SUMS OF FUNCTIONS OF DIGITS

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1. Introduction. We generalize in several directions a paper by Porges **(2)** who considered the integer $F(A)$ obtained from the positive integer A by taking the sum of the squares of the digits of A. Porges showed that if $A > 99$, then $F(A) < A$, so that under iteration of $F(A)$ all the positive integers are divided into a finite number of classes, called orbits in the terminology of Isaacs **(1)**, each containing a finite cycle. For his $F(A)$ Porges showed there are only two orbits: one with the 1-cycle: $1 \to 1$; and the other with the interesting 8-cycle: $4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$.

Consider the set Z of non-negative integers and choose as a base of enumeration any desired integer $B \geqq 2$ (not necessarily $B = 10$). Then only the "digits" $0, 1, 2, \ldots, B-1$ are needed, in suitable multiplicity, to represent any A of Z. Suppose there is given an arbitrary function assigning to each digit a the value $P(a)$ in Z. (In Porges' example the special function used is $P(a) = a^2$.) Each A in Z has a unique representation to the base B, hence if $F(A)$ is defined to be the sum of the values of $P(a)$, summed over all the digits of A, then not only is $F(A)$ well-defined, but also $F(A)$ is an integer of Z, so $F(F(A))$ is meaningful and continued iteration is possible.

More precisely, let a amd a_i be restricted to the set $0, 1, 2, \ldots, B-1$ and let a_i' be restricted to the subset $1, 2, \ldots, B-1$. Then any integer A in the range $B^k \leqq A < B^{k+1}$, $k > 0$, has a unique representation

$$A = a_k' B^k + \sum_0^{k-1} a_i B^i.$$

After $P(a)$ has been given, we make the definitions

$$F(a) = P(a), \qquad F(A) = P(a_k') + \sum_0^{k-1} P(a_i),$$

and thus obtain the type of function which suggested the title of this paper.

We propose to study the growth of the function $F(A)$ and to exhibit certain regularities in the behaviour of $F(A)$ despite the arbitrariness of $P(a)$. For example, it proves easy to demonstrate (Theorem 1) the *existence* of an integer C such that $F(C) \geqq C$ and $F(A) < A$ for every $A > C$. Then a more detailed analysis is presented, using an auxiliary constant S, to construct an algorithm (Theorem 2) for the *evaluation* of C. As an aid in finding the value of S, certain other constants J and L are introduced and they provide further interesting sidelights (Theorems 3, 4, 5) on the behaviour of $F(A)$.

Received January 2, 1957; in revised form January 5, 1960.

https://doi.org/10.4153/CJM-1960-032-9 Published online by Cambridge University Press

These general results are applied to the special case $P(a) = a^t$ with considerable effectiveness (Theorems 6, 7, 8).

A preliminary study is made of the orbit- and cycle-numbers resulting from the iteration of $F(A)$ and the finiteness of these numbers is assured. The teasing irregularities of these numbers are shown by selected tables.

Finally, a brief section is presented concerning products of functions of digits.

2. Existence of C. If proving the existence of C is the only concern, we may assume merely that $P(z)$ is a complex function for which $P(a)$ is defined for every a. Define $F(A)$ as above.

THEOREM 1. *To any real* $\epsilon > 0$ *there corresponds an integer* $C = C(\epsilon)$ *such that* $|F(C)| \geqq \epsilon C$ *and such that* $|F(A)| < \epsilon A$ *for every* $A > C$.

Proof. Let P be the maximum value of $|P(a)|$. Since $B^k/(k+1)$ is increasing and unbounded for $k = 0, 1, 2, \ldots,$ there exists $K = K(\epsilon, P)$ such that $B^k/(k+1) > P/\epsilon$ when $k > K$. If $B^k \leqq A < B^{k+1}$, then $|F(A)| \leqq (k+1)P < \epsilon B^k \leqq \epsilon A$ for all $k > K$. Also $|F(0)| \geqq 0$. Hence C exists, $0 \leqq C < B^{K+1}$.

In the sequel our intention to study iteration of $F(A)$ leads us to insist that the values of $P(a)$ be in Z and to avoid painful details we discuss only the case $\epsilon = 1$. As an aside, note that by the usual interpolation formula there exists a polynomial $P_1(x)$ with rational coefficients and degree at most $B - 1$ which will take on for the set $\{a\}$ the prescribed values $\{P(a)\}$. However, it may be convenient to use polynomials of degree higher than $B - 1$, but of simpler structure, as in the case $P(a) = a^t$ when $t \geqq B$.

3. Algorithm for C. Let $H(A) = F(A) - A$ and $H_i(a) = P(a) - aB^i$ for $i \geqq 0$. If $B^k \leqq A < B^{k+1}$, then for $k > 0$,

$$H(A) = H_k(a_k') + \sum_0^{k-1} H_i(a_i).$$

The properties defining C when $\epsilon = 1$ may now be restated:

$$H(C) \geqq 0, \; H(A) < 0 \text{ for every } A > C.$$

Let m_i be the maximum value of a for which $H_i(a)$ is a maximum, and let m_i' be the maximum value of a' for which $H_i(a')$ is a maximum. Then in the range $B^k \leqq A < B^{k+1}$ when $k > 0$, the maximum value U_k of $H(A)$ is given by $U_k = H(M_k)$, where

$$M_k = m_k'B^k + \sum_0^{k-1} m_i B^i;$$

when $k = 0$, $U_0 = H(m_0')$. Define $U_{-1} = H(0)$.

Define the integer S by the conditions $U_S \geqq 0$ and $U_k < 0$ for every $k > S$. The existence of S follows immediately from $U_{-1} = P(0) \geqq 0$ and from

Theorem 1, since $H(A) < 0$ for $A > C$ implies $U_k < 0$ for every $k > K$. Hence $-1 \leqq S \leqq K$. (In the next section we give much improved estimates of S.) These observations establish the following

LEMMA. *If* $S = -1$, $C = 0$. *If* $S \geqq 0$, $B^S \leqq M_S \leqq C < B^{S+1}$.

To determine the exact value of C when S is known and $S \geqq 0$, consider

$$U_S = H_S(m'_S) + \sum_0^{S-1} H_i(m_i).$$

Determine a maximum c_S' such that

$$H_S(c'_S) + U_S - H_S(m'_S) \geqq 0. \tag{1}$$

This selection is possible with $B - 1 \geqq c_S' \geqq m_S' \geqq 1$, for at least the choice $c_S' = m_S'$ makes (1) hold, since $U_S \geqq 0$.

Next (assuming $S > 0$) determine a maximum c_{S-1} such that

$$H_{S-1}(c_{S-1}) + H_S(c'_S) + U_S - H_S(m'_S) - H_{S-1}(m_{S-1}) \geqq 0.$$

This choice is possible with $B - 1 \geqq c_{S-1} \geqq m_{S-1}$, for at least the choice $c_{S-1} = m_{S-1}$ is valid, because of the previous step (1).

Proceed recursively from $i + 1$ to i, $S > i \geqq 0$, choosing a maximum c_i such that

$$\begin{aligned} H_i(c_i) + H_{i+1}(c_{i+1}) + \ldots + H_S(c'_S) + U_S \\ - H_S(m'_S) - \ldots - H_{i+1}(m_{i+1}) - H_i(m_i) \geqq 0. \end{aligned} \tag{2}$$

This choice is possible with $B - 1 \geqq c_i \geqq m_i$, for at least $c_i = m_i$ is a valid choice, because of the previous step in the algorithm.

THEOREM 2. *For* $S \geqq 0$, *let*

$$Q = c'_S B^S + \sum_0^{S-1} c_i B^i.$$

Then $Q = C$.

Proof. When $i = 0$, the inequality (2) shows that $H(Q) \geqq 0$. If $B^k \leqq A < B^{k+1}$ and $k > S$, then $H(A) \leqq U_k < 0$, by the definitions of U_k and S. If every digit of Q is $B - 1$, it follows that $C = Q = B^{S+1} - 1$.

Otherwise, suppose some digits of Q are less than $B - 1$. Then for each

$$A = a'_S B^S + \sum_0^{S-1} a_i B^i$$

in the range $Q < A < B^{S+1}$, there must be an index i, $S \geqq i \geqq 0$, such that either $B - 1 \geqq a_S' > c_S'$; or $a_S' = c_S'$ and $a_j = c_j$ when $j > i$, but $B - 1 \geqq a_i > c_i$.

In the first case, because of the maximum property of $H_i(m_i)$,

$$H(A) \leqq H_S(a'_S) + \sum_0^{S-1} H_i(m_i) = H_S(a'_S) + U_S - H_S(m'_S) < 0,$$

where the last strict inequality follows from $a_S' > c_S'$ and the maximum property of c_S' expressed in (1).

In the second case, because of the maximum property of $H_r(m_r)$,

$$H(A) \leqq H_S(c_S') + \ldots + H_{i+1}(c_{i+1}) + H_i(a_i) + \sum_0^{i-1} H_r(m_r)$$
$$= H_i(a_i) + H_{i+1}(c_{i+1}) + \ldots + H_S(c_S') + U_S - H_S(m_S') - \ldots$$
$$- H_i(m_i) < 0,$$

where the last strict inequality follows from $a_i > c_i$ and the maximum property of c_i expressed in (2).

Since we have shown $H(Q) \geqq 0$ and $H(A) < 0$ for every $A > Q$, it follows that $Q = C$.

In the following example $B = 4$. The table shows $P(a)$, $H_i(a)$ and U_i with a double underline for $H_i(m_i)$ and, if there is a distinction, a single underline for $H_i(m_i')$. All entries are written in the usual way with base 10.

TABLE I

EXAMPLE 1.		B^i	1	4	16	64	256	1024
a	$P(a)$	i	0	1	2	3	4	5
0	100	$H_i(0)$	100	100	100	$\underline{\underline{100}}$	$\underline{\underline{100}}$	$\underline{\underline{100}}$
1	50	$H_i(1)$	49	46	34	-14	$\underline{-206}$	$\underline{-974}$
2	200	$H_i(2)$	$\underline{\underline{198}}$	$\underline{\underline{192}}$	$\underline{\underline{168}}$	$\underline{72}$	-312	-1848
3	10	$H_i(3)$	7	-2	-38	-182	-758	-3062
		U_i	198	390	558	630	452	-216

With the aid of the later Corollary 5.1, we may see from this table that $S = 4$. Then starting from $M_4 = B^4 + 2B^2 + 2B + 2$, the algorithm of Theorem 2 is the following. Replacing $m_4 = 1$ by $a = 2$ gives $H = 346$, but by $a = 3$ gives $H = -100$, hence $c_4 = 2$. Next, replacing $m_3 = 0$ by $a = 3$ gives $H = 346 - 100 - 182 = 64$, hence $c_3 = 3$. No further replacements are possible: $c_2 = m_2$, $c_1 = m_1$, $c_0 = m_0$. Thus $C = 2B^4 + 3B^3 + 2B^2 + 2B + 2$.

4. Growth properties of $F(A)$**.** In this section we obtain further properties of $H(A) = F(A) - A$ and since our chief concern is what happens to $H(A)$ as A increases, we describe these as growth properties of $F(A)$.

Let R be the maximum value of $(P(a') - P(0))/a'$.

If $R < 1$, define $J = 0$. If $1 \leqq R$, define J by $B^{J-1} \leqq R < B^J$.

THEOREM 3. *If* $i \geqq J$, $m_i = 0$. *If* $i < J$, $m_i = m_i'$.

Proof. Note that $H_i(0) - H_i(a') = P(0) - P(a') + a'B^i > 0$ holds if $B^i > R$, hence for $i \geqq J$. But when $i < J$, suppose $R = (P(m') - P(0))/m'$ and note that $H_i(0) - H_i(m') \leqq 0$.

COROLLARY 3.1. $S \geqq J - 1$.

Proof. If $J = 0$, the statement $S \geqq -1$ is trivial. If $J > 0$ and $i < J$, then it follows from Theorem 3 that $H_i(m_i') \geqq H_i(0) = P(0) \geqq 0$. Hence for $k < J$, $U_k \geqq 0$, therefore $S \geqq J - 1$.

COROLLARY 3.2. *There exists an integer J_1 such that for $i \geqq J_1$, $m_i' = 1$; and if $i < J_1$, then $m_i' > 1$.*

Proof. The proof exactly parallels that of Theorem 3, starting with R_1 as the maximum value of $(P(a') - P(1))/(a' - 1)$ for all $a' > 1$, and defining $J_1 = 0$, if $R_1 < 1$; but otherwise, defining J_1 by $B^{J_1 - 1} \leqq R_1 < B^{J_1}$.

Example 1 provides an illustration of these results wherein $J = 3$, $J_1 = 4$.

COROLLARY 3.3. *The following relations hold:*

$$U_{i+1} - U_i = H_{i+1}(m'_{i+1}), \qquad 0 \leqq i < J; \tag{3}$$

$$U_{i+1} - U_i = H_{i+1}(m'_{i+1}) + P(0) - H_i(m'_i), \qquad J \leqq i. \tag{4}$$

Proof. In the sums representing U_{i+1} and U_i, the terms with index $j \leqq i - 1$ are the same, hence

$$U_{i+1} - U_i = H_{i+1}(m'_{i+1}) + H_i(m_i) - H_i(m'_i).$$

When $0 \leqq i < J$, the second part of Theorem 3 shows $H_i(m_i) = H_i(m_i')$ which establishes (3). When $J \leqq i$, the first part of Theorem 3 shows $H_i(m_i) = H_i(0) = P(0)$ which establishes (4).

COROLLARY 3.4. *Let J_2 be the maximum of J and J_1. If $i \geqq J_2$, then $U_{i+1} - U_i = P(0) - B^i(B - 1)$.*

Proof. From $i \geqq J_2 \geqq J$, relation (4) holds. From $i \geqq J_2 \geqq J_1$, Corollary 3.2 shows $m_{i+1}' = m_i' = 1$, hence

$$H_{i+1}(m'_{i+1}) - H_i(m'_i) = P(1) - B^{i+1} - (P(1) - B^i),$$

thus (4) reduces to the stated form.

THEOREM 4. *For $i \geqq 0$, $B^i(B - 1) \leqq H(m_i') - H_{i+1}(m_{i+1}') \leqq B^i(B - 1)^2$.*

Proof. From the maximum property of $H_i(m_i')$ it follows that

$$\begin{aligned} H_i(m'_i) - H_{i+1}(m'_{i+1}) &\geqslant H_i(m'_{i+1}) - H_{i+1}(m'_{i+1}) \\ &= m'_{i+1}B^i(B - 1) \geqq B^i(B - 1). \end{aligned}$$

From the maximum property of $H_{i+1}(m_{i+1}')$ it follows that

$$H_i(m'_i) - H_{i+1}(m'_{i+1}) \leqq H_i(m'_i) - H_{i+1}(m'_i) = m'_iB^i(B - 1) \leqq B^i(B - 1)^2.$$

COROLLARY 4.1. $m_{i+1}' \leqq m_i'$.

https://doi.org/10.4153/CJM-1960-032-9 Published online by Cambridge University Press

Proof. In the displayed steps of the proof of Theorem 4, note that

$$m'_{i+1}B^i(B-1) \leqq H_i(m'_i) - H_{i+1}(m'_{i+1}) \leqq m'_iB^i(B-1).$$

Define L to be the minimum integer such that $U_{L+1} < U_L$ and such that if $J > 0$, then $L \geqq J - 1$; but if $J = 0$, then $L \geqq J$.

We appeal to Corollary 3.4, with i sufficiently large, to show that L must exist. (The existence of L may be shown also by the existence of S and by Corollary 3.1, except for the case $J = 0$ and $S = -1$.)

THEOREM 5. *If* $i \geqq L$, then $U_{i+1} < U_i$.

Proof. The proof is by induction on i with the case L serving as the base for the induction. When $i \geqq J + 1$, it follows from (4) and Theorem 4 that

$$\begin{aligned} U_{i+1} - U_i = H_{i+1}(m'_{i+1}) + P(0) - H_i(m'_i) &\leqq P(0) - B^i(B-1) \\ < P(0) - B^{i-1}(B-1)^2 &\leqq P(0) + H_i(m'_i) - H_{i-1}(m'_{i-1}) \\ &= U_i - U_{i-1}. \end{aligned}$$

When $J = 0$ this completes the proof, since $i - 1 \geqq L \geqq J$ implies $i \geqq J+1$.

When $J > 0$ the above argument is valid except for the one possibility $i - 1 = L = J - 1$. But then using $P(0) = H_{J-1}(0)$, the second part of Theorem 3, and (3), we may modify the last displayed line to read

$$H_{J-1}(0) + H_J(m'_J) - H_{J-1}(m'_{J-1}) \leqq H_J(m'_J) = U_J - U_{J-1},$$

which completes the proof.

COROLLARY 5.1. *If* $E \geqq L$ *and if* $U_E \geqq 0$ but $U_{E+1} < 0$, *then* $E = S$.

Proof. Theorem 5 shows $U_k \leqq U_{E+1} < 0$ for every $k > E$. Hence $E = S$.

As an application of this corollary note in Example 1 that $L = 3$, $U_4 > 0$, $U_5 < 0$, consequently $S = 4$.

COROLLARY 5.2. *If* $J > 0$ *and* $i < J - 1$, *then* $U_{i+1} \geqq U_i$.

Proof. If $U_0 < U_{-1}$, then $P(m') - m' < P(0)$ implies $R < 1$ and $J = 0$. So the hypothesis $J > 0$ implies $U_0 \geqq U_{-1}$. Since $i < J - 1$, $i + 1 \leqq J - 1$, and $R = (P(m') - P(0))/m' \geqq B^{J-1} \geqq B^{i+1}$ which implies $P(m') - m'B^{i+1} \geqq P(0)$. Then for $J - 1 > i \geqq 0$, relation (3) holds, so that

$$U_{i+1} - U_i = H_{i+1}(m'_{i+1}) \geqq H_{i+1}(m') = P(m') - m'B^{i+1} \geqq P(0) \geqq 0.$$

Corollary 5.2 indicates that when $J > 0$, the condition $L \geqq J - 1$ is necessary if we are to have $U_{L+1} < U_L$. Thus the search for S, initiated in Corollary 3.1 and made explicit in Corollary 5.1, should begin at this point $L \geqq J - 1$.

However, when $J = 0$, the added condition $L \geqq J$ plays a different role. For $J = 0$ implies $R < 1$, hence $U_0 = P(m_0') - m_0' < P(0) = U_{-1}$, but this does not imply $U_1 < U_0$ as the following example shows.

TABLE II

EXAMPLE 2.		B^i	1	4	16	64	256	1024
a	$P(a)$	i	0	1	2	3	4	5
0	100	$H_i(0)$	100	100	100	100	100	100
1	90	$H_i(1)$	89	86	74	26	−166	−934
2	80	$H_i(2)$	78	72	48	−48	−432	−1968
3	70	$H_i(3)$	67	58	22	−122	−698	−3002
		U_i	89	186	274	326	234	−434

In Example 2, $J = 0$ and $U_0 = 89 < 100 = U_{-1}$. However, $L = 3$ and $S = 4$. Starting from $M_4 = B^4$ we find by the algorithm of Theorem 2 that $C = B^4 + 3B^3 + 1$.

5. The case $P(a) = a^t$**.** If $P(a) = a^t$ where t is a fixed positive integer, there are two trivial cases. If $t = 1$, it is obvious that $C = B - 1$ for every B. If $B = 2$, it is obvious that $C = 1$ for every t.

THEOREM 6. *If* $P(a) = a^t$, $t > 1$, $B > 2$, *then* $0 < J \leqq S \leqq t$ *and* S *may be determined by:* $B^S \leqq J(B-1)^t - (B^J - 2) < B^{S+1}$.

Proof. Since $P(0) = 0$, $R = (B-1)^{t-1}$, so that J is determined by $B^{J-1} < (B-1)^{t-1} < B^J$. The condition $t > 1$ implies $J > 0$. Moreover, $(B-1)^{t-1} < B^{t-1}$, hence $J \leqq t - 1$.

Since $t > 1$, $H_i(x) = x^t - xB^i$ is concave upward for $x > 0$. Hence m_i' is either 1 or $B - 1$. Note that

$$H_i(1) - H_i(B-1) = 1 + (B-2)B^i - (B-1)^t.$$

When $i \leqq J - 1$,

$$(B-2)B^i < (B-1)B^{J-1} < (B-1)^t,$$

so that $m_i' = B - 1$. When $i \geqq J + 1$,

$$(B-2)B^i \geqq (B-2)B^{J+1} > (B-2)(B-1)^t \geqq (B-1)^t,$$

so $m_i' = 1$. From Theorem 3, it follows that $m_i = m_i' = B - 1$ for $i \leqq J - 1$, and $m_i = 0$ for $i \geqq J$. The only undecided case is m_J' which is either 1 or $B - 1$.

From the preceding results

$$(5)\quad \begin{cases} U_{J-1} = \sum_0^{J-1} H_i(B-1) = \sum_0^{J-1} ((B-1)^t - (B-1)B^i) \\ \qquad\qquad = J(B-1)^t - (B^J - 1); \\ U_J \quad = H_J(m_J') + U_{J-1} \geqq H_J(1) + U_{J-1}; \\ U_i \quad = H_i(1) + U_{J-1}, \qquad \text{for} \qquad i > J. \end{cases}$$

Using (5) and $(B-1)^{t-1} \geqq B^{J-1}+1$, we may show $U_J \geqq 0$ as follows:

$$\begin{aligned} U_J &\geqq H_J(1) + U_{J-1} = J(B-1)^t - 2(B^J - 1) \\ &\geqq J(B-1)(B^{J-1}+1) - 2(B^J-1) = B^{J-1}((J-2)B - J) \\ &\qquad\qquad + J(B-1) + 2. \end{aligned}$$

If $J = 1$ or 2 the last expression is 0. If $J \geqq 3$, the last expression is positive, for $B > 2$ implies $(J-2)B \geqq J$. Hence $U_J \geqq 0$, so $S \geqq J$ (a bit more than Corollary 3.1).

If $i > t$, then $B^i \geqq B^{t+1}$; and also from $J \leqq t-1$, we have $i > J+1$. We combine these observations with (5) to see that if $i > t$, then

$$\begin{aligned} U_i &= 1 - B^i + J(B-1)^t - (B^J - 1) \\ &< (t-1)(B-1)^t - B^{t+1} < (B-1)^t + (t+1)(B-1)^t - B^{t+1} \\ &< ((B-1)^{t+1} + (t+1)(B-1)^t + \ldots + 1) - B^{t+1} \\ &= (B-1+1)^{t+1} - B^{t+1} = 0. \end{aligned}$$

Since $U_i < 0$ for $i > t$, it follows that $S \leqq t$.

In the proof that $S \geqq J$ we showed that $H_J(1) + U_{J-1} \geqq 0$ which implies $B^J \leqq 1 + U_{J-1}$. From (5) we have $U_i = 1 + U_{J-1} - B^i$ when $i > J$, hence we see that S (with $U_S \geqq 0$ and $U_k < 0$ for all $k > S$) is determined by

$$B^S \leqq 1 + U_{J-1} < B^{S+1}.$$

This result together with (5) completes the proof of Theorem 6.

In general, to find C we must next apply the algorithm of Theorem 2. However, in many cases we can say considerably more, as the following theorem indicates.

THEOREM 7. *If* $P(a) = a^t$, $t > 1$, $B > 2$, *then* $C < (t-1)B^t$. *If* $B > T = (1-(1-t^{-1})^{1/t})^{-1}$ (*which includes all* $B \geqq t^2$) *then*

$$C = (t-1)B^t - 1.$$

Proof. From Theorem 6, $S \leqq t$, hence $C < B^{t+1}$. Suppose that $C < (t-1)B^t$ is false. Then $B^{t+1} > C \geqq (t-1)B^t$ implies $B \geqq t$ and

$$C = c_t' B^t + \sum_0^{t-1} c_i B^i$$

with $B - 1 \geqq c_t' \geqq t-1$. But then it follows that

$$\begin{aligned} C &\geqq c_t'(B-1+1)^t + c_{t-1}B^{t-1} \\ &= c_t'((B-1)^t + t(B-1)^{t-1} + \ldots + 1) + c_{t-1}B^{t-1} \\ &> c_t'(B-1)^t + c_t' t(B-1)^{t-1} + c_{t-1}B^{t-1} \end{aligned}$$

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$$> (t-1)(B-1)^t + (c_t')^t + (c_{t-1})^t$$

$$\geqq (c_t')^t + \sum_0^{t-1} (c_i)^t = F(C).$$

The inequality $C > F(C)$ is a contradiction of one of the defining properties of C. Therefore $C < (t-1)B^t$ is true, as stated in the first part of Theorem 7.

It is natural to ask for $B \geqq t$ whether $Q = (t-1)B^t - 1$ will serve as C. Since $F(Q) = (t-2)^t + t(B-1)^t$, the inequality $F(Q) \geqq Q$ will hold if $t(B-1)^t \geqq (t-1)B^t$. This is readily brought to the form

$$B > T = (1 - (1 - t^{-1})^{1/t})^{-1}.$$

Since $(1 - t^{-2})^t > 1 - t^{-1} > (1 - t^{-1})^t$, it follows that $t^2 > T > t$. These observations complete the proof of Theorem 7.

In the remaining cases the method of Theorem 2 is available for finding C. At least one general observation can be made about the result.

THEOREM 8. *For $P(a) = a^t, t > 1, B > 2$, C has the property that $c_i = B - 1$ for $i < J$; and either $c_i = B - 1$ or $c_i \leqq t - 2$ for $J \leqq i \leqq S$.*

Proof. Recall from the proof of Theorem 6 that $m_i = m_i' = B - 1$ for $i \leqq J - 1$. Since $c_i \geqq m_i$ it follows that $c_i = B - 1$ for $i \leqq J - 1$.

The rest of the theorem is trivial if $B \leqq t$, and is known from Theorem 7 if $B > T$. In what follows assume $B > t$.

If $S = t$, it follows from $C < (t-1)B^t$, that $c_t \leqq t - 2$. Since $S \leqq t$, it remains to discuss c_i for the cases $J \leqq i \leqq S$ where $i < t$.

Since $i < t < B$, note that

$$\frac{(B-1)^t - (t-1)^t}{(B-t)} = \sum_0^{t-1} (B-1)^j (t-1)^{t-1-j}$$

$$\geqq \sum_0^{t-1} (B-1)^j \binom{t-1}{j}$$

$$= (B - 1 + 1)^{t-1} = B^{t-1} \geqq B^i.$$

Hence $H_i(B-1) = (B-1)^t - (B-1)B^i \geqq (t-1)^t - (t-1)B^i = H_i(t-1)$. Because of the concave upward property of $H_i(x)$ the inequality $H_i(B-1) \geqq H_i(t-1)$ indicates that the choice of c_i in the range $t - 1 \leqq c_i < B - 1$ would be a contradiction of the requirement in the algorithm of Theorem 2 that c_i be maximal satisfying (1) or (2). Consequently c_i must be limited to the values stated in the theorem.

The following tables illustrate Theorems 6, 7, 8 by showing C for $P(a) = a^t$ for all $B \geqq 3$ when $t = 2, 3, 4, 5$.

TABLE III

$t = 2$	
B	C
$B \geqq 3$	$B^2 - 1$

TABLE IV

$t = 3$	
B	C
3	$2B^2 - 1$
4, 5	$B^3 - 1$
6, 7	$B^3 + B^2 - 1$
$B \geqq 8$	$2B^3 - 1$

TABLE V

$t = 4$	
B	C
3	$B^3 - 1$
4, 5	$B^4 - 1$
6	$B^4 + B^3 - 1$
7 to 11	$2B^4 - 1$
12, 13	$2B^4 + B^3 - 1$
14	$2B^4 + 3B^3 - 1$
$B \geqq 15$	$3B^4 - 1$

TABLE VI

$t = 5$	
B	C
3	$B^4 - 1$
4	$B^5 - 1$
5	$B^5 + B^4 - 1$
6, 7, 8	$2B^5 - 1$
9	$2B^5 + B^4 - 1$
10 to 19	$3B^5 - 1$
20	$3B^5 + B^4 - 1$
21	$3B^5 + 2B^4 - 1$
22	$3B^5 + 3B^4 - 1$
$B \geqq 23$	$4B^5 - 1$

The effectiveness of the algorithm for finding C may be illustrated by an example such as $B = 10$, $t = 100$. The necessary comparisons are in this case successfully made with a table of logarithms.

	Test	*Decision*
(1)	$10^{J-1} < 9^{99} < 10^J$	$J = 95$
(2)	$10^S \leqq U_{94} + 1 = 95 \cdot 9^{100} - 10^{95} + 2 < 10^{S+1}$ (Remember from (5) that $U_{97} = H_{97}(1) + U_{94}$.)	$S = 97$
(3)	$c^{100} - c \cdot 10^{97} + U_{94} \geqq 0$	$c_{97}' = 2$
(4)	$c^{100} - c \cdot 10^{96} + 2^{100} - 2 \cdot 10^{97} + U_{94} \geqq 0$	$c_{96} = 5$
(5)	$c^{100} - c \cdot 10^{95} + 5^{100} - 5 \cdot 10^{96} + 2^{100} - 2 \cdot 10^{97} + U_{94} \geqq 0$	$c_{95} = 1$

Theorem 8 guarantees $c_i = 9$ for $0 \leqq i < J$, so the algorithm closes, and $C = 2 \cdot B^{97} + 5 \cdot B^{96} + B^{95} + (B^{95} - 1)$.

6. Orbits of F-related integers. Return now to the general function $P(a)$ requiring only that $P(a)$ is a non-negative integer. This modest restriction not only allows the number C to be determined as in Theorem 2, but also allows the function $F(A)$ to be iterated.

Define $F^{(0)}(A) = A$ and $F^{(k+1)}(A) = F(F^{(k)}(A))$. Integers X and Y are said to be F-related if and only if there exist non-negative integers k and m such that $F^{(k)}(X) = F^{(m)}(Y)$. Being F-related is an equivalence relation dividing all non-negative integers into N disjoint sets of F-related integers. Following Isaacs **(1)** call each such set an orbit and denote the orbit containing A by $\{A\}$.

THEOREM 9. *For $F(A)$ the number N is finite.*

Proof. The existence of C implies that each orbit $\{A\}$ contains at least one integer K with $K \leqq C$, for otherwise the sequence $F^{(n)}(A)$ for $n = 0, 1, 2, \ldots$ (all of whose members belong to $\{A\}$) would be an infinite decreasing sequence of non-negative integers. The existence of such a K for each orbit $\{A\}$ shows that $1 \leqq N \leqq C + 1$.

COROLLARY 9.1. *At least one orbit must be infinite.*

An improved estimate of the value of N may be obtained by noting that the value of $F(A)$ does not depend on the order of the digits of A. For if A_1 is obtained from A merely by permuting the digits (but keeping $a_k' > 0$, of course), then $F(A_1) = F(A)$. Consequently many numbers less than C are apt to be F-related.

Let C^* be the number of integers A, $1 \leqq A \leqq C$, which can be written

$$A = a_k' B^k + \sum_0^{k-1} a_i B^i, \qquad B - 1 \geqq a_k' \geqq a_{k-1} \geqq \ldots \geqq a_1 \geqq a_0 \geqq 0.$$

Then an improved estimate for N is given by $1 \leqq N \leqq C^* + 1$.

From $C < B^{S+1}$ and properties of the binomial coefficients it follows that

$$C^* \leqq \binom{B+S+1}{S+1} - (S+2).$$

The work of Isaacs shows for the iteration of a much more general function G, that each orbit of G-related numbers has at most one "cycle" and various incoming "branches." The word "cycle" has the usual meaning—namely, for $F(A)$ it will mean the existence of a period number p (minimal and positive) and an initial point q such that

$$F^{(i+p)}(A) = F^{(i)}(A) \text{ for all } i \geqq q.$$

If $F^{(m)}(X) = Y$, $m \geqq 1$, then X is called an "antecedent" of Y. If $m = 1$, X is an "immediate antecedent" of Y. If $X \neq Y$, X is a "proper antecedent" of Y. If $F(X) = U$ is in the cycle part of $\{A\}$, but X itself is not in the cycle, then X and all its antecedents constitute a "branch" of $\{A\}$.

THEOREM 10. *For $F(A)$ each orbit $\{A\}$ has a unique cycle.*

Proof. If the orbit $\{A\}$ is non-cyclic, then for all n sufficiently large $F^{(n)}(A) > C$; however, for such n, $F^{(n+1)}(A) < F^{(n)}(A)$ and a contradiction is reached, for we cannot have an infinite decreasing sequence of integers $> C$. Thus each orbit $\{A\}$ must have a finite cycle.

To show that this cycle depends on $\{A\}$ and not on the representative A, we reproduce Isaacs' proof. Suppose U and U' are both in $\{A\}$ and that each is a member of some cycle of $\{A\}$. The first hypothesis implies the existence of k and m so that $F^{(k)}(U) = F^{(m)}(U') = U''$. The second hypothesis now shows that U'' is in the cycle containing U and also in the cycle containing U'. In other words, $\{A\}$ has only one cycle.

COROLLARY 10.1. *Let W be the maximum value of $F(A)$ for $A \leqq C$. Then the period p of the cycle of $\{A\}$ is bounded by $1 \leqq p \leqq W+1$.*

Proof. In the proof of Theorem 9 we showed that $\{A\}$ contains at least one K with $K \leqq C$. Then $F^{(n)}(K) \leqq W$ for all $n \geqq 0$. For either $C < F^{(n)}(K) \leqq W$, whence $F^{(n+1)}(K) < F^{(n)}(K)$ by the definition of C, thus $F^{(n+1)}(K) < W$; or $0 \leqq F^{(n)}(K) \leqq C$, whence $F^{(n+1)}(K) \leqq W$ by the definition of W. Not only is the existence of a cycle of $\{A\}$ newly evident, but also the maximum number of elements in the cycle is the complete set $0 \leqq X \leqq W$, hence $1 \leqq p \leqq W+1$.

COROLLARY 10.2. *Each element U of the cycle part of $\{A\}$ has the property $U \leqq W$ and at least one member U satisfies $U \leqq C$.*

A simple example in which the maximums of both N and p are attained is given by $B = 2$, $P(0) = 1$, $P(1) = 0$, wherein $C = 0$, $W = 1$, and there is just one orbit: $N = 1 = C + 1$, with $p = 2 = W + 1$.

There seem to be few additional general statements to be made about the orbits, cycles, and branches, for by varying $P(a)$ properly, we may construct bizarre situations which contradict proposed generalizations.

REMARK 1. *Not every orbit need be infinite.* For if $P(q) = q$, but $P(a) > q$ when $a \neq q$, then $\{q\}$ contains only q.

REMARK 2. *If $P(a) = 1$ for some $a \neq 0$, then every $Y > 1$ has a proper antecedent.* For $F(A) = Y$ has a solution

$$A = a \sum_{0}^{Y-1} B^i$$

and $A > Y$.

REMARK 3. *If $P(a) = 0$ for some a, and if $A \neq 0$, then $F(A)$ has infinitely many immediate antecedents.* For since $P(a) = 0$,

$$A_m = AB^m + a \sum_{0}^{m-1} B^i$$

has $F(A_m) = F(A)$, for $m = 1, 2 \ldots$. And since $A \neq 0$, the A_m are distinct (even if $a = 0$).

REMARK 4. *If $P(a) > 0$ for all a, then each Y has at most a finite number of immediate antecedents.* For note that if x_a denotes the number of digits of A which are equal to a, then $F(A)$ may be written

$$F(A) = \sum_{0}^{B-1} x_a P(a).$$

Then the assumption $P(a) > 0$ for every a and the restrictions $x_a \geqq 0$ mean that $F(A) = Y$ is a linear Diophantine form problem with at most a finite number of solutions: $x_0, x_1, \ldots, x_{B-1}$. (Of course, there may be *no* solution.) Corresponding to each such solution set there are only a finite number of integers A resulting from permissible permutations of the sets of digits. $\Big($*Permissible* means at least one $x_{a'} > 0$ and $a_k' > 0$ where

$$k = \sum_{0}^{B-1} x_a.\Big)$$

Example 3. Suppose $B = 10$ and $P(0) = P(2) = P(4) = 18$, $P(6) = 8$, $P(8) = 6$, $P(1) = P(3) = P(5) = 5$, $P(7) = 9$, $P(9) = 7$. It is easy to find $J = 0$, $S = 1$, $M_1 = 20$, $C = 27$, $W = 36$. Then $\{1, 3, 5\}$ is a finite orbit with $p = 1$; and $\{6, 8\}$ and $\{7, 9\}$ are finite orbits each with $p = 2$. All other integers belong to either $\{23\}$, $\{26\}$ or $\{27\}$, all of which are infinite orbits, each with $p = 1$. Hence $N = 6$. These results follow from Corollary 10.2 and Remark 4.

Example 4. Suppose $P(a) = q$ for every a. If $0 \leqq q < B/2$, then $N = 1$ with $p = 1$ and $F(C) = C = q$. If $B^S/S \leqq q < B^{S+1}/(S+2)$, $S \geqq 1$, then $N = 1$ with $p = 1$ and $F(C) = C = (S+1)q$. If $B^S/(S+1) \leqq q < B^S/S$, $S \geqq 1$, then $N = 2$ and both orbits have $p = 1$: one (infinite) contains $F(C) = C = (S+1)q$, the other (finite) contains $F(U) = U = Sq$.

8. Orbits for $P(a) = a^t$**.** When the previous discussion is applied to the case $P(a) = a^t$, a few additional comments may be made.

Remark 1 applies with $q = 0$. Hence $\{0\}$ contains only 0.

Remark 2 applies with $a = 1$. Hence if $Y > 1$, Y has a proper antecedent. Because $P(0) = 0$, $F(B^i) = 1$, so $Y = 1$ also has a proper antecedent. Note that the orbit $\{1\}$ has $p = 1$.

Remark 3 applies. Hence each $A \neq 0$ has infinitely many immediate antecedents.

Let N_i indicate the number of orbits of F-related integers with period i. Then $N = \sum N_i$.

For $t = 1$ and any B, $N = N_1 = B$. For $C = B - 1$ implies $N \leqq C + 1 = B$ and $P(a) = a$ shows each $\{a\}$ has $p = 1$. Note that the corresponding $F(A) = \sum a_i$ is the function met in arithmetic in the process called "casting-out $(B-1)$'s" and has the useful property $F(A) \equiv A \bmod B - 1$.

For $B = 2$ and any t, $N = N_1 = 2$. For $C = 1$ shows $N \leqq 2$ and each of $\{0\}$ and $\{1\}$ has $p = 1$. By the same argument $N_1 \geqq 2$ for every t and every B.

If $t = 2$ and B is odd, then N_1 is even and $N_1 \geqq 4$. From Section 5, when $t = 2$, $C = B^2 - 1$, and hence by Corollary 10.2 each 1-cycle must contain either $U = b$ or $U = aB + b$. If $F(b) = b^2 = b$, then $b = 0$ or 1, the cases noted in the previous paragraph. If $F(aB + b) = a^2 + b^2 = aB + b$, then it follows that

$$F((B-a)B + b) = (B-a)^2 + b^2 = B^2 - 2aB + (aB + b) = (B-a)B + b.$$

Also $B - a \neq a$, because B is odd. Hence 1-cycles of this type occur in pairs, thus N_1 is even. Furthermore, at least one choice of a and b is always available: $a = b = (B+1)/2$. Hence $N_1 \geqq 4$.

Perhaps the best way to show the teasing irregularity of the orbit and cycle numbers of F-related integers when $P(a) = a^t$ is to append the following brief tables.

TABLE VII

$t = 2$

B	N_1	N_2	N_3	Others	N
3	4	1			5
4	2				2
5	4		1		5
6	2			$N_8 = 1$	3
7	6			$N_4 = 2$	8
8	4	2	1		7
9	4	1	1		6
10	2			$N_8 = 1$	3
11	4		2		6
12	4	2	1	$N_{10} = 1$	8
13	8	3			11
14	2	1		$N_9 = 1$	4
15	4	1	3	$N_5 = N_9 = 1 = N_7$	11
16	2			$N_6 = 1$	3

TABLE VIII

$t = 3$

B	N_1	N_2	N_3	Others	N
3	3			$N_4 = 1$	4
4	10				10
5	4		1		5
6	5			$N_5 = 1$	6
7	8	4	2	$N_4 = N_9 = 1$	16
8	7			$N_5 = 1$	8
9	9	2		$N_5 = N_{11} = 1$	13
10	6	2	2		10

Added in proof:

During 1959-60, as part of an NSF Undergraduate Research Project, Joseph C. Ferrar made use of the Michigan State University MISTIC to check and extend Tables VII and VIII. Thanks to this work several corrections have been made in Table VII. The extended tables for $t = 2$ show B from 17 to 32 and for $t = 3$ show B from 11 to 16.

Space allows explanation of just one of these entries.

When $B = 10$ and $t = 3$, then $C = 1999$. From the discussion following Corollary 9.1, there are

$$\binom{12}{3} = 220$$

numbers from 0 to 999 and

$$\binom{11}{3} = 165$$

numbers from 1111 to 1999 which need to be considered. The results are as follows:

$N_1 = 6$: the 1-cycles being 0; 1; 153; 370; 371; 407;

$N_2 = 2$: the 2-cycles being 136, 244, and 919, 1459;

$N_3 = 2$: the 3-cycles being 55, 250, 133, and 160, 217, 352.

Then by Corollary 10.2 each non-negative integer is a member of one and only one of these $N = 10$ orbits.

9. Products of functions of digits. Use the previous notation for a, a' and A and suppose $P(a)$ is a rational integer $P(a) \geqq 0$. Define

$$G(a) = P(a), \qquad G(A) = P(a_k') \prod_0^{k-1} P(a_i).$$

The question suggested by Theorem 1 (for $\epsilon = 1$) is whether there exists an integer D for which $G(D) \geqq D$ and $G(A) < A$ for every $A > D$.

Let M indicate the maximum value of $P(a)$ and let M' indicate the maximum value of $P(a')$.

Case 1. *If $M' \geqq B$, then D does not exist.*

Proof. If $P(b') = M'$, then

$$A = b' \sum_0^k B^i$$

has $G(A) = (M')^{k+1} \geqq B^{k+1} > A$ for every k.

Case 2. *If $M' = 0$, then $D = 0$.*

Proof. If $A > 0$, $P(a_k') = 0$, so $G(A) = 0 < A$. And $G(0) = P(0) \geqq 0$.

Case 3. *If $0 < M' < B$ and $M \geqq B + 1$, then D does not exist.*

Proof. The hypotheses imply $P(0) = M$. Then $A = b'B^k$ has $G(A) = M'M^k \geqq (B+1)^k > B^{k+1} > A$, for all k sufficiently large.

Case 4. *If $M < B$, then D exists.*

Proof. Note $M' \leqq M$. If $B^k \leqq A < B^{k+1}$ and if $k \geqq (B-1)(B-2) = k_1$ then $G(A) \leqq M'M^k \leqq (B-1)^{k+1} < B^k \leqq A$. For from the assumption $k \geqq k_1$ it follows that $B - 1 \leqq 1 + k/(B-1) < (1 + 1/(B-1))^k = (B/(B-1))^k$. Since $G(0) \geqq 0$, D exists and is in the range $0 \leqq D < B^{k_1}$.

Case 5. *If $M' < B$, if $M = B$, and if $P(a') \geqq a'$ for any a', then D does not exist.*

Proof. The hypotheses imply $P(0) = B$. Hence if $A = a'B^k$, then $G(A) = P(a')B^k \geqq a'B^k = A$, for every k.

Case 6. *If $M \leqq B$ and $P(a') < a'$ for every a', then $D = 0$.*

Proof. If $B^k \leqq A < B^{k+1}$, then

$$G(A) = P(a'_k) \prod_0^{k-1} P(a_i) < a'_k B^k \leqq A$$

for every k. Since $G(0) \geqq 0$, $D = 0$.

Since these six cases exhaust the possible situations, the only "interesting" cases (having $D > 0$) arise when $1 \leqq M' \leqq M < B$ and $P(a') \geqq a'$ for at least one a'. For these cases the actual value of D and the orbits of G-related integers and their cycles may be determined by methods similar to those in §§ 3 and 6.

In particular, the choice $P(a) = a^t$ leads to an "interesting" case only when $t = 1$, and then there are $B - 1$ orbits, each infinite and of period 1.

The author thanks the referee for his stimulating criticisms and suggestions.

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