma_1$ such that $|dw^p|(x_\varepsilon) \leq \varepsilon_*$. If p is large enough, then x_ε is the desired point. This completes the proof of Theorem 2.3.

Using Theorem 2.3 we complete the proof of Theorem 2.1.

Proof of Theorem 2.1, (i) *and* (iii) *for the case* $0 < \alpha < 1$. To show (i) suppose $\Gamma(u, \alpha, \beta)$ is empty. Let λ be a smooth function defined on the real line such that $\lambda(t) = 0$ for $t \leq 0$, $\lambda'(t) > 0, \lambda''(t) \geq 0$ for $t \geq 0$ and $\lambda(t) = t$ for $t \geq c > 0$. Taking c arbitrarily we may assume that $v := \lambda(u)$ satisfies $\Delta v \geq \beta v^{\alpha+1}$ on $\{v > \delta\} \neq \phi$ with $\delta := \lambda(c) > 0$. Set $w := -1/(1+v)^p$ with $p = \alpha/2 > 0$ and $\varepsilon_* := \min\{\sup_M w - w(\delta), 1\} > 0$. By Theorem 2.3 for any $\varepsilon > 0$ with $0 < \varepsilon < \varepsilon_*$, there exists a point $x_\varepsilon \in M$ such that (1) $\sup_M w - \varepsilon < w(x_\varepsilon)$, (2) $|\text{grad } w|(x_\varepsilon) < \varepsilon$, (3) $\Delta w(x_\varepsilon) < \varepsilon$. Since $\Delta w = p\Delta v/(1+v)^{p+1} - p(p+1)|\text{grad } v|^2/(1+v)^{p+2}$

and $\Delta v(x_\varepsilon) \geq \beta v^{\alpha+1}(x_\varepsilon)$, we get the following :

$$\left(\frac{v(x_\varepsilon)}{1+v(x_\varepsilon)}\right)^{2p+1} \beta \leq \left(\frac{2p+1}{p^2}\right)\varepsilon.$$

This implies $v^* := \sup_M v < +\infty$ and so there exists $C > 0$ independent of ε such that $v(x_\varepsilon)^{2p+1} \leq C\varepsilon$. Letting $\varepsilon \to 0$ we obtain $v^* = 0$, which implies $u \equiv 0$ on $\{u > 0\} \neq \phi$. This is a contradiction. To show (iii) suppose $u^*(\alpha,\beta)$ is finite and there exists a point $z \in M$ with $u^*(\alpha,\beta) < u(z)$. Then u satisfies that $\Delta u \geq \beta u^{\alpha+1}$ on $\{u > u^*(\alpha,\beta)\} \neq \phi$. By the similar argument as in the proof of (i) we can conclude $u \equiv u^*(\alpha,\beta)$ on $\{u > u^*(\alpha,\beta)\} \neq \phi$, which is a contradiction. This completes the proof of Theorem 2.1.

Remark 2.4. For a given smooth monotone increasing function $h(r)$ satisfying $c_1 = \int_1^{+\infty} dr/rh(r) < +\infty$, the example stated in the introduction is constructed as follows: Take a function $f(r)$ such that (1) $f(0) = 0$, $f'(0) = 1$, $f(r) > 0$, $f'(r) > 0$, $f''(r) \geq 0$ if $r > 0$, and (2) $f(r) = c_2(\exp(r^2 h(r)))'$ with $c_2 > 0$ (is determined later) if $r \geq 1$. We define a complete metric g on $([0,+\infty) \times S^1, (r,\theta))$ by $g = dr^2 + f(r)^2 d\theta^2$. Since $\Delta u(r) = (f(r)u_r(r))_r/f(r)$, $\Delta u \equiv 1$ and $\sup u \leq c_1$ if $u(r) := \int_0^r \{(\int_0^t f(s)ds/f(t)\}dt$. Setting $v(r) =$ the volume of $S^1 = S^1 \times \{r\}$ relative to g, $(\log(f(r)/v(r)))_r \equiv 0$ for any $r > 0$ (cf. [G-W]). Put $c_2 \equiv f(r)/v(r) > 0$. Since $V_0(r) = \int_0^r v(t)dt$, setting $h_0(r) = (\log V_0(r))/r^2$, it can be easily verified that $h(r)/2 \leq h_0(r) \leq 2h(r)$ for any $r \gg 0$.

§3. Applications

The maximum principle is closely related to several problems in differential geometry and geometric function theory; i.e., Liouville type theorem and Ahlfors-Schwarz type lemma for maps, minimal submanifolds in $\mathbf{R}^n$, conformal deformation of Riemann structure, etc. By replacing curvature condition by volume growth condition to assure the maximum principle, a few results related to those problems can be stated without proof. With respect to the details of the proof and the related topics the reader should see the references cited in the end of each statement.

THEOREM 3.1. *Let (M,g) be a complete Riemannian manifold of dimension $m \geq 1$. Suppose (M,g) admits a point x_1 (resp. x_2) such that*

$$\liminf_{r\to+\infty} \frac{\log V_{x_1}(r)}{r} < +\infty \qquad \left(resp. \quad \liminf_{r\to+\infty} \frac{\log V_{x_2}(r)}{\log r} < +\infty\right)$$

Then for a non-negative continuous function K_1 (resp. K_2) on M satisfying $K_1(x) \geq c_*/d(x_1, x)$, $x \in M \setminus B_{x_1}(r_*)$ (resp. $K_2(x) \geq c_*/d(x_2, x)^2$, $x \in M \setminus B_{x_2}(r_*)$) for $r_* \gg 0$ and $c_* > 0$, there exists no non-zero smooth solution $u \geq 0$ satisfying $\Delta u \geq K_1 u^\sigma$ (resp. $\Delta u \geq K_2 u^\sigma$) on M if $\sigma \geq 2$. In particular, there exists no non-zero smooth solution $u \geq 0$ with $\sup_M u < +\infty$ satisfying $\Delta u \geq K_1 u^\sigma$ (resp. $\Delta u \geq K_2 u^\sigma$) on M if $\sigma \geq 0$. Furthermore there exists no non-zero smooth solution $v \geq 0$ satisfying $\Delta \log v \geq K_1 v$ (resp. $\Delta \log v \geq K_2 v$) on M (cf. [C-L], Theorems 2.1 and 5.1, [L], [N], §3, [R-R-S], Theorems 3.1, 4.1 and 4.2, and [Y-2], Theorems 1 and 5).

Remark 3.2.   In Theorem 3.1 if $m \geq 3$, then the non-negativity condition of K_1 (resp. K_2) can be weakened in the following way: *Under the hypothesis of Theorem 3.1 there exists a constant $\varepsilon > 0$ such that there is no positive smooth solution satisfying $\Delta u \geq K_1 u^\sigma$ (resp. $\Delta u \geq K_2 u^\sigma$) provided that $\sigma \geq 2$ and $K_1 \geq -\varepsilon$ (resp. $K_2 \geq -\varepsilon$) on M* (cf. § 1, Theorem 1.1 and [R-R-S], Proposition 3.5).

THEOREM 3.3.   *Under the volume growth condition $(*)$ of Theorem 2.1 the following holds :*

(i) *Suppose $h : (M, g) \to (N, g_N)$ is an isometric immersion into an Hadamard manifold (N, g_N) of dimension $n > m$ such that the mean curvature H of h satisfies $H_* := \sup_M |H| < +\infty$, and $h(M) \subset B_{y_*}(R)$, $y_* \in N$. Then $R \geq \frac{1}{H_*}$* (cf. [Kr], Theorem 3.1).

(ii) *Suppose $f : M \to \mathbf{R}^n$ is a smooth map whose energy density $e(f)$ satisfies $e(f) \geq c > 0$ on M, and there exists a unit vector ξ at the origin of $\mathbf{R}^n$ such that $\langle \xi, f(x)/\|f(x)\| \rangle \geq \delta > 0$ for any $x \in M$; i.e., $f(M)$ lies inside of a non-degenerate cone of $\mathbf{R}^n$. Then there exists a point $y \in M$ such that the tension field $\tau(f)$ of f does not vanish at y* (cf. [B-K], Theorem 3 and [Om], Theorem B).

COROLLARY 3.4.   *Under the volume growth condition $(*)$ of Theorem 2.1, M can not be isometrically immersed as a minimal submanifold into a bounded set of a Riemannian manifold admitting a smooth strictly convex function. In particular, if $f : M \to \mathbf{R}^n$ is a minimal isometric immersion, then $f(M)$ does not lie inside of any non-degenerate cone of $\mathbf{R}^n$* (cf. [Ks], Proposition 2 and Remarks).

THEOREM 3.5.   *Let (N, g_N) be a Riemannian manifold with sectional curvature bounded above by a constant K and let $B_R(y_*)$ be inside the cut*

locus of $y_* \in N$ ($R < \pi/2\sqrt{K}$ if $K > 0$). Under the volume growth condition (∗) of Theorem 2.1 suppose $h : M \to N$ is a smooth map with $|\tau(h)| \leq \tau_*$, $\tau_* \in [0, +\infty)$, and $h(M) \subset B_R(y_*)$. Then setting $e_* := \inf_M e(h)$ for the energy density $e(h)$ of h, the following holds (cf. [C-X], Theorem 3.1 and [R-R], Theorem 2.17) :

(i) $\qquad R \geq \sqrt{K} \tan^{-1}(2\sqrt{K}e_*/\tau_*)$ *when $K > 0$*

(ii) $\qquad R \geq 2e_*/\tau_*$ *when $K = 0$*

(iii) $\qquad R \geq \sqrt{-K} \tanh^{-1}(2\sqrt{-K}e_*/\tau_*)$ *when $K < 0$*

COROLLARY 3.6. *Under the volume growth condition (∗) of Theorem 2.1, suppose M is a submanifold in $\mathbf{R}^n$ with parallel mean curvature and the image of M of the Gauss map $\mathcal{G} : M \to \mathbf{G}_{m,n-m}$ lies in the geodesic ball $B_R(y_*) \subset \mathbf{G}_{m,n-m}$, $R < \pi/2\sqrt{K}$ and $y_* \in \mathbf{G}_{m,n-m}$, ($K = 1$ if $\mathbf{G}_{m,n-m}$ is the sphere, otherwise $K = 2$), then M should be minimal (cf. [C-X], Theorem 3.3, [H-J-K], §4, Theorems 5 and 7, and [R-R], Corollary 2.24).*

THEOREM 3.7. *Let (M, ω_M) be a complete Hermitian manifold of dimension m whose scalar curvature bounded from below by $-K_1$ and let N be a complex manifold of the same dimension with a volume form V_N whose Ricci form is negative definite and satisfies $\wedge^m(\sqrt{-1}/2)\partial\bar\partial \log V_N \geq K_2 V_N$. Suppose (M, ω_M) satisfies the volume growth condition (∗) of Theorem 2.1 and $f : M \to N$ is a holomorphic map whose Jacobian does not vanish at some point. Then $K_1 > 0$ and $\sup f^* V_N / V_M \leq K_1^m / m^m K_2$, where V_M is the volume form relative to ω_M (cf. [M-Y], §1, [Ry], [Y-3]).*

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