MONOTONE FUNCTIONS MAPPING THE SET OF RATIONAL NUMBERS ON ITSELF

(IN MEMORIAM FELIX A. BEHREND)

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1.

The functions f defined by

$$f(x) = \frac{x}{cx+1-c}$$

or by

$$f(x) = \frac{x-1}{cx-1}$$

for c rational and less than $+1$ map the set of rational numbers between 0 and 1 one-to-one onto itself; and they are the only fractional linear functions with this property. Miss Tekla Taylor recently raised the question * whether these are the only differentiable functions with the stated property. In the present note we show, by two different constructions, that the answer is negative; in each case much freedom remains, which could be used to make the functions in question have various additional properties.

2.

Let P denote the set of all rational numbers, R the set of all real numbers, and C the set of all complex numbers.

THEOREM 1. *There is a function $f: R \to R$ with the following properties.*

(i) *f is differentiable and monotone increasing in R, in fact $f'(x) \geqq 1$ for all real x;*

(ii) *$f(P) = P$, that is to say, f maps the set of rational numbers onto itself;*

(iii) *f is not (entire) linear, that is to say, to all $a, b \in R$ there is an $x \in R$ such that $f(x) \neq ax + b$;*

* Oral communication. A related but simpler problem, proposed by D. G. Northcott and communicated to us by I. D. Macdonald, is solved in a note by Peter M. Neumann in INVARIANT [the journal of the Oxford University Invariant Society] **1**, 9—11 (1961). Subsequently one of us jointly with H. A. Heilbronn obtained a more general result (not published).

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(iv) *f is "locally linear" at every rational point, in the sense that to each $\rho \in P$ there are numbers α, β, $\delta \in P$, $\delta > 0$, such that for all $x \in [\rho - \delta, \rho + \delta]$,*

$$f(x) = \alpha x + \beta.$$

An immediate consequence of (iv) is that f is not properly fractional linear (that is of the form $f(x) = (ax + b)/(cx + d)$ with $c \neq 0$) in any real open interval.

THEOREM 2. *There is a function $f : C \to C$ with the following properties.*

(i) *f is differentiable on C, that is to say, an entire function;*

(ii) *$f(R) = R$ and $f(P) = P$, that is to say, f maps the sets of real numbers and of rational numbers onto themselves;*

(iii) *f is monotone increasing on the real line, in fact, $f'(x) \geqq 1$ for all $x \in R$;*

(iv) *f is not a polynomial.*

The montoneity of these functions f on R implies that they map R, and thus also P, one-to-one. Miss Taylor's question is answered by the function φ defined by

$$\varphi(x) = \frac{f(x) - f(0)}{f(1) - f(0)},$$

where f is the function either of Theorem 1 or of Theorem 2.

The proof of Theorem 1 is quite simple and short and occupies § 3. The proof of Theorem 2 is more elaborate, as it requires the construction of an analytic function, not just a real once differentiable function; it is given in § 4.

3. Proof of Theorem 1

Let

$$P = \{\rho_0, \rho_1, \rho_2, \cdots\}$$

be an enumeration of the rational numbers. It is possible to define, by simultaneous induction, integers $\lambda(n) \geqq 0$, closed intervals $I_n \subset R$ of positive length and with irrational endpoints, rational numbers α_n, β_n, and differentiable functions $f_n : R \to R$, for $n = 0, 1, 2, \cdots$, such that, let us say

$$\lambda(0) = 0, I_0 = [\rho_0 - \sqrt{2}, \rho_0 + \sqrt{2}], \alpha_0 = 2, \beta_0 = 0, f_0(x) = 2x$$

and such that, further, with the abbreviation

$$J_n = I_0 \cup I_1 \cup \cdots \cup I_{n-1},$$

we have for $n = 1, 2, 3, \cdots$

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(1) $$I_n \cap J_n = \emptyset,$$

(2) $$f_n(x) = f_{n-1}(x) \quad \text{for all} \quad x \in J_n,$$

(3) $$f_n(x) = \alpha_n x + \beta_n \quad \text{for all} \quad x \in I_n,$$

(4) $$\alpha_n \neq \alpha_m \quad \text{for} \quad m < n,$$

(5) $$|f'_n(x) - f'_{n-1}(x)| < 2^{-n} \quad \text{for all} \quad x \in R.$$

Finally, we stipulate that

(a) if $n = 2, 4, 6, \cdots$ then $\lambda(n)$ is the least integer $\lambda \geqq 0$ such that $\rho_\lambda \notin J_n$, and I_n is so chosen that $\rho_{\lambda(n)} \in I_n$;

(b) if $n = 1, 3, 5, \cdots$ then $\lambda(n)$ is the least integer $\lambda \geqq 0$ such that $\rho_\lambda \notin f_{n-1}(J_n)$, and then I_n, α_n, β_n are so chosen that $\rho_{\lambda(n)} \in f_n(I_n)$.

We first remark that from the definition of f_0 and from (5) we have

(6) $$1 < f'_n(x) < 3 \quad \text{for} \quad n = 0, 1, 2, \cdots \quad \text{and all} \quad x \in R;$$

moreover, still by (5), $\lim_{n\to\infty} f'_n$ exists uniformly in R. Also, by repeated application of (2)

$$f_n(\rho_0) = f_0(\rho_0) \quad \text{for} \quad n = 0, 1, 2, \cdots,$$

so that $\lim_{n\to\infty} f_n(\rho_0)$ (trivially) exists. Hence

$$\lim_{n\to\infty} f_n = f$$

exists on R and is differentiable, and

$$f'(x) = \lim_{n\to\infty} f'_n(x) \geqq 1 \quad \text{for all} \quad x \in R.$$

This proves (i).

Next, if $\rho \in P$, then there is an integer $m \geqq 0$ such that $\rho \in I_m$, by (a). Then for all $n \geqq m$,

$$f_n(\rho) = f_m(\rho) = \alpha_m \rho + \beta_m \in P,$$

by repeated application of (2), (3), and the choice of α_m, $\beta_m \in P$; thus

$$f(\rho) = \lim_{n\to\infty} f_n(\rho) = f_m(\rho) \in P,$$

and it follows that $f(P) \subseteqq P$.

Again, if $\sigma \in P$, then there is an integer $k \geqq 0$ such that $\sigma \in f_k(I_k)$, by (b). Thus there is $\rho \in I_k$ such that

$$\sigma = \alpha_k \rho + \beta_k,$$

and as $\alpha_k \neq 0$ — an obvious consequence of (3) and (6) — we have $\rho \in P$. Again we have as before $f_n(\rho) = f_k(\rho)$ for $n \geqq k$, and

$$f(\rho) = \lim_{n\to\infty} f_n(\rho) = f_k(\rho) = \sigma,$$

and it follows that $P \subseteqq f(P)$, completing the proof of (ii). Property (iii) is a consequence of (4), and (iv) follows from the fact that the endpoints of all I_n have been chosen irrational. This completes the proof of Theorem 1.

4. Proof of Theorem 2

We adopt the convention that x ranges over the real numbers, z over the complex numbers, and m, n over the non-negative integers.

Let two sequences of (not necessarily distinct) rational numbers

$$\pi_0 = 0, \pi_1, \pi_2, \cdots \quad \text{and} \quad a_0 = 2, a_1, a_2, \cdots \tag{7}$$

be chosen so that for $m \geqq 1$

$$-m \leqq \pi_m \leqq m \quad \text{and} \quad 0 < a_m \leqq (2m)^{-m^2-m-2}. \tag{8}$$

We define polynomials $p_0, p_1, p_2, \cdots$ and $f_0, f_1, f_2, \cdots$ by

$$p_m(z) = z^{m^2+1}(z-\pi_1)(z-\pi_2)\cdots(z-\pi_m),$$

$$f_n = \sum_{m=0}^{n} a_m p_m.$$

Then we have, for $m \geqq 1$ and $|z| \leqq m$,

$$|a_m p_m(z)| \leqq a_m m^{m^2+1}(2m)^m \leqq 2^{-m^2} \leqq 2^{-m}, \tag{9}$$

$$\begin{aligned} |a_m p'_m(z)| &\leqq a_m\big((m^2+1)m^{m^2}(2m)^m + m^{m^2+1}m(2m)^{m-1}\big) \\ &\leqq a_m m^{m^2}(2m)^{m+2} \leqq 2^{-m^2} \leqq 2^{-m}. \end{aligned} \tag{10}$$

Hence we may define a function $f: C \to C$ by

$$f = \lim_{n\to\infty} f_n = \sum_{m=0}^{\infty} a_m p_m,$$

and as (9) ensures the uniform convergence in every circle, this function is entire. Also, by (10), we have

$$f' = \lim_{n\to\infty} f'_n = \sum_{m=0}^{\infty} a_m p'_m.$$

We note that the only powers of the variable that actually occur in p_m are among those with exponents $m^2+1, m^2+2, \cdots, m^2+m+1$, and the last of these has coefficient 1; hence the different p_m contribute distinct powers of the variable to f, and in the power series expansion of f about the origin, infinitely many powers occur with non-zero coefficients. It follows that f is not a polynomial.

Next we note that $p_m, p_{m+1}, p_{m+2}, \cdots$ all vanish at $z = \pi_m$, whence

$$f(\pi_m) = f_n(\pi_m) \quad \text{for} \quad m \leqq n + 1; \tag{11}$$

this is a rational number, and if we ensure that all rational numbers occur among the π_m, then we shall have $f(P) \subseteqq P$.

However, before carrying this out, we prove that f is monotone on the real line; to this end we show by induction that

$$f'_n(x) \geqq 1 + 2^{-n} \quad \text{for all} \quad x \in R. \tag{12}$$

As $f_0 = p_0$ is defined by $f_0(z) = 2z$, (12) is true for $n = 0$. Let now $m \geqq 0$ be fixed and assume (12) is valid for $n = m$. Then, if $|x| \leqq m + 1$, we apply (10) and obtain

$$f'_{m+1}(x) = f'_m(x) + a_{m+1}p'_{m+1}(x) \geqq 1 + 2^{-m} - 2^{-m-1} = 1 + 2^{-m-1}.$$

If $|x| > m + 1$, then $p'_{m+1}(x) > 0$ since p_{m+1} is a monic polynomial of odd degree whose roots are real and contained in the interval $[-m - 1, m + 1]$. Hence

$$f'_{m+1}(x) = f'_m(x) + a_{m+1}p'_{m+1}(x) \geqq 1 + 2^{-m} + 0 \geqq 1 + 2^{-m-1}.$$

This proves (12) for $n = m + 1$, and thus (12) is true for all n. The monotoneity of f follows at once:

$$f'(x) = \lim_{n \to \infty} f'_n(x) \geqq 1 \quad \text{for all} \quad x \in R.$$

It only remains to specialize the sequences (7) of rational numbers, subject to (8), so that

$$f(P) = P. \tag{13}$$

Let again, as in the proof of Theorem 1,

$$P = \{\rho_0, \rho_1, \rho_2, \cdots\} \tag{14}$$

be an enumeration of the rational numbers, which we now choose so that

$$|\rho_m| \leqq m \quad \text{for all} \quad m \geqq 0.$$

Recall that $\pi_0 = 0$ and $a_0 = 2$, so that p_0 and f_0 are already given. The coefficients a_m for $m \geqq 1$ can be chosen arbitrarily; let us put

$$a_m = (2m)^{-m^2-m-2} \quad \text{for} \quad m \geqq 1.$$

We define π_n inductively as follows. Suppose $\pi_0, \pi_1, \cdots, \pi_{n-1}$ have been determined, where n is fixed, $n \geqq 1$; then p_m and f_m are defined for $m < n$.

Case 1. Let $n = 3k + 1$. We put $\pi_n = \rho_k$. Then $|\pi_n| \leqq k < n$, as required by (8). We thus ensure that all rational numbers occur among the π_n, and thus that

(15) $$f(P) \subseteqq P.$$

Case 2. Let $n = 3k+2$; we then define both π_n and π_{n+1}. Determine $\xi \in R$ from the equation

$$f_{n-1}(\xi) = \rho_k.$$

There is precisely one such ξ, and as $|\rho_k| \leqq k < n$, and as $f_{n-1}(0) = 0$ and $f'_{n-1}(x) > 1$ for all $x \in R$ (see (12)), also $|\xi| < n$.

Case 2a. Let $\xi \in P$. Then we put $\pi_n = \pi_{n+1} = \xi$. Then $|\pi_n| \leqq n$ and $|\pi_{n+1}| \leqq n+1$, as required by (8); and, by (11),

(16) $$f(\pi_n) = f_{n-1}(\pi_n) = \rho_k.$$

Case 2b. Let ξ be irrational. Define a function F by

$$F(x) = \frac{f_{n-1}(x) - \rho_k}{a_n x^{2n-1} p_{n-1}(x)} + x.$$

This function exists and is continuous in a neighbourhood of ξ, that is for $|x - \xi| < \delta$ with a suitable $\delta > 0$; and

$$|F(\xi)| = |\xi| < n.$$

Hence there is a $\rho \in P$ such that $|\rho| < n$ and $|F(\rho)| < n$. Note that also $F(\rho) \in P$. We put

$$\pi_n = F(\rho), \qquad \pi_{n+1} = \rho.$$

Then $|\pi_n| \leqq n$ and $|\pi_{n+1}| \leqq n+1$, as required by (8); and

$$\pi_n = F(\rho) = \frac{f_{n-1}(\rho) - \rho_k}{a_n \rho^{2n-1} p_{n-1}(\rho)} + \rho,$$

$$\rho_k = f_{n-1}(\rho) + a_n \rho^{2n-1} p_{n-1}(\rho)(\rho - \pi_n) = f_{n-1}(\rho) + a_n p_n(\rho) = f_n(\rho);$$

thus, using (11),

(17) $$f(\pi_{n+1}) = f_n(\pi_{n+1}) = f_n(\rho) = \rho_k.$$

Now (16) and (17) combine to show that every rational number is of the form

$$\rho_k = f(\pi_{3k+2}) \quad \text{or} \quad \rho_k = f(\pi_{3k+3}),$$

and so

$$P \subseteqq f(P).$$

In conjunction with (15) this shows (13) and completes the proof of Theorem 2.

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