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Singular perturbations of the unicritical polynomials with two parameters

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Abstract. In this paper, we study the dynamics of the family of rational maps with two parameters

$$f_{a,b}(z) = z^n + \frac{a^2}{z^n - b} + \frac{a^2}{b},$$

where $n \geq 2$ and $a, b \in \mathbb{C}^*$. We give a characterization of the topological properties of the Julia set and the Fatou set of $f_{a,b}$ according to the dynamical behavior of the orbits of the free critical points.

1. *Introduction*

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map defined on the Riemann sphere $\widehat{\mathbb{C}}$. The Fatou set $F(f)$ of f is defined to be the set of points at which the family of iterates of f forms a normal family, in the sense of Montel. The complement of the Fatou set is called the *Julia set*, which we denote by $J(f)$. A connected component of the Fatou set is called a *Fatou component*. According to Sullivan's theorem, every Fatou component of a rational map is eventually periodic and there are only five kinds of periodic Fatou components: attracting domains, super-attracting domains, parabolic domains, Siegel disks and Herman rings. The *critical points* of f are defined to be the points at which f is not univalent in any neighborhood, and the *critical values* of f are defined to be the images of the critical points.

An interesting and important problem in complex dynamics is to describe the topology of the Julia sets of rational maps, such as the connectivity and local connectivity. For a polynomial, it was proved by Fatou that the Julia set is connected if and only if the orbits of the finite critical points are bounded. In [**20**], Qiu and Yin obtained a sufficient and necessary condition for the Julia set of a polynomial to be a Cantor set. For rational maps, the Julia sets may exhibit more complex topological structures. Pilgrim and Tan proved that if the Julia set of a hyperbolic (more generally, geometrically finite) rational map is disconnected, then, with the possible exception of a finite number of periodic components and their countable collection of preimages, every Julia component is either a single point or a Jordan curve [**17**, Theorem 1.2].

For the general rational maps, it is difficult to describe the topological properties of the corresponding Julia sets. However, for some special families of rational maps, the topological properties of the Julia sets can be studied well. For example, the *McMullen maps* $F_\lambda(z) = z^n + \lambda/z^d$ with $n, d \geq 2$ and $\lambda \in \mathbb{C}^*$ have been studied extensively by Devaney and his collaborators in a series of papers (see [**3**, **5**–**7**]). Specifically, it was proved in [**6**] that if the orbits of the critical points of F_λ are all attracted to ∞, then the Julia set of F_λ is either a Cantor set, a Sierpiński curve, or a Cantor set of circles. In particular, the Julia set of F_λ is a Cantor set of circles if $1/n + 1/d < 1$ and $\lambda \neq 0$ is small. If the orbits of the free critical points of F_λ are bounded, then F_λ has no Herman rings [**24**] and, actually, the corresponding Julia set is connected [**7**]. Since the McMullen family exhibits extremely rich dynamics, this family has also been studied in [**21**] and [**18**]. As a variation of F_λ, the *generalized McMullen map* $F_{\lambda,\eta}(z) = z^n + \lambda/z^d + \eta$ also attracts much interest. Some additional dynamical phenomenon happens for this family since the parameter space becomes $\mathbb{C}^2$, which is two-dimensional. For a comprehensive study on $F_{\lambda,\eta}$, see [**2**, **10**, **12**, **25**] and the references therein. There also exist some other special families of rational maps which have been well studied. For example, see [**9**, **11**, **22**].

Note that, for McMullen maps and generalized McMullen maps, the point at infinity is always a super-attracting fixed point and the origin is always a pole. If the parameter is close to the origin, then each of these maps can be seen as a perturbation of the unicritical polynomial $P_n(z) = z^n$. In this paper, we are interested in the problem of finding a two-dimensional family of rational maps, such that the point at infinity and the origin are both super-attracting fixed points. For this, we consider the following family of rational maps

$$f_{a,b}(z) = z^n + \frac{a^2}{z^n - b} + \frac{a^2}{b}, \tag{1.1}$$

where $n \geq 2$ and $a, b \in \mathbb{C}^*$. If $a = 0$, then $f_{a,b}$ degenerates to the unicritical polynomial P_n. Therefore, the map $f_{a,b}$ with $(a, b) \in (\mathbb{C}^*, \mathbb{C}^*)$ can be also seen as a perturbation of the simple polynomial P_n. This perturbation is essentially different from that of McMullen maps and the generalized McMullen maps since $f_{a,b}$ not only keeps the dynamics of P_n near the point at infinity but also keeps the dynamics near the origin.

A straightforward calculation (see (2.1)) shows that $f_{a,b}$ has two super-attracting fixed points 0 and ∞. We use B and T to denote the *immediate* super-attracting basins of ∞ and 0, respectively. Then $B \cap T = \emptyset$. The map $f_{a,b}$ has $4n - 2$ critical points (counted with multiplicity) since the degree of $f_{a,b}$ is $2n$. The local degrees of 0 and ∞ are both n. Hence this leaves $2n$ more critical points. The forward orbits of 0 and ∞ are trivial since they are both fixed by $f_{a,b}$. We call the remaining $2n$ critical points the *free* critical points. In §2.1 we will show that these free critical points can be divided into two pairs, and each pair has a unique critical value. We call these two critical values the *free* critical values and denote them by v_+ and v_-. The dynamics of $f_{a,b}$ are determined by the orbits of these two free critical values.

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1.1. *Statement of the main results.* We use $\mathcal{A}(0)$ and $\mathcal{A}(\infty)$ to denote the super-attracting basins of $f_{a,b}$ containing 0 and ∞, respectively. In this paper, the notation $\sharp(\cdot)$ is used to denote the cardinal number of a finite set. Since the dynamics of $f_{a,b}$ is determined by the free critical orbits, we give the characterization of the Julia set and the Fatou set of $f_{a,b}$ by dividing the results into several cases.

THEOREM 1.1. *Suppose that the two free critical values v_+ and v_- of $f_{a,b}$ are attracted by $\mathcal{A}(\infty)$ (similar results hold for $\mathcal{A}(0)$). Then we have the following three cases.*

(1) *If $\sharp(B \cap \{v_+, v_-\}) \geq 1$, then B is completely invariant and the Julia set $J(f_{a,b}) = \partial B$ is disconnected. Moreover, except for B, other Fatou components of $f_{a,b}$ are Jordan domains.*
(2) *If $\sharp(B \cap \{v_+, v_-\}) = 0$ and $\sharp(U \cap \{v_+, v_-\}) \leq 1$ for each component U of $\mathcal{A}(\infty)$, then the Julia set $J(f_{a,b})$ is connected.*
(3) *If $\sharp(B \cap \{v_+, v_-\}) = 0$ and $v_+, v_- \in U$ for some component U of $\mathcal{A}(\infty)$, then all the preimages of U are annuli and other Fatou components of $f_{a,b}$ are simply connected.*

Actually, we conjecture that the third case in Theorem 1.1 cannot happen†. However, for McMullen maps [**6**] and the generalized McMullen maps [**25**], this case happens (for example, when the Julia set is a Cantor set of circles‡). For specific examples of the first two cases in Theorem 1.1, see §7.1 and Figure 2.

THEOREM 1.2. *Suppose that $\mathcal{A}(\infty)$ (respectively, $\mathcal{A}(0)$) attracts exactly one free critical value. Then the Julia set of $f_{a,b}$ is connected.*

We will give some typical examples for Theorem 1.2 in §§7.2 and 7.3. See Figures 3 and 4. The proof of Theorem 1.2 will be divided into two main cases: the first is where each of $\mathcal{A}(0)$ and $\mathcal{A}(\infty)$ attracts exactly one free critical value, and the second is where $\mathcal{A}(0) \cup \mathcal{A}(\infty)$ attracts exactly one free critical value.

Finally, we consider the case where the two free critical values are neither attracted by $\mathcal{A}(0)$ nor $\mathcal{A}(\infty)$ and we have the following theorem.

THEOREM 1.3. *Suppose that $v_+, v_- \notin \mathcal{A}(0) \cup \mathcal{A}(\infty)$.*

(1) *If there exists a Fatou component that contains exactly one free critical value, then $J(f_{a,b})$ is connected.*
(2) *If there exists a Fatou component that contains two free critical values, then the Julia set is disconnected.*

The two cases stated in Theorem 1.3 do, indeed, happen. For specific examples of these cases, see §7.4 and Figure 5.

A subset of the Riemann sphere $\widehat{\mathbb{C}}$ is called a *Cantor set of circles* (or *Cantor circles* for short) if it consists of an uncountable number of closed Jordan curves and is homeomorphic to $\mathcal{C} \times \mathbb{S}^1$, where $\mathcal{C}$ is the Cantor middle third set and $\mathbb{S}^1$ is the unit circle. It is known that Julia sets of the Cantor circles can appear in the McMullen family and the generalized

† One possible reason is that the Fatou component centered at the origin (i.e., T) is fixed and another reason is that v_+, v_- cannot be too close if they are attracted by ∞. See (2.2) and Figure 1.

‡ Although we will prove that the Julia set of $f_{a,b}$ cannot be a Cantor set of circles in Theorem 1.4, we still cannot rule out the third case in Theorem 1.1.

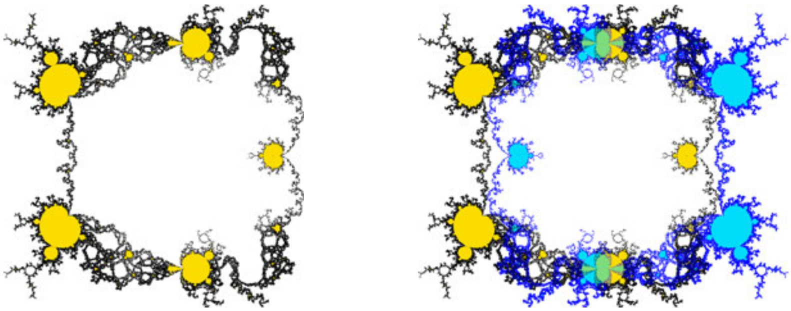

FIGURE 1. A slice of the parameter space (i.e., a-plane) of $f_{a,b}$, where $n = 3$ and $b = 1/25$. The colorful part in the left-hand picture indicates the parameters such that the free critical value v_+ is not attracted by 0 and ∞, while the white regions denote the parameters such that v_+ is attracted by 0 or ∞. The right-hand picture indicates the dynamical behaviors of the both critical values and the colorful set is the non-escaping locus of $f_{a,b}$.

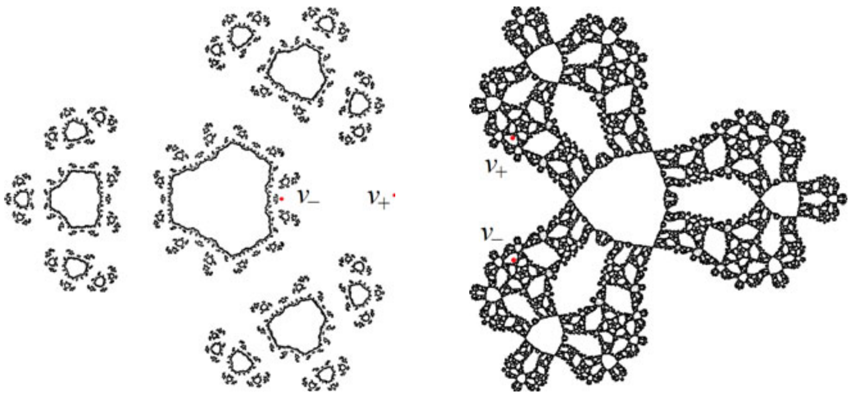

FIGURE 2. The Julia sets of f_a, where $a = 0.16$ and $a = 0.1625i$, respectively. These two Julia sets correspond to the first two cases in Theorem 1.1. The one on the left is disconnected and the one on the right is connected.

McMullen family. Although some of the Fatou components of $f_{a,b}$ may be doubly connected, we can still prove the following theorem.

THEOREM 1.4. *For any $a, b \in \mathbb{C}^*$ and $n \geq 2$, the Julia set of $f_{a,b}$ can never be a Cantor set of circles.*

As a remark, we would like to point out that there exists a family of rational maps such that 0 and ∞ are both super-attracting fixed points, but every Julia set is a Cantor set of circles. For example, see [**9**] and [**19**].

1.2. *Organization of the paper.* The paper is organized as follows. In §2, the family $f_{a,b}$ is introduced and some basic properties of $f_{a,b}$ are presented. We will also prepare some useful lemmas in this section, which are necessary in the proofs of our theorems. In §3, we describe the Julia set of $f_{a,b}$ for the case where two free critical points are attracted by one of the super-attracting fixed points 0 and ∞ and prove Theorem 1.1. In §4, we discuss the case where the super-attracting fixed point (∞ or 0) attracts exactly one free critical value and prove Theorem 1.2. In §5, we deal with the case where the two free critical values do not belong to the attracting basins of the two super-attracting fixed points and prove Theorem 1.3. We will prove the non-existence of a Cantor set of circles in §6. In the last section, we give some examples to show that the various cases stated in the above theorems (except the third case in Theorem 1.1), actually happen. The parameters that correspond to the examples are chosen from the slice shown in Figure 1.

We would like to provide more information on the parameter space of $f_{a,b}$ based on Figure 1. Unlike McMullen maps, our family $f_{a,b}$ essentially has *two* free critical orbits. Therefore, we cannot draw the whole parameter space on the complex plane $\mathbb{C}$. A way to study the high dimensional parameter space is to study its slices, and usually the slices with complex dimension one. As captioned in Figure 1, we choose the slice $b = 1/25$ and all the examples in this paper come from this slice. For the left-hand picture, the colorful set indicates the parameters $a \in \mathbb{C}^*$ such that the free critical value v_+ is not attracted by 0 and ∞. There also exists a set in this slice corresponding to the parameters $a \in \mathbb{C}^*$ such that the free critical value v_- is not attracted by 0 and ∞, which is drawn in the right-hand picture (mostly the blue parts and homeomorphic to the left-hand picture). These two sets have some overlaps which are also drawn in different colors on the right-hand picture. The different colors denote different dynamical behaviors of v_+ and v_-. For example, the green parts correspond to the set of $a \in \mathbb{C}^*$ such that neither v_+ nor v_- are attracted by 0 and ∞.

2. *Preliminaries*

In this section, we prepare some preliminary results. We first study the symmetric distribution of the critical points and the symmetric dynamical behavior of $f_{a,b}$. Then we consider the topological properties of the immediate basins of 0 and ∞, respectively. Finally, we present some classical results in complex dynamics. In the rest of this paper, we always assume that $n \geq 2$ is an integer.

2.1. *Dynamical symmetry.*

As pointed out in the introduction, the rational map

$$f_{a,b}(z) = z^n + \frac{a^2}{z^n - b} + \frac{a^2}{b} = \frac{z^n}{b} \cdot \frac{bz^n + a^2 - b^2}{z^n - b} \tag{2.1}$$

has two super-attracting fixed points, 0 and ∞. These two points are also critical points of $f_{a,b}$ with multiplicity $n - 1$. A direct calculation shows that the collection of poles of $f_{a,b}$ is

$$\{\xi_k = \omega^k \sqrt[n]{b} : 1 \leq k \leq n\} \quad \text{where } \omega = e^{2\pi i/n}.$$

Besides the origin, the other zeroes of $f_{a,b}$ can be written as

$$\left\{\zeta_k = \omega^k \sqrt[n]{(b^2 - a^2)/b} : 1 \leq k \leq n\right\}.$$

Recall that B and T are the immediate super-attracting basins of ∞ and 0, respectively. Therefore, each ξ_k or ζ_k is contained in the Fatou set. For $1 \leq k \leq n$, we use B_k to denote the Fatou component containing ξ_k and T_k to denote the Fatou component containing ζ_k, respectively. Since the degree of $f_{a,b}$ is $2n$ and the local degrees of $f_{a,b}$ at ∞ and 0

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are both n, we have $f_{a,b}^{-1}(B) = B \cup \bigcup_{k=1}^{n} B_k$ and $f_{a,b}^{-1}(T) = T \cup \bigcup_{k=1}^{n} T_k$. Note that B_k (respectively, T_k) may be disjoint from B (respectively, T).

By calculating the derivative of $f_{a,b}$, one can show that there are $2n$ critical points for $f_{a,b}$ given by

$$c_k^+ = \omega^k \sqrt[n]{b+a} \quad \text{and} \quad c_k^- = \omega^k \sqrt[n]{b-a} \quad \text{where } 1 \le k \le n.$$

However, there are only two critical values for these critical points. They are

$$v_+ = f_{a,b}(c_k^+) = (b+a)^2/b \quad \text{and} \quad v_- = f_{a,b}(c_k^-) = (b-a)^2/b. \tag{2.2}$$

In this paper, we call c_k^+, c_k^- the *free* critical points, and v_+, v_- the *free* critical values of $f_{a,b}$. The dynamics of $f_{a,b}$ are determined by the orbits of these two free critical values. Since the local degree of $f_{a,b}$ is two at every free critical point, $f_{a,b}^{-1}(v_\pm) = \{c_k^\pm : 1 \le k \le n\}$.

In the rest of this paper, we use M to denote the connected component of $\widehat{\mathbb{C}} \setminus \overline{B}$ that contains the origin and N to denote the connected component of $\widehat{\mathbb{C}} \setminus \overline{T}$ that contains the point at infinity. Since $0 \in T$, $\infty \in B$ and $T \cap B = \emptyset$, we have $B \subset N$ and $T \subset M$.

Let U be a subset of $\widehat{\mathbb{C}}$ and $a \in \mathbb{C}$. We denote $aU := \{az : z \in U\}$. The proof of the following lemma is straightforward.

LEMMA 2.1. *Let ω be a complex number satisfying $\omega^n = 1$ and suppose that U is a Fatou component of $f_{a,b}$.*

(1) *$f_{a,b}(\omega z) = f_{a,b}(z)$ and ωU is also a Fatou component of $f_{a,b}$.*

(2) *The four sets B, T, M and N have n-fold symmetry: i.e., $z \in B$ (respectively T, M and N) if and only if $\omega z \in B$ (respectively T, M and N).*

For a Fatou component U, different from B and T, we claim that either U has n-fold symmetry or the sets $\omega^k U$ with $\omega = e^{2\pi i/n}$ and $1 \le k \le n$ are pairwise disjoint. To prove this, we first need the following lemma.

LEMMA 2.2. *Suppose $\gamma \subset \mathbb{C}$ is a continuous closed curve which separates 0 from ∞. Then $\sigma\gamma \cap \gamma \neq \emptyset$ for all $\sigma = e^{2\pi i\theta}$, where $\theta \in [0, 1)$.*

Proof. Since γ is compact, there exist two points z_1 and z_2 on γ such that

$$d(z_1, 0) = \min_{z\in\gamma} d(z, 0) \quad \text{and} \quad d(z_2, 0) = \max_{z\in\gamma} d(z, 0),$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance in $\mathbb{C}$. Then the curve γ is contained in an annulus $\mathbb{A}(r_1, r_2) = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$, where $r_1 = d(z_1, 0)$ and $r_2 = d(z_2, 0)$. The lemma is clearly true for $r_1 = r_2$: i.e., when γ is a round circle. Hence we can assume that $r_1 < r_2$. By the definition of r_1 and r_2, the curve γ connects the two boundary components of $\mathbb{A}(r_1, r_2)$. For any $\sigma = e^{2\pi i\theta}$ with $\theta \in [0, 1)$, we have $\sigma z_1 \in \{z : |z| = r_1\}$ and $\sigma z_2 \in \{z : |z| = r_2\}$. If none of them lies on the curve γ, then they must lie in different components of $\mathbb{C} \setminus \gamma$. Since γ is a closed curve in $\mathbb{A}(r_1, r_2)$ surrounding 0, any curve that connects the boundary circles $\{z : |z| = r_1\}$ and $\{z : |z| = r_2\}$ must intersect γ. Hence, the subarc of $\sigma\gamma$ that connects σz_1 and σz_2 must intersect γ. This implies $\sigma\gamma \cap \gamma \neq \emptyset$. □

LEMMA 2.3. *Let U be a Fatou component of $f_{a,b}$ which is different from B and T. Suppose that z_0, $\omega^{j_0}z_0 \in U$, where ω satisfies $\omega^n = 1$ and $\omega^{j_0} \neq 1$. Then $\omega^j z_0$ belongs to U for any integer j. In particular, U has n-fold symmetry and surrounds the origin.*

Proof. Suppose z_0, $\omega^{j_0}z_0 \in U$ and $\omega^j z_0 \notin U$ for some j. Let γ_1 be a smooth curve in U that connects z_0 and $\omega^{j_0}z_0$. Then $\gamma_2 := \omega^{j_0}\gamma_1$ is contained in $\omega^{j_0}U$, which is also a Fatou component, by Lemma 2.1(1). Since $\omega^{j_0}z_0 \in \gamma_1 \cap \gamma_2 \subset U \cap \omega^{j_0}U$, we obtain $U = \omega^{j_0}U$. Note that $\omega^{2j_0}z_0 \in \gamma_2 \subset \omega^{j_0}U$: this means that U contains $\omega^{2j_0}z_0$. By induction, it follows that U contains all points $\omega^{ij_0}z_0$ and curves γ_i for $i \in \mathbb{N}$, where $\gamma_i = \omega^{ij_0}\gamma_1$.

By the definition of ω, there must exist a smallest positive integer k such that $\omega^{kj_0} = 1$. Let $\gamma = \bigcup_{i=1}^{k} \gamma_i$. Then $\gamma \subset U$ and it is a continuous closed curve. Moreover, 0 and ∞ lie in the different components of $\mathbb{C} \setminus \gamma$ since γ is obtained by taking the union of the curves that are produced by rotating γ_1 around the origin. For any $j \in \mathbb{N}$, let $\beta = \omega^j\gamma$. Then $\beta \subset \omega^j U$. By Lemma 2.2, $\emptyset \neq \beta \cap \gamma \subset U \cap \omega^j U$. Since $\omega^j U$ is also a Fatou component, $U = \omega^j U$. The proof is complete. □

We have the following immediate corollary.

COROLLARY 2.4. *Let U be a Fatou component of $f_{a,b}$ which is different from B and T. Then either U has n-fold symmetry and surrounds the origin or ωU, $\omega^2 U$, ..., $\omega^n U = U$ are pairwise disjoint, where $\omega = e^{2\pi i/n}$.*

In this paper, we need to prove that some domains are simply connected or doubly connected and the following lemma is useful.

LEMMA 2.5. (Riemann–Hurwitz formula, [**1**, §5.4, pp. 85–89]) *Let f be a rational map defined from $\widehat{\mathbb{C}}$ to itself. Assume that:*

(1) *V is a domain in $\widehat{\mathbb{C}}$ with a finite number of boundary components;*
(2) *U is a component of $f^{-1}(V)$; and*
(3) *there are no critical values of f on ∂V.*

Then there exists an integer $d \geq 1$ such that f is a branched covering map from U onto V with degree d and

$$\chi(U) + \delta_f(U) = d\chi(V),$$

where $\chi(\cdot)$ denotes the Euler characteristic and $\delta_f(U)$ denotes the total number of the critical points of f in U (counted with multiplicity).

Remark. Let D be a domain in $\widehat{\mathbb{C}}$. Then $\chi(D) = 2$ if and only if D is the Riemann sphere $\widehat{\mathbb{C}}$; $\chi(D) = 1$ if and only if D is simply connected; and $\chi(D) = 0$ if and only if D is doubly connected (i.e., an annulus).

2.2. *The general topological properties of B and T.* If a simply connected domain $U \subset \mathbb{C}^*$ does not contain any critical values, then $f_{a,b}^{-1}(U)$ consists of exactly $2n$ connected components and each of them is simply connected. For the case where U contains a free critical value, we have the following lemma.

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LEMMA 2.6. *Let $U \subset \mathbb{C}^*$ be a simply connected domain which contains exactly one free critical value of $f_{a,b}$. Then $f_{a,b}^{-1}(U)$ consists of exactly n components and each of them is simply connected. Moreover, each component of $\bigcup_{m=0}^{\infty} f_{a,b}^{-m}(U)$ is simply connected.*

Proof. Let V be a connected component of $f_{a,b}^{-1}(U)$. Without loss of generality, suppose that $v_+ \in U$. Then $f_{a,b} : V \setminus f_{a,b}^{-1}(v_+) \to U \setminus \{v_+\}$ is a covering map. Note that $U \setminus \{v_+\}$ is an annulus: this means that $V \setminus f_{a,b}^{-1}(v_+)$ is also an annulus, by Lemma 2.5. Therefore, the cardinal number $\sharp(V \cap f_{a,b}^{-1}(v_+)) \leq 1$ and V is simply connected. Moreover, $\sharp(V \cap f_{a,b}^{-1}(v_+)) \leq 1$ means that $f_{a,b}^{-1}(U)$ consists of at least n components since $\{c_k^+ : 1 \leq k \leq n\}$ is contained in $f_{a,b}^{-1}(v_+)$. Note that the degree of $f_{a,b}$ is $2n$: it follows that $f_{a,b}^{-1}(U)$ consists of exactly n connected components and each of them is simply connected. The second part of the lemma follows in a similar fashion. □

PROPOSITION 2.7. *Suppose that B (respectively, T) does not contain any free critical values.*

(1) *B (respectively, T) and each component of $f_{a,b}^{-1}(B)$ (respectively, $f_{a,b}^{-1}(T)$) is simply connected.*

(2) *If, in addition, every component of $\mathcal{A}(\infty)$ (respectively, $\mathcal{A}(0)$) contains at most one free critical value, then every component of $\mathcal{A}(\infty)$ (respectively, $\mathcal{A}(0)$) is simply connected.*

Proof. (1) By [**13**, Theorem 8.9], B is either simply connected or infinitely connected. However, by the assumption, B is super-attracting and contains no critical values except the fixed point ∞ itself. This means that B must be simply connected. The proof that each component of $f_{a,b}^{-1}(B)$ is simply connected is similar to the proof of Lemma 2.6 and we omit it. A similar result follows when we replace B by T and $\mathcal{A}(\infty)$ by $\mathcal{A}(0)$.

(2) This is an immediate corollary of Lemma 2.6. □

PROPOSITION 2.8. *If B contains at least one free critical value, then B is completely invariant and $J(f_{a,b}) = \partial B$. In particular, if B contains two free critical values, then $J(f_{a,b})$ is disconnected. The same conclusion holds for T.*

Proof. Without loss of generality, suppose that $v_+ \in B$. If there exists a critical point $c_{k_0}^+$ such that $c_{k_0}^+ \notin B$, then $c_k^+ \notin B$ for all $1 \leq k \leq n$, by Lemma 2.1(2). Therefore, v_+ has $2n$ preimages outside B (counted with multiplicity). Since $f_{a,b}(B) = B$, this means that v_+ has at least $3n$ preimages, which is a contradiction since the degree of $f_{a,b}$ is $2n$. Therefore, each c_k^+ is contained in B. Since $f_{a,b}^{-1}(v_+) = \{c_k^+ : 1 \leq k \leq n\}$, we have $f_{a,b}^{-1}(B) = B = f_{a,b}(B)$: i.e., B is completely invariant. The assertion $J(f_{a,b}) = \partial B$ follows by [**13**, Corollary 4.12].

If B contains two free critical values, then B contains $3n$ critical points (counted with multiplicity). This means that B cannot be simply connected. Hence B is infinitely connected and the Julia set of $f_{a,b}$ is disconnected. □

2.3. *Some useful definitions and lemmas.* We will prepare some useful lemmas in this subsection. These lemmas focus on the connectivity and local connectivity of the Julia sets and some regular properties of the Fatou components. For the connectivity of the Julia set of a rational map, we think the following criterion is useful.

LEMMA 2.9. *Suppose that f is a rational function which has no Herman rings and each Fatou component contains at most one critical value. Then the Julia set of f is connected.*

Proof. Since f has no Herman rings, each periodic Fatou component of f is either a (super-) attracting basin, a parabolic basin or a Siegel disk, by Sullivan's classification theorem. Let $\mathcal{U}$ be the union of all the periodic Fatou components of f. Then the Fatou set of f is equal to $\bigcup_{m\geq 0} f^{-m}(\mathcal{U})$. The Julia set is connected if and only if all Fatou components are simply connected. Let U be a periodic Fatou component of f. Consider the orbit

$$\cdots \xrightarrow{f} U_{m+1} \xrightarrow{f} U_m \xrightarrow{f} \cdots \xrightarrow{f} U_2 \xrightarrow{f} U_1 \xrightarrow{f} U_0 = U.$$

Assume that U_m is simply connected for some $m \geq 0$. If U_m does not contain any critical values, then U_{m+1} is simply connected, by Lemma 2.5. If U_m contains exactly one critical value v, then $f : U_{m+1} \setminus f^{-1}(v) \to U_m \setminus \{v\}$ is a covering map. Since $U_m \setminus \{v\}$ is an annulus, this means that $U_{m+1} \setminus f^{-1}(v)$ is also, and thus U_{m+1} is simply connected. By induction, every $U_{m'}$ is simply connected if $m' \geq m$. This means that, in order to prove the Julia set of f is connected, it is sufficient to prove that each periodic Fatou component is simply connected. We divide the arguments into three cases.

Case 1: If U is a periodic Siegel disk, then U is simply connected, by the definition.

Case 2: Suppose that U is a periodic attracting basin with period $p \geq 1$. Let $z_0 \in U$ be the attracting periodic point and define $z_i = f^{\circ i}(z_0) \in f^{\circ i}(U)$ for $1 \leq i < p$. Then there exists a small disk D_0, centered at z_0, such that $f^{\circ p}(\overline{D}_0) \subset D_0$ and ∂D_0 does not contain any points in the critical orbits of f. For $m \geq 0$ and $0 \leq i < p$, we use D_{mp+i} to denote the connected component of $f^{-(mp+i)}(D_0)$ containing z_{p-i} (we denote $z_p = z_0$ for convenience). Then for $0 \leq i < p$,

$$D_i \subset D_{p+i} \subset D_{2p+i} \subset \cdots \subset D_{mp+i} \subset \cdots . \tag{2.3}$$

Note that each D_{mp+i} contains at most one critical value and ∂D_0 does not contain any points in the critical orbits of f. We conclude that each D_{mp+i} is simply connected by an argument similar to the one presented in the first part of the proof. Since $U = \bigcup_{m\geq 0} D_{mp}$ and given (2.3), it follows that U is simply connected.

Case 3: Suppose that U is a periodic parabolic basin with period $p \geq 1$. Let $z_0 \in \partial U$ be the parabolic periodic point† such that $f^{\circ mp}(z)$ is attracted to z_0 for any $z \in U$. Define $z_i = f^{\circ i}(z_0) \in f^{\circ i}(\partial U)$ for $1 \leq i < p$. According to the dynamics of f on the parabolic basins, one can choose a small disk $D_0 \subset U$ such that $z_0 \in \partial D_0$, $\overline{D}_0 \subset U \cup \{z_0\}$, $f^{\circ p}(\overline{D}_0) \subset D_0 \cup \{z_0\}$ and ∂D_0 does not contain any points in the critical orbits of f. By an argument similar to that of Case 2, it is easy to see that U is simply connected. □

† There exists a parabolic basin whose boundary contains two or more parabolic periodic points.

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We remark that Peherstorfer and Stroh proved a similar result to Lemma 2.9 in [**15**, Theorem 4.2], where they required that each Fatou component contains at most one critical *point* (counted without multiplicity).

A rational map is called *hyperbolic* if its critical points are all attracted by the attracting cycles. It is called *subhyperbolic* if each of its critical points is either attracted by an attracting cycle or eventually periodic. A rational map is called *geometrically finite* if its critical points in the Julia set are eventually periodic. Clearly, a hyperbolic rational map must be subhyperbolic, and a subhyperbolic rational map must be geometrically finite.

For the local connectivity of the Julia set of a rational map, the following theorem was proved by Tan and Yin.

THEOREM 2.10. [**23**, Theorem A and Lemma 2.1] *Let f be a geometrically finite rational map.*

(1) *If the Julia set of f is connected, then it is locally connected.*
(2) *The boundary of a simply connected Fatou component of f is locally connected.*

A *Jordan domain* is a component of the complement of a Jordan curve in $\widehat{\mathbb{C}}$. Pilgrim proved the following result on Jordan domains.

LEMMA 2.11. [**16**, Proposition 2.8] *If f is a rational map and U is a Jordan domain whose closure contains at most one critical value v of f, then every component V of $f^{-1}(U)$ is also a Jordan domain. If $v \in \partial U$, then $f|_{\overline{V}} : \overline{V} \to \overline{U}$ is a homeomorphism.*

A set $K \subset \mathbb{C}$ is said to be *full* if it is compact and connected, and if its complement is non-empty and connected. A full set is said to be *non-degenerate* if it is not a single point. Pilgrim proved the following lemma.

LEMMA 2.12. [**16**, Proposition 2.5] *Let K be a non-degenerate full subset of $\mathbb{C}$ whose boundary is locally connected. Let U be a bounded component of $\widehat{\mathbb{C}} \setminus \partial K$ (if any). Then U and $\widehat{\mathbb{C}} \setminus U$ are both Jordan domains.*

For subhyperbolic rational maps, Morosawa established a useful lemma to prove the boundary of a Fatou component is a Jordan curve [**14**, Lemma 6]. In this paper, we will use its general form as stated in the following lemma.

LEMMA 2.13. *Let f be a rational map and U a Fatou component of f with locally connected boundary such that $f(U) = U$. If there is a component W of $\widehat{\mathbb{C}} \setminus \overline{U}$ and a Fatou component V of f such that $V \cup f^{-1}(V) \subset W$, then the boundary of U is a Jordan curve.*

Proof. Without loss of generality, we assume that $\infty \in U$. By the assumption, each component of $\widehat{\mathbb{C}} \setminus U$ is a non-degenerate full subset of $\mathbb{C}$. Suppose that $\widehat{\mathbb{C}} \setminus \overline{U}$ has exactly one component W. Then $\overline{W}$ is a non-degenerate full subset of $\mathbb{C}$. Since ∂U is locally connected, it follows, by Lemma 2.12, that W and $\widehat{\mathbb{C}} \setminus \overline{W}$ are both Jordan domains. By $f(U) = U$ and a fundamental property of the Julia set [**1**, Theorem 4.2.7(i), p. 71], for

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arbitrary $z \in V \subset W$ but at most two points,

$$J(f) \subset \overline{\bigcup_{n=1}^{\infty} f^{-n}(z)} \subset \overline{W}. \tag{2.4}$$

This means that $U = \widehat{\mathbb{C}} \setminus \overline{W}$ and ∂U is a Jordan curve.

Suppose that $\widehat{\mathbb{C}} \setminus \overline{U}$ contains at least two components. Let W' be a component of $\widehat{\mathbb{C}} \setminus \overline{U}$ that is different from W. Since ∂U is locally connected, it follows that W and W' are Jordan domains, by Lemma 2.12. Since $\partial W' \subset \partial U$ and $f(U) = U$, it follows that $f(\partial W')$ is contained in ∂U and it is a continuous closed curve. If $f(W') \cap W \neq \emptyset$, then $W \subset f(W')$. This means that there is a Fatou component V' in W' such that $f(V') = V$. However, this contradicts the assumption that $V \cup f^{-1}(V) \subset W$. Hence $f(W') \cap W = \emptyset$ and $f^{-1}(W) \subset W$. Similar to the first case, for arbitrary $z \in V$ but at most two points, we have (2.4). Therefore, $U = \widehat{\mathbb{C}} \setminus \overline{W}$ and ∂U is a Jordan curve. $\square$

3. *Both free critical values escape to the same basin*

In this section, we consider the case where both free critical orbits are attracted by one of the basins of 0 and ∞ or, equivalently, $v_+, v_- \in \mathcal{A}(0)$ or $\mathcal{A}(\infty)$. By Sullivan's classification theorem on the Fatou components of rational maps, the Fatou set $F(f_{a,b})$ of $f_{a,b}$ is equal to $\mathcal{A}(0) \cup \mathcal{A}(\infty)$ and the Julia set is $J(f_{a,b}) = \widehat{\mathbb{C}} \setminus (\mathcal{A}(0) \cup \mathcal{A}(\infty))$.

Proof of Theorem 1.1. Without loss of generality, we suppose that two free critical orbits of $f_{a,b}$ are attracted to the point at infinity.

(1) If B contains at least one free critical value, then B is completely invariant and $J(f_{a,b}) = \partial B$, by Proposition 2.8. Since v_+ and v_- are attracted by ∞, this means that B contains all the free critical points since B is completely invariant. In particular, B contains $3n$ critical points (counted with multiplicity). Note that B is either simply connected or infinitely connected. If B is simply connected, then, by Lemma 2.5, each point in B has $3n$ preimages. This is impossible since the degree of $f_{a,b}$ is $2n$. Hence B is infinitely connected and the Julia set of $f_{a,b}$ is disconnected.

By the assumption, we know that the Fatou set of $f_{a,b}$ is $B \cup \mathcal{A}(0)$. Since two free critical orbits are attracted by ∞, the basin $\mathcal{A}(0)$ contains exactly one critical value 0. In order to prove that each component of $\mathcal{A}(0)$ is a Jordan domain, by Lemma 2.11, it is sufficient to prove that T is simply connected. Note that N is the component of $\widehat{\mathbb{C}} \setminus \overline{T}$ containing ∞. Clearly, $B = f_{a,b}^{-1}(B) \subset N$. By Theorem 2.10 and Lemma 2.13, ∂T is a Jordan curve and hence each component of $\mathcal{A}(0)$ is a Jordan domain.

(2) Suppose that $\sharp(B \cap \{v_+, v_-\}) = 0$ and every connected component of $\mathcal{A}(\infty)$ contains at most one free critical value. According to Proposition 2.7, every component in $\mathcal{A}(0) \cup \mathcal{A}(\infty)$ is simply connected. Thus $J(f_{a,b}) = \widehat{\mathbb{C}} \setminus (\mathcal{A}(0) \bigcup \mathcal{A}(\infty))$ is connected.

(3) Let $U(v_+, v_-)$ be the connected component of $\mathcal{A}(\infty) \setminus B$ that contains v_+ and v_-. By Lemma 2.7(1), B is simply connected. Since $U(v_+, v_-)$ is mapped to B by a conformal map, it follows that $U(v_+, v_-)$ is also simply connected. Since $U(v_+, v_-)$ contains free critical values, there exists a component V of $f_{a,b}^{-1}(U(v_+, v_-))$ such that V contains at least one free critical point.

If V contains exactly one free critical point, say c_i^+ for some i (the similar argument can be applied to c_i^-), then V is simply connected and each set of the form $e^{2k\pi i}V$, where $1 \le k < n$, is simply connected. Moreover, by the symmetry stated in Corollary 2.4, each one of these sets contains exactly one free critical point of the form $e^{2k\pi i}c_i^+$. However, there exists another Fatou component V', such that $c_i^- \in V'$ and $f_{a,b}(V') = U(v_+, v_-)$. This is a contradiction since $U(v_+, v_-)$ cannot have $2n+2$ preimages (counted with multiplicity). Hence this case is impossible.

If V contains at least two free critical points, there are two cases. The first case: V contains two free critical points c_i^+ and c_j^+ for $1 \le i \ne j \le n$ (a similar argument can be applied to the case when V contains c_i^- and c_j^-). Then V contains c_k^+ for all $1 \le k \le n$ by the symmetry and $V = f_{a,b}^{-1}(U(v_+, v_-))$. In particular, V contains all the free critical points. This means that V is an annulus, by Lemma 2.5. Moreover, all the preimages of V are doubly connected. The second case: V contains two free critical points c_i^+ and c_j^- for $1 \le i, j \le n$. If V contains at least three free critical points, then we are back to the first case. Hence we assume that V contains exactly two free critical points c_i^+ and c_j^-. Then $\omega^k V$, where $1 \le k \le n$ and $\omega = e^{2\pi i/n}$, are pairwise disjoint and each $\omega^k V$ is an annulus by Lemma 2.5. Moreover, all the preimages of $\omega^k V$ are doubly connected.

We have proved that all components of $\bigcup_{m=1}^{\infty} f_{a,b}^{-m}(U(v_+, v_-))$ are doubly connected. By Proposition 2.7, the rest of the Fatou components are simply connected. $\square$

4. *One of the basins attracts exactly one free critical value*

In this section, we consider the case where $\mathcal{A}(\infty)$ attracts exactly one free critical orbit of $f_{a,b}$. We show that in this case the Julia set is always connected.

4.1. *Polynomial-like mappings.*

In order to prove Theorem 1.2, we need the polynomial-like mapping theory introduced by Douady and Hubbard in [**8**].

Definition. A triple (U, V, f) is called a *polynomial-like mapping* of degree $d \ge 2$ if U and V are simply connected plane domains such that $\overline{U} \subset V$, and $f : U \to V$ is a holomorphic proper mapping of degree d. The *filled Julia set* $K(f)$ of a polynomial-like mapping f is defined as

$$K(f) = \{z \in U : f^{\circ k}(z) \in U \ \forall k \ge 0\}.$$

The *Julia set* of the polynomial-like mapping f is defined as $J(f) = \partial K(f)$.

Two polynomial-like mappings (U_1, V_1, f_1) and (U_2, V_2, f_2) of degree d are said to be *hybrid equivalent* if there exists a quasiconformal homeomorphism h defined from a neighborhood of $K(f_1)$ onto a neighborhood of $K(f_2)$, which conjugates f_1 to f_2 and satisfies the complex dilatation $\mu_h = 0$ on $K(f_1)$. The following theorem was proved by Douady and Hubbard in [**8**].

THEOREM 4.1. (The straightening theorem, [**8**, Theorem 1, p. 296])

(1) *Every polynomial-like mapping (U, V, f) of degree $d \ge 2$ is hybrid equivalent to a polynomial of degree d.*

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(2) *If $K(f)$ is connected, then the polynomial is uniquely determined up to conjugation by an affine map.*

COROLLARY 4.2. *Suppose (U, V, f) is a polynomial-like mapping of degree $d \geq 2$. Then $K(f)$ and $J(f)$ are connected if and only if all critical points of f are contained in $K(f)$.*

Proof. By Theorem 4.1(1), there exists a quasiconformal mapping h, defined from a neighborhood W_f of $K(f)$ to a neighborhood of the filled Julia set $K(P)$ of a polynomial P with degree d, which satisfies $P \circ h = h \circ f$ in W_f.

Suppose that all critical points of f are contained in $K(f)$. Then all the $d - 1$ finite critical points of P are contained in $K(P)$. By [**4**, Theorem 4.1, p. 66], the Julia set $J(P)$ of P is connected. Therefore, $J(f) = h^{-1}(J(P))$ and hence $K(P)$ are both connected.

Suppose that $K(f)$ is connected. Then $K(P)$ is also connected. Still by [**4**, Theorem 4.1, p. 66], the filled Julia set $K(P)$ contains all the $d - 1$ finite critical points of P. This means that $K(f)$ contains exactly $d - 1$ critical points. Since (U, V, f) is a polynomial-like mapping of degree $d \geq 2$, by the Riemann–Hurwitz formula stated in Lemma 2.5, it follows that U contains exactly $d - 1$ critical points since U and V are both simply connected. Therefore, all critical points of f are contained in $K(f)$. □

4.2. *Only one escaping free critical orbit.* The proof of Theorem 1.2 is based on the consideration of several cases. In this subsection, we assume that only one free critical value is attracted by $\mathcal{A}(0) \cup \mathcal{A}(\infty)$. The following lemma is used to prove the non-existence of Herman rings in the family $f_{a,b}$.

LEMMA 4.3. [**26**, Main theorem] *If a rational map f has only one critical orbit in its Julia set, then f has no Herman rings.*

Now we prove Theorem 1.2 in the following case.

THEOREM 4.4. *Suppose that $\mathcal{A}(0) \cup \mathcal{A}(\infty)$ attracts exactly one free critical value. Then the Julia set of $f_{a,b}$ is connected.*

Proof. Without loss of generality, we assume that $v_+ \in \mathcal{A}(\infty)$ and $v_- \notin \mathcal{A}(0) \cup \mathcal{A}(\infty)$. The proof is divided in the following two cases: (1) $v_+ \in \mathcal{A}(\infty) \setminus B$ and (2) $v_+ \in B$.

(1) Suppose that $v_+ \in \mathcal{A}(\infty) \setminus B$. Then each Fatou component of $f_{a,b}$ contains at most one critical value. By Lemma 4.3, $f_{a,b}$ has no Herman rings. By Lemma 2.9, this means that $J(f_{a,b})$ is connected.

(2) Suppose that $v_+ \in B$. Then B is completely invariant and $J(f_{a,b}) = \partial B$ by Proposition 2.8. We use the dynamics of $f_{a,b}$ to construct a polynomial-like mapping and prove that its Julia set is connected and quasiconformally conjugate to the Julia set of $f_{a,b}$.

Since ∞ is a super-attracting fixed point, we can choose a small simply connected neighborhood D_0 of ∞ such that $v_+ \notin \overline{D}_0$, $f_{a,b}(\overline{D}_0) \subset D_0$ and ∂D_0 is a Jordan curve which is disjoint from all the critical orbits. For $m \geq 0$, let D_m be the connected component of $f_{a,b}^{-m}(D_0)$ containing D_0. Then

$$D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_m \subset \cdots$$

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and $B = \bigcup_{m \geq 0} D_m$. Since $v_+ \in B$, there must exist $m_0 \geq 1$ such that $v_+ \in D_{m_0} \setminus \overline{D}_{m_0-1}$. By the Riemann–Hurwitz formula, we conclude that both D_{m_0} and $V := \widehat{\mathbb{C}} \setminus \overline{D}_{m_0}$ are simply connected. In particular, since ∂D_0 is a Jordan curve which is disjoint from all the critical orbits, this means that D_{m_0} and V are Jordan domains. We clearly see that $v_- \in V$, $T \subset V$ and $f_{a,b}^{-1}(V) \subset V$.

Let U be the connected component of $f_{a,b}^{-1}(V)$ containing T. We claim that $U = f_{a,b}^{-1}(V)$. Otherwise, by the symmetry, $f_{a,b}^{-1}(V)$ has exactly $n+1$ simply connected components: i.e., U and $U_1, U_2, \ldots, U_n$, where $f_{a,b}|_U : U \to V$ is of degree n and $\{U_1, U_2, \cdots, U_n\}$ is invariant under the rotation $z \mapsto e^{2\pi i/n} z$. Since $\overline{V} \cup D_{m_0} = \widehat{\mathbb{C}}$, it follows that $f_{a,b}^{-1}(D_{m_0}) = \widehat{\mathbb{C}} \setminus f_{a,b}^{-1}(\overline{V}) = \widehat{\mathbb{C}} \setminus (\overline{U} \cup \bigcup_{k=1}^{n} \overline{U}_k)$ is connected. Therefore $f_{a,b}^{-1}(D_{m_0}) = D_{m_0+1}$, by definition. Note that D_{m_0+1} contains $2n-1$ critical points (counted with multiplicity) and consider the map $f_{a,b} : D_{m_0+1} \to D_{m_0}$, by the Riemann–Hurwitz formula in Lemma 2.5: so

$$\deg(f_{a,b}|D_{m_0+1}) = 2 - (n+1) + (2n-1) = n.$$

This is a contradiction since $\deg(f_{a,b}|D_{m_0+1}) = 2n$. Therefore, $U = f_{a,b}^{-1}(V)$.

Note that U contains $2n-1$ critical points (counted with multiplicity), and hence U is simply connected. Now we have a polynomial-like mapping $(U, V, f_{a,b})$ with degree $2n$. Since $v_- \notin \mathcal{A}(\infty)$, it follows that all the critical orbits of $(U, V, f_{a,b})$ are contained in U. Therefore, the Julia set of $(U, V, f_{a,b})$ is connected, by Corollary 4.2. Since $\widehat{\mathbb{C}} \setminus U \subset \mathcal{A}(\infty)$, this means that the Julia set of the polynomial-like mapping $(U, V, f_{a,b})$ is homeomorphic to that of the rational map $f_{a,b}$. Therefore, the Julia set of the rational map $f_{a,b}$ is connected. □

4.3. *Two free critical orbits are attracted to* 0 *and* ∞*, respectively.* In this subsection, we consider the case where the two free critical orbits are attracted by two super-attracting basins $\mathcal{A}(0)$ and $\mathcal{A}(\infty)$, respectively. We first prove the following Theorem 4.5 and then give some specific examples in Figure 4.

THEOREM 4.5. *Suppose that $v_+ \in \mathcal{A}(\infty)$ and $v_- \in \mathcal{A}(0)$. Then we have the following four cases:*

(1) *if $v_+ \in B$ and $v_- \in T$, then $J(f_{a,b})$ is a Jordan curve;*

(2) *if $v_+ \in B$ and $v_- \notin T$, then $J(f_{a,b})$ is connected and each component of $\mathcal{A}(0)$ is a Jordan domain;*

(3) *if $v_+ \notin B$ and $v_- \in T$, then $J(f_{a,b})$ is connected and each component of $\mathcal{A}(\infty)$ is a Jordan domain; and*

(4) *if $v_+ \notin B$ and $v_- \notin T$, then $J(f_{a,b})$ is connected.*

Similar results follow when $v_- \in \mathcal{A}(\infty)$ and $v_+ \in \mathcal{A}(0)$.

Proof. (1) Suppose that $v_+ \in B$ and $v_- \in T$. By Proposition 2.8, both B and T are completely invariant and the Julia set $J(f_{a,b}) = \partial B = \partial T$. Note that $B = f_{a,b}^{-1}(B) \subset N$, where N is the connected component of $\widehat{\mathbb{C}} \setminus \overline{T}$ containing ∞. Therefore, ∂T is a Jordan curve by Lemma 2.13: i.e., the Julia set $J(f_{a,b})$ is a Jordan curve. Actually, by a theorem of Sullivan [**4**, Theorem 2.1, p. 102], $J(f_{a,b})$ is a quasicircle.

(2) The proof of the connectivity of $J(f_{a,b})$ is similar to the one given in Case 2 of Theorem 4.4 (by constructing a polynomial-like mapping). The proof of the regularity of the boundaries of the Fatou components is similar to the one given in the second part of the proof of Theorem 1.1(1).

(3) Consider the conjugacy $1/f_{a,b}(1/z)$; the result follows immediately by (2).

(4) Since $v_+ \in \mathcal{A}(\infty) \setminus B$ and $v_- \in \mathcal{A}(0) \setminus T$, according to Proposition 2.7, every connected component in $\mathcal{A}(0) \cup \mathcal{A}(\infty)$ is simply connected. Thus the Julia set $J(f_{a,b})$ is connected. □

Proof of Theorem 1.2. By Theorems 4.4 and 4.5, it follows that if $\mathcal{A}(\infty)$ (respectively, $\mathcal{A}(0)$) attracts exactly one free critical value, then the Julia set of $f_{a,b}$ is connected. □

5. *Both free critical values do not escape*

In this section, we consider the case that v_+ and v_- do not belong to $\mathcal{A}(0) \cup \mathcal{A}(\infty)$. Hence each Fatou component in $\mathcal{A}(0) \cup \mathcal{A}(\infty)$ is simply connected.

Proof of Theorem 1.3. (1) By hypothesis, there is a Fatou component that contains exactly one free critical value. Then every Fatou component contains at most one critical value and, by Lemmas 2.9 and 4.3, the Julia set $J(f_{a,b})$ is connected.

(2) Suppose that the Fatou component $U(v_+, v_-)$ contains two free critical values. Hence $f_{a,b}$ is geometrically finite. By the classification theorem of periodic Fatou components, $U(v_+, v_-)$ is either an attracting basin (including super-attracting basin) or a parabolic basin. This means that $U(v_+, v_-)$ is either simply connected or infinitely connected. If $U(v_+, v_-)$ is infinitely connected then $J(f_{a,b})$ is disconnected. Instead, if $U(v_+, v_-)$ is simply connected then it follows, with an argument similar to the proof of Theorem 1.1(3), that all its preimages are doubly connected. □

6. *The Julia set cannot be a Cantor set of circles*

Proof of Theorem 1.4. Suppose that there exist $a, b \in \mathbb{C}^*$ and $n \geq 2$ such that the Julia set of $f_{a,b}$ is a Cantor set of circles. By the definition, $f_{a,b}$ has exactly two simply connected Fatou components and the rest of the Fatou components are all doubly connected. Note that B and T are both super-attracting basins. They are either simply connected or infinitely connected. The latter case is impossible since $J(f_{a,b})$ is a Cantor set of circles. Therefore B and T must be simply connected.

We claim that B cannot be completely invariant and all the components of $f_{a,b}^{-1}(B) \setminus B$ are doubly connected. In fact, if $f_{a,b}^{-1}(B) = B$, then ∂B is the Julia set of $f_{a,b}$ (see [**13**, Corollary 4.12, p. 48]), which contradicts the assumption that $J(f_{a,b})$ is a Cantor set of circles. Note that $f_{a,b}(T) = T$ so $f_{a,b}^{-1}(B) \cap T = \emptyset$. Since all the Fatou components of $f_{a,b}$ are annuli except B and T, this means that all the components of $f_{a,b}^{-1}(B) \setminus B$ are doubly connected. If B contains no free critical values then, by Proposition 2.7(1), each component of $f_{a,b}^{-1}(B)$ is simply connected. Therefore, in any case, $J(f_{a,b})$ is not a Cantor set of circles.

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7. *Examples*

In this section, we study some examples that correspond to the various assumptions in Theorems 1.1, 1.2 (corresponding to Theorems 4.4 and 4.5) and 1.3.

As stated in the introduction, in order to study the parameter space of $f_{a,b}$, one can study its slices instead. Since we just need to find some specific examples, we will fix these in a particular slice ($n = 3$ and $b = 1/25$)

$$f_a(z) := z^3 + \frac{25a^2}{25z^3 - 1} + 25a^2 \quad \text{where } a \in \mathbb{C}^*. \tag{7.1}$$

The parameter space of this slice has been drawn in Figure 1, in the introduction. Moreover, by (2.2),

$$v_+ = \frac{(1+25a)^2}{25} \quad \text{and} \quad v_- = \frac{(1-25a)^2}{25}. \tag{7.2}$$

7.1. *Examples corresponding to Theorem 1.1.*

PROPOSITION 7.1. *In the slice $b = 1/25$ with $n = 3$, if $|a| \geq 1$, then $v_+, v_- \in B$.*

Proof. By (7.2), if $|a| \geq 1$,

$$|v_+| \geq 20|a|^2 \quad \text{and} \quad |v_-| \geq 20|a|^2. \tag{7.3}$$

Therefore, if $|z| \geq 20|a|^2$,

$$|f_a(z)| \geq |z|^3 - \left|\frac{25a^2}{25z^3 - 1}\right| - 25|a|^2 \geq |z|^3 - \frac{5|a|^2}{|z|^3} - 25|a|^2 \geq 2|z|.$$

This means that $\{z \in \mathbb{C} : |z| \geq 20|a|^2\}$ is contained in the Fatou set of f_a and hence contained in B. By (7.3), we know that $v_+, v_- \in B$. □

Remark. It is clear that the condition on a in Proposition 7.1 is not optimal. There exist many smaller a such that $v_+, v_- \in B$. In Figure 2, we choose $a = 0.16$ and generate a disconnected Julia set such that $v_+, v_- \in B$. A reason to choose a small a is to guarantee that the reader could see a clearer structure of the Julia set. Otherwise, the Julia set will look like a circle decorating some discrete points if a is large.

For the second case of Theorem 1.1, let $a = 0.1625i$. We then have $v_\pm = -0.620\,156\,25 \pm 0.325i$. By a direct calculation,

$$f_a^{\circ 11}(v_\pm) \approx (3.41 \mp 1.38i) \times 10^{13}.$$

By a proof similar to that of Proposition 7.1, one can see that $v_\pm$ tends to ∞ under the iterations of f_a. The positions of the free critical values (points marked red) and the connectivity of the Julia set of f_a can be seen from the picture on the right in Figure 2. This provides an example corresponding to the second case of Theorem 1.1.

For the third case of Theorem 1.1, as stated in the introduction, we conjecture that this case cannot happen, although it happens for McMullen maps and the generalized McMullen maps.

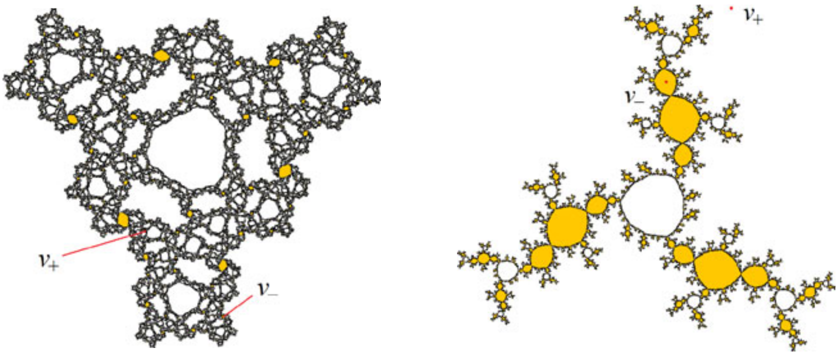

FIGURE 3. The Julia sets of f_a, where $a = -0.1175 + 0.1293i$ (left) and $a = 0.1716 + 0.1206i$ (right), which correspond to the two cases in the proof of Theorem 4.4. The Julia set on the right is quasiconformally homeomorphic to the Julia set of a polynomial with degree 6.

7.2. *Examples corresponding to Theorem 4.4.* The proof of Theorem 4.4 was divided into two cases. For the first case, we choose $a = -0.1175 + 0.1293i$. Then

$$v_+ = -0.267\,806 - 0.501\,0375i \quad \text{and} \quad v_- = 0.202\,194 - 1.018\,2375i.$$

By a direct calculation,

$$f_a^{\circ 9}(v_+) \approx (-2.57 - 2.39i) \times 10^{11}.$$

As in the proof of Proposition 7.1, one can see that $v_+ \in \mathcal{A}(\infty)$. One can check that v_+ is not contained in B (see the picture on the left in Figure 3). Moreover, a direct calculation shows that the orbit of v_- under f_a is bounded and is attracted by an attracting cycle with period three.

For the second case, let $a = 0.1716 + 0.1206i$. Then

$$v_+ = 0.755\,755 + 1.275\,948i \quad \text{and} \quad v_- = 0.069\,355 + 0.793\,548i.$$

By a direct calculation,

$$f_a^{\circ 4}(v_+) \approx (1.55 - 1.86i) \times 10^{13}.$$

As before, one can see that $v_+ \in \mathcal{A}(\infty)$. One can verify that $v_+ \in B$ (see the picture on the right in Figure 3). Moreover, a direct calculation shows that the orbit of v_- under f_a is bounded and is attracted by an attracting cycle with period two.

7.3. *Examples corresponding to Theorem 4.5.* There are four cases in Theorem 4.5. We collect four examples in Table 1 and the corresponding Julia sets are drawn in Figure 4. The positions of the free critical values can be observed from the pictures directly.

7.4. *Examples corresponding to Theorem 1.3.* There are two cases in Theorem 1.3. We now consider the examples corresponding to these two cases. For the first case, let $a = 0.1413i$. We have $v_\pm = -0.459\,142\,25 \pm 0.2826i$. By a direct calculation, one can verify that the orbit of $v_\pm$ under f_a is bounded and is attracted by an attracting fixed point $\approx -0.467\,630\,76 \pm 0.280\,705\,88i$.

TABLE 1. The parameters and the orbits corresponding to Theorem 4.5. They all satisfy $v_+ \in \mathcal{A}(\infty)$ and $v_- \in \mathcal{A}(0)$. The corresponding Julia sets are drawn in Figure 4.

Parameters	$f_a^{\circ k}(v_+)$	$f_a^{\circ l}(v_-)$
$a = 0.13$	$f_a^{\circ 7}(v_+) \approx 4.3 \times 10^{18}$	$f_a^{\circ 4}(v_-) \approx 8.2 \times 10^{-15}$
$a = -0.026 + 0.176i$	$f_a^{\circ 6}(v_+) \approx (1.1 + 2.1i) \times 10^{28}$	$f_a^{\circ 4}(v_-) \approx (2.5 - 3.4i) \times 10^{-13}$
$a = 0.095 + 0.078i$	$f_a^{\circ 5}(v_+) \approx (-1.1 - 5.0i) \times 10^{9}$	$f_a^{\circ 4}(v_-) \approx (6.7 + 0.1i) \times 10^{-11}$
$a = -0.1 + 0.097i$	$f_a^{\circ 5}(v_+) \approx (2.0 - 0.6i) \times 10^{21}$	$f_a^{\circ 7}(v_-) \approx (2.3 + 6.5i) \times 10^{-8}$

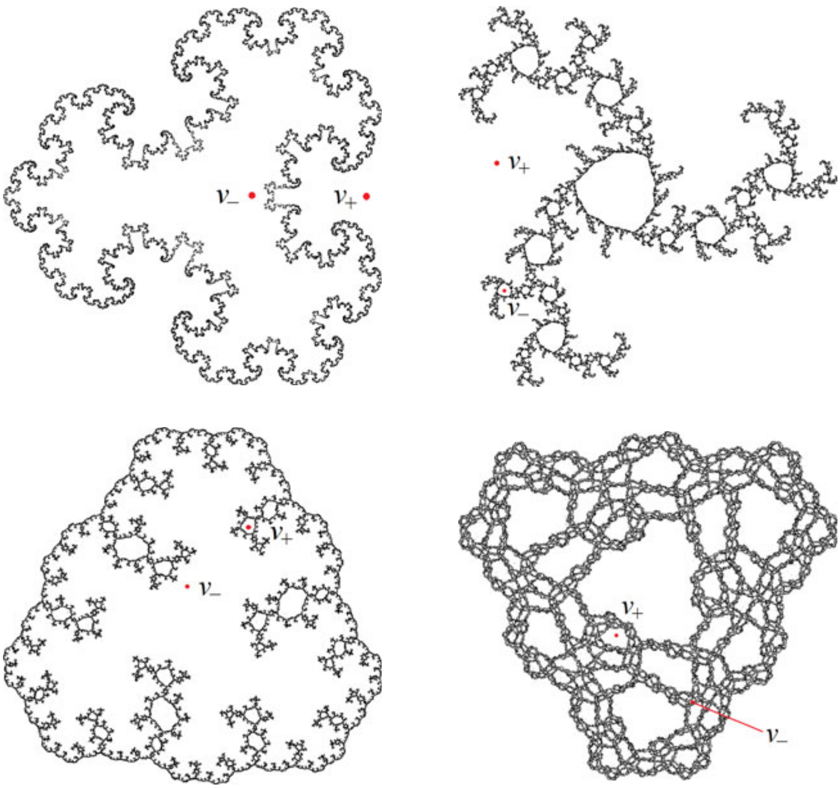

FIGURE 4. The Julia sets of f_a, where $a = 0.13$, $a = -0.026 + 0.176i$, $a = 0.095 + 0.078i$ and $a = -0.1 + 0.097i$, respectively (from the upper left-hand corner to lower right-hand corner), which correspond to the four cases in Theorem 4.5.

For the second case, let $a = 0.0142 + 0.1413i$. We have $v_+ = -0.425\,701\,25 + 0.382\,923i$ and $v_- = -0.482\,501\,25 - 0.182\,277i$. By a direct calculation, one can verify that the orbit of v_+ and v_- under f_a both are attracted by an attracting fixed point $\approx -0.418\,981\,79 + 0.371\,367\,47i$.

See Figure 5 for their Julia sets and the positions of the free critical values.

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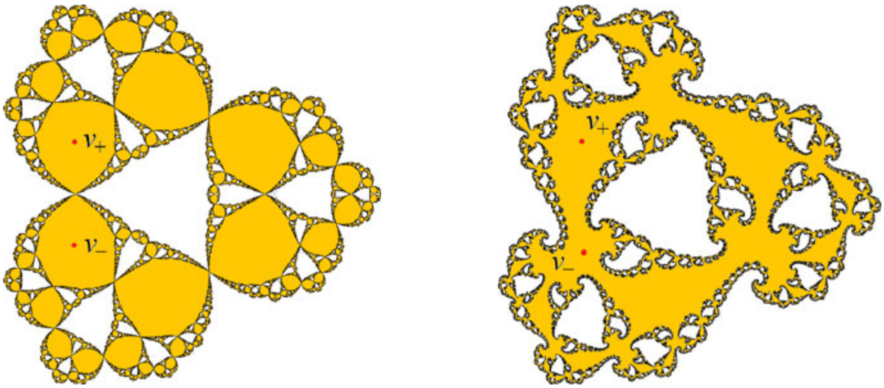

FIGURE 5. The Julia sets of f_a, where $a = 0.1413i$ (left) and $a = 0.0142 + 0.1413i$ (right), which correspond to the two cases in Theorem 1.3. The Julia set on the left is connected while the one on the right is disconnected.

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