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OPTIMALITY AND DUALITY IN CONTINUOUS-TIME NONLINEAR FRACTIONAL PROGRAMMING

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Abstract

Optimality conditions via subdifferentiability and generalised Charnes-Cooper transformation are obtained for a continuous-time nonlinear fractional programming problem. Perturbation functions play a key role in the development. A dual problem is presented and certain duality results are obtained.

1. Introduction

Continuous-time programming originated from the bottleneck problems studied by Bellman [1]. Levinson [7] and Tyndall [14] established duality theorems for continuous-time linear programming problems. Hanson and Mond [5], Farr and Hanson [3], Kaul and Kaur [6], Singh [11, 12], Singh and Farr [13] and several other authors considered the continuous-time nonlinear programming problem and obtained optimality conditions and duality results. Recently Zalmai [15–18] considered the following continuous-time nonlinear programming problem:

$$\text{Maximise } \phi(z) = \int_0^T f(z(t), t)\, dt$$

$$\text{subject to } g(z(t), t) \leq 0 \text{ almost everywhere in } [0, T], \quad z \in Z^0$$

where Z^0 is a nonempty convex subset of the Banach space $L_\infty^n[0, T]$, ϕ is a concave real valued function defined on Z^0 and $g(z(t), t) = v(z)(t)$

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with v a function defined on Z^0 and taking values into the normed space $V_1^m[0, T]$ and g is convex in its first argument, for all t in $[0, T]$. Zalmai [15] obtained optimality conditions and duality results for the above problem by defining a perturbation function and using its subgradients.

In [18] Zalmai presented a continuous-time analogue of a duality formulation due to Craven and Mond [8, 9] for a class of homogeneous fractional programming problems. Recently Zalmai [19] presented optimality and duality theory for a class of continuous-time generalised (minmax) fractional programming problems. However, in this work, Zalmai assumed differentiability of functions involved and instead of using Charnes-Cooper type transformation, a parametrisation approach was used.

In this paper, we obtain optimality conditions and duality results for a general class of continuous-time nonlinear fractional programming via subdifferentiability and generalised Charnes-Cooper transformation. The development in Section 2 is analogous to the work of Schaible [10]; however the main portion of the paper (Sections 3 and 4) is based on the works of Geoffrion [4] and Zalmai [15]. A perturbation function is defined in Section 3 (Definition 3.3), which plays a key role in the development. For convenience of notation, we define also

$$Z = \{z \in Z^0 | g(z(t), t) \leq 0 \text{ almost everywhere in } [0, T]\}.$$

2. Continuous-time nonlinear fractional programming

We consider the following problem:

$$\text{Maximise } \phi(z) = \frac{\int_0^T f(z(t), t)\, dt}{\int_0^T h(z(t), t)\, dt}$$

subject to $g(z(t), t) \leq 0$ almost everywhere in $[0, T]$ where $z \in Z^0$; Z^0 and g are the same as in Section 1, $-f$, h are convex (in their first argument) real valued functions defined on $L_\infty^n[0, T] \times [0, T]$ and $f(z(t), t) \geq 0$, $h(z(t), t) > 0$ for all z in Z^0 and $t \in [0, T]$. Nonnegativity of f is not such a severe restriction. Since we are maximising, even the boundedness of f from below will produce a function that will be nonnegative. Here, by maximise $\phi(z)$, we mean there exist $\bar{z} \in L_\infty^n[0, T]$ such that $\phi(\bar{z}) = \max \phi(z)$ and $\phi(\bar{z})$ is finite.

We apply the generalised Charnes-Cooper transformation,

$$u = \left[\int_0^T h((z), t)\, dt\right]^{-1}, \tag{2.1}$$

$$s(t) = \left[\int_0^T h(z(t), t)\, dt\right]^{-1} z(t) \tag{2.2}$$

to Problem (FP) and the following transformed problem is obtained.

(FP)$'$ Maximise $\displaystyle\int_0^T uf\left(\frac{s(t)}{u}, t\right) dt$

$$\text{subject to} \quad ug\left(\frac{s(t)}{u}, t\right) \le 0 \quad \text{almost everywhere in } [0, T],$$

$$\int_0^T uh\left(\frac{s(t)}{u}, t\right) dt \le 1, \tag{2.2$'$}$$

$$u > 0,$$

$$s \in L_\infty^n[0, T], \qquad u \in R.$$

In (2.2)$'$ we have weakened the constraint from equality in (2.2) to inequality in order to have the feasible set of (FP)$'$ be convex. However, for our development to work, we need u to be strictly positive rather than being nonnegative, making the feasible set of (FP)$'$ to be nonclosed. This in turn causes some optimality theory to be inapplicable to our problem. Let

$$W^0 = \left\{(s, u) | s \in L_\infty^n[0, T],\ u \in R,\ u = \left[\int_0^T h(z(t), t)\, dt\right]^{-1}, \right.$$
$$\left. s(t) = uz(t),\ z \in Z^0\right\}.$$

$$W = \{s, u) | s \in L_\infty^n[0, T],\ u \in R,\ u > 0, s/u \in Z^0\};$$

R_+ = the set of positive reals.

Lemma 2.1. *W is a convex subset of $L_\infty^n[0, T] \times R$.*

Proof. Let (s_1, u_1), $(s_2, u_2) \in W$, $0 \leqq \lambda \leqq 1$. Then $u_1 > 0$, $u_2 > 0$, $s_1/u_1 \in Z^0$, $s_2/u_2 \in Z^0$. Therefore for $0 \leqq \lambda \leqq 1$, $(1-\lambda)u_1 + \lambda u_2 > 0$, and

$$\frac{(1-\lambda)s_1 + \lambda s_2}{(1-\lambda)u_1 + \lambda u_2} = \frac{(1-\lambda u_1}{(1-\lambda)u_1 + \lambda u_2} \cdot \frac{s_1}{u_1} + \frac{\lambda u_2}{(1-\lambda)u_1 + \lambda u_2} \cdot \frac{s_2}{u_2}$$
$$= p_1 \frac{s_1}{u_1} + p_2 \frac{s_2}{u_2} \in Z^0$$

where

$$\frac{(1-\lambda)_1}{(1-\lambda)u_1+\lambda u_2}, \qquad p_2=\frac{\lambda u_2}{(1-\lambda)u_1+\lambda u_2},$$

$p_1+p_2=1$, and $0\le p_i\le 1$, $i=1,2$. Hence W is a convex set.

Now we proceed to show that the function $\psi_1\colon W\to R$ defined by

$$\psi_1(s,u)=\int_0^T uf\left(\frac{s(t)}{u},t\right)dt$$

is a concave function in its first argument, and the functions $v_1\colon W\to V_1^m[0,T]$ defined by

$$v_1(s,u)(t)=ug\left(\frac{s(t)}{u},t\right)$$

and $v_2\colon W\to R$ defined by

$$v_2(s,u)=\int_0^T uh\left(\frac{s(t)}{u},t\right)dt-1$$

are convex functions in their first arguments.

LEMMA 2.2. *The function $\psi_1\colon W\to R$ defined by*

$$\psi(s,u)=\int_0^T uf(s(t)/u,t)\,dt$$

is a concave function.

PROOF. Let $(s_1,u_1)\in W$, $(s_2,u_2)\in W$, $0\le\lambda\le 1$.

Then $u_1>0$, $u_2>0$, $s_1/u_1\in Z^0$, $s_2/u_2\in Z^0$ and

$$\begin{aligned}
&\psi_1((1-\lambda)(s_1,u_1)+\lambda(s_2,u_2))\\
&\quad=\psi_1((1-\lambda)s_1+\lambda s_2,(1-\lambda)u_1+\lambda u_2)\\
&\quad=\int_0^T((1-\lambda)u_1+\lambda u_2)f\left(\frac{((1-\lambda)s_1+\lambda s_2)(t)}{(1-\lambda)u_1+\lambda u_2},t\right)dt\\
&\quad=\int_0^T((1-\lambda)u_1+\lambda u_2)f\left(\frac{(1-\lambda)u_1}{(1-\lambda)u_1+\lambda u_2}\frac{s_1(t)}{u_1}+\frac{\lambda u_2}{(1-\lambda)u_1+\lambda u_2}\frac{s_2(t)}{u_2},t\right)dt\\
&\quad\ge\int_0^T((1-\lambda)u_1+\lambda u_2)\left[\frac{(1-\lambda)u_1}{(1-\lambda)u_1+\lambda u_2}f\left(\frac{s_1(t)}{u_1},t\right)\right.\\
&\qquad\qquad\left.+\frac{\lambda u_2}{(1-\lambda)u_1+\lambda u_2}f\left(\frac{s_2(t)}{u_2},t\right)\right]dt\\
&\quad=(1-\lambda)\int_0^T u_1f\left(\frac{s_1(t)}{u_1},t\right)dt+\lambda\int_0^T u_2f\left(\frac{s_2(t)}{u_2},t\right)dt\\
&\quad=(1-\lambda)\psi_1(s_1,u_1)+\lambda\psi_1(s_2,u_2).
\end{aligned}$$

Hence ψ_1 is a concave function in its first argument.

Lemma 2.3. *The function* $v_1: W \to V_1^m[0, T]$ *defined by* $v_1(s, u)(t) = ug(s(t)/u, t)$ *and* $v_2: W \to R$ *defined by* $v_2(s, u) = \int_0^T uh(\frac{s(t)}{u}, t)\,dt - 1$ *are convex functions in* s.

The proof of the above lemma is very similar to the proof of Lemma 2.2.

In the next lemma we shall show that Problem (FP)$'$ obtained from (FP) by means of the generalised Charnes-Cooper transformation is equivalent to (FP) in the sense of having optimal solutions.

Lemma 2.4. *Problem* (FP) *has an optimal solution if and only if* (FP)$'$ *has one and the optimal solutions of* (FP) *and* (FP)$'$ *are connected by* (2.1) *and* (2.2).

Proof. Let (FP) have an optimal solution $z^*(t)$. We assert that

$$u^* = \left[\int_0^T h(z^*(t), t)\,dt\right]^{-1}, \tag{2.3}$$

$$s^*(t) = \left[\int_0^T h(z^*(t), t)\,dt\right]^{-1} z^*(t) \tag{2.4}$$

is an optimal solution of (FP)$'$. The feasibility of $(s^*(t), u^*)$ for Problem (FP)$'$ clearly follows from the feasibility of $z^*(t)$ for Problem (FP). Let $(s(t), u)$ be any other feasible solution for (FP)$'$ then $z(t) = s(t)/u$ is feasible for (FP) and

$$\frac{\int_0^T f(z^*(t), t)\,dt}{\int_0^T h(z^*(t), t)\,dt} \geq \frac{\int_0^T f(z(t), t)\,dt}{\int_0^T h(z(t), t)\,dt},$$

which implies that

$$\frac{\int_0^T f\left(\frac{s^*(t)}{u^*}, t\right)}{\int_0^T h\left(\frac{s^*(t)}{u^*}, t\right)} \geq \int_0^T uf\left(\frac{s(T)}{u}, t\right)\,dt,$$

because of the feasibility of $(s(t), u)$ and the assumption that $f(z(t), t) \geq 0$ for all $z \in Z^0$. On using (2.3) and (2.4), we get

$$\int_0^T u^* f\left(\frac{s^*(t)}{u^*}, t\right)\,dt \geq \int_0^T uf\left(\frac{s(t)}{u}, t\right)\,dt$$

which shows that $(s^*(t), u^*)$ is an optimal solution of (FP)$'$; conversely let Problem (FP)$'$ have an optimal solution $(s^*(t), u^*)$. We shall show that

$z^*(t) = s^*(t)/u^*$ is an optimal solution of (FP). Clearly $z^*(t)$ is feasible for Problem (FP). Let $z(t)$ be any other feasible solution for (FP). Then (2.1) and (2.2) give a feasible solution $(s(t), u)$ of (FP)$'$ and

$$\int_0^T u^* f\left(\frac{s^*(t)}{u^*}, t\right) dt \geq \int_0^T u f\left(\frac{s(t)}{u}, t\right) dt.$$

The last inequality when combined with (2.1) and (2.2) gives

$$\frac{\int_0^T f(z^*(t), t)\, dt}{\int_0^T h(z^*(t), t)\, dt} \geq \frac{\int_0^T f(z(t), t)\, dt}{\int_0^T h(z(t), t)\, dt}$$

because of the feasibility of $(s^*(t), u^*)$ for (FP)$'$ and the assumption that $f(z(t), t) \geq 0$ for all $z \in Z^0$. Hence $z^*(t)$ is an optimal solution of (FP). Thus Problems (FP) and (FP)$'$ are equivalent with respect to their sets of optimal solutions.

REMARK 2.1. We assume that $\sup_{z \in Z}\{f(z(t), t)\} > 0$.

3. Optimality conditions

Following Zalmai [15], we define optimality conditions, optimal multipliers for (FP), perturbation function and stability for (FP)$'$.

DEFINITION 3.1 (Optimality conditions). A pair $(\bar{z}, \bar{w}) \in L_\infty^n[0, T] \times L_\infty^m[0, T]$, $\bar{z} \in Z^0$, is said to satisfy the optimality conditions for Problem (FP) if and only if

(i) $\bar{z}$ maximises

$$\left[\frac{\int_0^T f(z(t), t)\, dt}{\int_0^T h(z(t), t)\, dt} - \int_0^T \bar{w}(t) g(z(t), t)\, dt\right]$$

over Z^0,

(ii) $\bar{w}(t) g(\bar{z}(t), t) = 0$ almost everywhere in $[0, T]$,

(iii) $\bar{w}(t) \geq 0$ almost everywhere in $[0, T]$,

(iv) $g(\bar{z}(t), t) \leq 0$ almost everywhere in $[0, T]$.

DEFINITION 3.2 (Optimal multiplier). An element $\bar{w} \in L_\infty^m[0, T]$ is said to be an optimal multiplier for Problem (FP) if and only if $(\bar{z}, \bar{w})$ satisfies the optimality conditions for some $\bar{z} \in Z^0 \subseteq L_\infty^n[0, T]$.

DEFINITION 3.3. We say $p\colon V_1^m[0, T] \to R$ is a perturbation function for Problem (FP)$'$ if and only if

$$p(y) = \sup_{(s,u)\in W}\left[\int_0^T uf\left(\frac{s(t)}{u}, t\right) dt \;\middle|\; \begin{array}{l} g\left(\frac{s(t)}{u}, t\right) \le y(t) \text{ almost everywhere} \\ \text{in } [0, T] \text{ for } y \in V_1^m[0, T] \\ \text{and } \int_0^T uh\left(\frac{s(t)}{u}, t\right) dt \le 1 \end{array}\right]$$

and $p(y) = -\infty$ if there does not exist (s, u) in W such that $g(s(t)/u, t) \le y(t)$ almost everywhere in $[0, T]$ and $\int_0^T uh(s(t)/u, t)\,dt \le 1$. It is instructive to note that only one of the constraints of (FP)$'$ is perturbed. Perturbing the second integral constraint involving h is still open for investigation. It is easy to see that $p(\cdot)$ is a nondecreasing function.

DEFINITION 3.4 (Subgradient of perturbation function). If the perturbation function p is finite at $\bar{y} \in V_1^m[0, T]$, we say that $w \in L_\infty^m[0, T]$ is a subgradient of p at $\bar{y}$ if and only if

$$p(y) - p(\bar{y}) \le \int_0^T w(t)(y(t) - \bar{y}(t))\,dt \quad \text{for all } y \in V_1^m[0, T].$$

We let

$$Y = \left[y \in V_1^m[0, T] \;\middle|\; \begin{array}{l} g\left(\frac{s(t)}{u}, t\right) \le y(t) \text{ almost everywhere in } [0, T], \\ \int_0^T uh\left(\frac{s(t)}{u}, t\right) dt \le 1 \text{ for some } (s, u) \in W \end{array}\right].$$

REMARK 3.1. We remark that the variable u, being strictly positive, is dropped from the constraint $ug(s(t)/u, t) \le 0$ without affecting the feasible set of (FP)$'$. This helps the construction of the set Y to be better adopted for the investigation to follow.

DEFINITION 3.5 (Stable problem). We say that Problem (FP)$'$ is stable if $p(0)$ is finite and there exists a positive number M such that $p(0) \ge p(y) - M\|y\|_1$ for all $y \in Y$.

LEMMA 3.1. (i) *The set Y is convex.*

(ii) *The function p is concave on Y.*

PROOF. This can be established along the lines of the finite-dimensional case as presented in Geoffrion [4].

LEMMA 3.2. *Suppose*

(1) *Problem* (FP) *has an optimal solution*;
(2) $\bar{w} \in L_\infty^m[0, T]$.

Then $\bar{w}$ *is an optimal multiplier for* (FP) *if and only if* $\bar{w} \in \partial p(0)$ *where* p *is perturbation function of* (FP)$'$ *and* $\partial p(0)$ *is the set of all subgradients of* p *at* 0.

PROOF. Suppose $\bar{w}$ is an optimal multiplier for (FP). Then by Definition 3.2, there exists a $\bar{z} \in Z^0 \subseteq L_\infty^n[0, T]$ such that $(\bar{z}, \bar{w})$ satisfies optimality conditions (i)–(iv) of Definition 3.1. By conditions (i) and (ii) of Definition 3.1, we have

$$\frac{\int_0^T f(z(t), t)\, dt}{\int_0^T h(z(t), t)\, dt} \leq \frac{\int_0^T f(\bar{z}(t), t)\, dt}{\int_0^T h(\bar{z}(t), t)\, dt} + \int_0^T \bar{w}(t) g(z(t), t)\, dt \quad \text{for all } z \in Z^0. \tag{3.1}$$

Let $\bar{u} = [\int_0^T h(\bar{z}(t), t)\, dt]^{-1}$, $\bar{s}(t) = [\int_0^T h(\bar{z}(t), t)\, dt]^{-1} \bar{z}(t)$. Then $(\bar{s}, \bar{u}) \in W$. For $(s, u) \in W$, $s/u \in Z^0$ and by (3.1),

$$\frac{\int_0^T f\left(\frac{s(t)}{u}, t\right) dt}{\int_0^T h\left(\frac{s(t)}{u}, t\right) dt} \leq \int_0^T \bar{u} f\left(\frac{\bar{s}(t)}{\bar{u}}, t\right) dt + \int_0^T \bar{w}(t) g\left(\frac{s(t)}{u}, t\right) dt.$$

Now proceeding as in Zalmai [15], it can be shown that $\bar{w} \in \partial p(0)$.

Conversely suppose $\bar{w} \in \partial p(0)$. Further suppose that $\bar{z}$ is an optimal solution of (FP). Then

$$\bar{u} = \left[\int_0^T h(\bar{z}(t), t)\, dt\right]^{-1}, \qquad \bar{s}(t) = \left[\int_0^T h(\bar{z}(t), t)\, dt\right]^{-1} \bar{z}(t) \tag{3.2}$$

is an optimal solution of (FP)$'$ by Lemma 2.4. Now we shall show that $(\bar{z}, \bar{w})$ satisfies optimality conditions (i)–(iv) of Definition 3.1 and then invoke Definition 3.2.

The conditions (ii), (iii) and (iv) of Definition 3.1. can be proved on similar lines as given by Zalmai [15]. To establish condition (i), we let $y(t) = g(s(t)/u, t) = v(s, u)(t)$ in the subgradient inequality

$$p(y) \leq p(0) + \int_0^T \bar{w}(t) y(t)\, dt \quad \text{for all } y \in V_1^m[0, T], \tag{3.3}$$

which holds as $\bar{w} \in \partial p(0)$. Therefore (3.3) becomes

$$\begin{aligned} p(v(s, u)) &\le p(0) + \int_0^T \bar{w}(t) g\left(\frac{s(t)}{u}, t\right) dt \\ &= \int_0^T \bar{u} f\left(\frac{\bar{s}(t)}{\bar{u}}, t\right) dt + \int_0^T \bar{w}(t) g\left(\frac{s(t)}{u}, t\right) dt \quad \text{for all } (s, u) \in W. \end{aligned} \tag{3.4}$$

For $z \in Z^0$, let $u = [\int_0^T h(z(t), t)\, dt]^{-1}$ and $s(t) = [\int_0^T h(z(t), t)\, dt]^{-1} z(t)$. Then $(s, u) \in W$ and $uh(s(t)/u, t) = 1$. Therefore by Definition 3.3 (of $p(\cdot)$),

$$\frac{\int_0^T f(z(t), t)\, dt}{\int_0^T h(z(t), t)\, dt} \le p(g(z(t), t)) \quad \text{for all } z \in Z^0. \tag{3.5}$$

Combining (3.2), (3.4) and (3.5), we have

$$\frac{\int_0^T f(z(t), t)\, dt}{\int_0^T h(z(t), t)\, dt} \le \frac{\int_0^T f(\bar{z}(t), t)\, dt}{\int_0^T h(\bar{z}(t), t)\, dt} + \int_0^T \bar{w}(t) g(z(t), t)\, dt.$$

By using condition (ii) of Definition 3.1, it can be easily seen that condition (i) holds.

The next result displays a relationship between stability and optimal multiplier functions, which needs the following lemma proved by Zalmai [15].

LEMMA 3.3. *Suppose F is a concave real-valued function defined on a convex set X of a real normed space with norm $\|\cdot\|$. Then F has a subgradient at $\bar{x}$ if and only if there exists a positive constant M such that $F(x) - F(\bar{x}) \le M\|x - \bar{x}\|$ for all $x \in X$.*

THEOREM 3.1. *Suppose Problem* (FP) *has an optimal solution. Then*

(a) *an optimal multiplier for* (FP) *exists if and only if* (FP)$'$ *is stable*;
(b) *$w \in L_\infty^m[0, T]$ is an optimal multiplier for* (FP) *if and only if $w \in \partial p(0)$.*

PROOF. (a) The proof follows by Lemma 3.1, Lemma 3.2, Lemma 3.3, Definition 3.5 and the fact that $p(0)$ is finite because of the existence of an optimal solution of (FP) and therefore of (FP)$'$. Part (b) is just a restatement of the conclusion of Lemma 3.2.

The next two theorems exhibit the equivalence of the saddlepoint of the Lagrangian function L and the optimality conditions for problem (FP) where

$$L: L_{\infty}^{n}[0, T] \times L_{\infty}^{m}[0, T] \to R$$

is defined by

$$L(z, w) = \frac{\int_0^T f(z(t), t)\,dt}{\int_0^T h(z(t), t)\,dt} - \int_0^T w(t) g(z(t), t)\,dt.$$

DEFINITION 3.6 (Saddlepoint). Suppose $\bar{z} \in Z^0$ and $\bar{w} \in L_{\infty}^{m}[0, T]$, $\bar{w}(t) \geq 0$ almost everywhere in $[0, T]$. We say $(\bar{z}, \bar{w})$ is a saddlepoint of the Lagrangian function L if and only if

$$L(z, \bar{w}) \leq L(\bar{z}, \bar{w}) \leq L(\bar{z}, w)$$

for all $w \in L_{\infty}^{m}[0, T]$, $w(t) \geq 0$ almost everywhere in $[0, T]$ and all $z \in Z^0$.

THEOREM 3.2. *A pair* $(\bar{z}, \bar{w})$ *is a saddlepoint of the Lagrangian function* L *if and only if it satisfies the optimality conditions* (i)-(iv) *of Definition* 3.1.

PROOF. Suppose $(\bar{z}, \bar{w})$ satisfies the optimality conditions (i)–(iv) of Definition 3.1. Then by condition (i), we have

$$L(z, \bar{w}) \leq L(\bar{z}, \bar{w}) \quad \text{for all } z \in Z^0. \tag{3.6}$$

On using conditions (ii) and (iv), we get

$$\begin{aligned} L(\bar{z}, w) \leq L(\bar{z}, \bar{w}) \quad &\text{for all } w \in L_{\infty}^{m}[0, T], \\ w(t) \geq 0 \quad &\text{almost everywhere in } [0, T]. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), it follows that $(\bar{z}, \bar{w})$ is a saddlepoint of the Lagrangian L.

Conversely, suppose $(\bar{z}, \bar{w})$ is a saddlepoint of the Lagrangian L. Then

$$\bar{w}(t) \geq 0 \quad \text{almost everywhere in } [0, T] \tag{3.8}$$

and

$$L(z, \bar{w}) \leq L(\bar{z}, \bar{w}) \leq L(\bar{z}, w) \tag{3.9}$$

for all $w \in L_{\infty}^{m}[0, T]$, $w(t) \geq 0$ almost everywhere in $[0, T]$ and for all $z \in Z^0$.

The condition (i) of Definition 3.1 follows from the first inequality in (3.9) and condition (iii) follows from (3.8). The second inequality in (3.9) gives

$$\int_0^T [w(t) - \bar{w}(t)] g(\bar{z}(t), t)\, dt \leq 0$$
$$\text{for all } w(t) \geq 0 \text{ almost everywhere in } [0, T].$$

The condition (iv) of Definition 3.1 now follows on similar lines as given by Zalmai [15]. Now, we know that conditions (iii) and (iv) of Definition 3.2 hold, therefore

$$\bar{w}(t) g(\bar{t}z(t), t) \leq 0 \quad \text{almost everywhere in } [0, T]. \tag{3.10}$$

If strict inequality holds in (3.10) over a subset D of $[0, T]$ with positive measure then

$$L(\bar{z}, \bar{w}) > L(\bar{z}, 0)$$

which contradicts the hypothesis that $(\bar{z}, \bar{w})$ is a saddlepoint of the Lagrangian L. Thus $w(t)g(z(t), t) = 0$ almost everywhere in $[0, T]$ which is condition (ii). Hence $(\bar{z}, \bar{w})$ satisfies the optimality conditions (i)–(iv) of Definition 3.1.

THEOREM 3.3 (Kuhn Tucker Saddlepoint Theorem). *Suppose Problem* (FP)$'$ *is stable. Then* $\bar{z} \in Z^0$ *is an optimal solution of* (FP) *if and only if there exists* $\bar{w} \in L_\infty^m[0, T]$, $\bar{w}(t) \geq 0$ *almost everywhere in* $[0, T]$ *such that* $(\bar{z}, \bar{w})$ *is a saddlepoint of the Lagrangian* L.

PROOF. Suppose $\bar{z}$ is an optimal solution of (FP). Since (FP)$'$ is stable; therefore by Theorem 3.1, part (a), there exists an optimal multiplier $\bar{w} \in L_\infty^m[0, T]$ which means that $(\bar{z}, \bar{w})$ satisfies the optimality conditions (i)–(iv) of Definition 3.1. Hence by Theorem 3.2, $(\bar{z}, \bar{w})$ is a saddlepoint of the Lagrangian L. Conversely, suppose $(\bar{z}, \bar{w})$ is a saddlepoint of the Lagrangian L. Then by (3.8) condition (iii) holds and by (3.9) conditions (ii) and (iv) of Definition 3.1 hold as seen in Theorem 3.2.

The condition (iv) shows that $\bar{z}(t)$ is a feasible solution of (FP) and the first inequality of (3.9) when combined with (3.8) and condition (ii) gives

$$\frac{\int_0^T f(z(t), t)\, dt}{\int_0^T h(z(t), t)\, dt} \leq \frac{\int_0^T f(\bar{z}(t), t)\, dt}{\int_0^T h(\bar{z}(t), t)\, dt} \quad \text{for every feasible solution } z \text{ of (FP).}$$

Hence $\bar{z}$ is an optimal solution of (FP).

4. Duality

We associate the following dual problem with Problem (FP)

(DFP) Minimise $\psi(w)$ subject to $w(t) \geq 0$ almost everywhere in $[0, T]$

where

$$\psi(w) = \sup \left\{ \frac{\int_0^T f(z(t), t)\,dt}{\int_0^T h(z(t), t)\,dt} - \int_0^T w(t) g(z(t), t)\,dt \right\}$$

and $w \in L_\infty^m[0, T]$.

Let Z denote the set of all feasible solutions of (FP) and D_z denote the set of all feasible solutions of (DFP). We prove the weak and strong duality results for (FP) and (DFP).

Theorem 4.1. *Suppose $z \in Z$ and $w \in D_Z$. Then $\psi(w) \geq \phi(z)$.*

Proof. By the definition of ψ,

$$\begin{aligned}
\psi(w) &= \sup_{z \in Z^0} \left\{ \frac{\int_0^T f(z(t), t)\,dt}{\int_0^T h(z(t), t)\,dt} - \int_0^T w(t) g(z(t), t)\,dt \right\} \\
&\geq \frac{\int_0^T f(z(t), t)\,dt}{\int_0^T h(z(t), t)\,dt} - \int_0^T w(t) g(z(t), t)\,dt \quad \text{for all } z \in Z^0 \\
&\geq \frac{\int_0^T f(z(t), t)\,dt}{\int_0^T h(z(t), t)\,dt} \quad \text{for all } z \in Z
\end{aligned}$$

The last inequality follows because for every $z \in Z$, $g(z(t), t) \leq 0$ almost everywhere in $[0, T]$ and $w \in D_z$ gives $w(t) \geq 0$ almost everywhere in $[0, T]$.

Corollary 4.1. *Suppose there exist $\bar{z} \in Z$ and $\bar{w} \in D_z$ such that $\psi(\bar{w}) = \phi(\bar{z})$. Then $\bar{z}$ and $\bar{w}$ are optimal solutions of Problems* (FP) *and* (DFP) *respectively.*

THEOREM 4.2. *Suppose Problem* (FP) *has an optimal solution* $\bar{z}$. *Then* $\bar{w} \in L_\infty^m[0, T]$ *is an optimal solution of the dual Problem* (DFP) *with* $\phi(\bar{z}) = \psi(\bar{w})$ *if and only if* $\bar{w} \in \partial p(0)$ *where* p *is the perturbation function of the transformed Problem* (FP)$'$.

PROOF. Suppose $\bar{w} \in \partial p(0)$. Then by arguing in a manner similar to Zalmai [15], we can establish that $\bar{w}(t) \geq 0$ almost everywhere in $[0, T]$, i.e. $\bar{w} \in D_z$. Since $\bar{z}$ is an optimal solution of (FP), therefore by Lemma 2.4, with $\bar{u} = [\int_0^T h(\bar{z}(t), t)\,dt]^{-1}$, $\bar{s}(t) = (\int_0^T h(\bar{z}(t), t)\,dt]^{-1} z(t)$, $(\bar{s}, \bar{u})$ is an optimal solution of (FP)$'$. Now proceeding as in the converse of Lemma 3.2, we get

$$\frac{\int_0^T f(z(t), t)\,dt}{\int_0^T h(z(t), t)\,dt} - \int_0^T \bar{w}(t) g(z(t), t)\,dt \leq \frac{\int_0^T f(\bar{z}(t), t)\,dt}{\int_0^T h(\bar{z}(t), t)\,dt} \quad \text{for all } z \in Z^0,$$

which gives $\psi(\bar{w}) \leq \phi(\bar{z})$. By Theorem 4.1, $\psi(\bar{w}) \geq \phi(\bar{z})$. It now follows from Corollary 4.1 that $\bar{w}$ is an optimal solution of the dual problem. Conversely suppose $\bar{w} \in L_\infty^m[0, T]$ is an optimal solution of the dual problem (DFP) with $\psi(\bar{w}) = \phi(\bar{z})$. Since $\bar{z}$ is an optimal solution of (FP), therefore by Lemma 2.4, with

$$\bar{s} = \left[\int_0^T h(\bar{z}(t), t)\,dt\right]^{-1} \bar{z}(t), \qquad \bar{u} = \left[\int_0^T h(\bar{z}(t), t)\,dt\right]^{-1},$$

$(\bar{s}, \bar{u})$ is an optimal solution of (FP)$'$. By the definition of $p(\cdot)$, $p(0) = \phi(\bar{z}) = \psi(\bar{w})$ which gives that

$$\frac{\int_0^T f(z(t), t)\,dt}{\int_0^T h(z(t), t)\,dt} - \int_0^T \bar{w}(t) g(z(t), t)\,dt \leq p(0). \tag{4.1}$$

Therefore for all $(s, u) \in W$, $y \in V_1^m[0, T]$,

$$g(s(t)/u, t) \leq y(t), \int_0^T u h(s(t)/u, t)\,dt \leq 1,$$

we have (by (4.1))

$$\int_0^T u f\left(\frac{s(t)}{u}, t\right) dt \leq \frac{\int_0^T f(z(t), t)\,dt}{\int_0^T h(z(t), t)\,dt} \leq p(0) + \int_0^T \bar{w}(t) y(t)\,dt.$$

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Hence

$$p(y) \leq p(0) + \int_0^T \bar{w}(t)y(t)\,dt. \tag{4.2}$$

Therefore by (4.2), and in view of the fact that $p(y) = -\infty$ if there exist no $(s, u) \in W$ such that

$$g\left(\frac{s(t)}{u}, t\right) \leq y(t), \qquad \int_0^T uh\left(\frac{s(t)}{u}, t\right) dt \leq 1,$$

$$p(y) \leq p(0) = \int_0^T \bar{w}(t)y(t)\,dt \quad \text{for all } y \in V_1^m[0, T].$$

Hence $\bar{w} \in \partial p(0)$.

THEOREM 4.3. *Suppose*

(1) $p(0)$ *is finite,*
(2) $\bar{w} \in \partial p(0)$.

Then $\bar{z} \in L_\infty^n[0, T]$ *is an optimal solution of the primal problem* (FP) *if and only if* $(\bar{z}, \bar{w})$ *satisfies the optimality conditions* (i), (ii) *and* (iv) *of Definition* 3.1.

PROOF. Suppose $\bar{z}$ is an optimal solution of the primal problem (FP). Then by Theorem 3.1 and Definition 3.2 we see that $(\bar{z}, \bar{w})$ satisfies condition (i), (ii) and (iv) of Definition 3.1.

Conversely suppose $(\bar{z}, \bar{w})$ satisfies conditions (i), (ii) and (iv) of Definition 3.1. Then condition (iii) of Definition 3.1 is satisfied as shown by Zalmai [15]. Hence by Theorem 3.3, $(\bar{z}, \bar{u})$ is a saddlepoint of the Lagrangian L.

Also $\bar{w} \in \partial p(0)$ gives that Problem (FP)$'$ is stable. Hence by Theorem 3.3, $\bar{z}$ is an optimal solution of (FP).

THEOREM 4.4. *Suppose Problem* (FP)$'$ *is stable. Then*

(1) *Problem* (DFP) *has an optimal solution.*
(2) *The optimal values of Problems* (FP) *and* (DFP) *are equal.*
(3) $\bar{w} \in L_\infty^m[0, T]$ *is an optimal solution of Problem* (DFP) *if and only if* $\bar{w} \in \partial p(0)$
(4) *Every optimal solution* $\bar{w}$ *of Problem* (DFP) *characterises the set of all optimal solutions* (*if there are any*) *of Problem* (FP) *as maximisers of*

$$\frac{\int_0^T f(z(t), t)\,dt}{\int_0^T h(z(t), t)\,dt} - \int_0^T \bar{w}(t)g((t), t)\,dt$$

over Z^0 *which also satisfy the constraints* $g(z(t), t) \leq 0$ *almost everywhere in* $[0, T]$ *and* $\bar{w} g(\bar{z}(t), t) = 0$ *almost everywhere in* $[0, T]$.

Proof. Since Problem $(\text{FP})'$ is stable, $p(0)$ is finite. By Lemma 3.1 and Lemma 3.3, p has a subgradient $\bar{w}$ at $y = 0$. Since $p(0)$ exists, Problem $(\text{FP})'$ has an optimal solution and consequently Problem (FP) has an optimal solution $\bar{z}$. Then by Theorem 4.2, conclusions (1), (2) and (3) hold. The conclusion (4) follows from (3) and Theorem 4.3.

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