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FATOU-RIESZ-THEOREMS IN GENERAL SEQUENCE SPACES

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1.

Consider a formal series $\sum_{n=0}^{\infty} a_n$ with partial sums $s_n = \sum_{n=0}^{n} a_k$ and the corresponding power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Throughout we will assume that f is analytic for $|z| < 1$, i.e. that $\limsup_{n\to\infty} |a_n|^{1/n} \leqq 1$. A classical theorem of Fatou-Riesz (see (**1**, **4**)) states that if $\lim_{n\to 0} a_n = 0$ and

$$(F-R)\colon f \text{ is analytic for } z=1, f(1)=0$$

then $\sum_{n=0}^{\infty} a_n$ is convergent to 0.

Jurkat-Peyerimhoff (**2**, **3**) obtained the following modification for absolute convergence:

If $\sum_{n=0}^{\infty} |a_{n+1} - a_n| < \infty$ and $(F-R)$ are satisfied then $\sum_{n=0}^{\infty} |a_n| < \infty$.

If we denote by

$$c_0 = \{a = (a_n)_0^\infty \mid \lim_{n\to\infty} a_n = 0\}$$

the space of null sequences and

$$b_v = \{a = (a_n)_0^\infty \mid \sum_{n=0}^{\infty} |a_{n+1} - a_n| < \infty\}$$

the space of sequences of bounded variation, then the above theorems may be equivalently formulated in the following way:

If $(F-R)$ holds then $(a_n)_0^\infty \in c_0$ implies $(s_n)_0^\infty \in c_0$.
If $(F-R)$ holds then $(a_n)_0^\infty \in b_v$ implies $(s_n)_0^\infty \in b_v$.

This formulation leads to the consideration of a general theorem of the following type:

Given a certain sequence space V. If $(F-R)$ holds then $(a_n)_0^\infty \in V$ implies $(s_n)_0^\infty \in V$.

The main problem now is to decide for which type of sequence spaces this general

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theorem is valid. Remarkably enough there is a sufficient condition of a purely algebraic nature which applies to a wide class of sequence spaces.

2.

We use the notations $a=(a_n)_0^\infty$, $s=(s_n)_0^\infty$ (where $s_n=\sum_{k=0}^{n} a_k$) and $e_k=(\delta_{nk})_0^\infty$ (having 1 in the k-th position and 0 otherwise). We also consider two-sided infinite sequences $g=(g_n)_{-\infty}^{\infty}$. For $\alpha>0$ let

$$\mathcal{K}_\alpha=\{g=(g_n)_{-\infty}^{\infty} \mid g_n=O(|n|^{-\alpha}), |n|\to\infty\}.$$

We consider the convolution product

$$b=g*a, \quad (b_n)_0^\infty=(g_n)_{-\infty}^{\infty}*(a_n)_0^\infty$$

defined by

$$b_n=\sum_{k=0}^{\infty} g_{n-k}a_k, \quad n=0,1,2,\ldots$$

(the sums being assumed to exist for all $n=0,1,2,\ldots$). We will consider a sequence space V which is a linear space over $\boldsymbol{C}$ and in addition satisfies the following axioms:

A_0: $e_0\in V$

B: for each $a\in V$ there exists $\alpha>0$, such that for every $g=(g_n)_{-\infty}^{\infty}\in\mathcal{K}_\alpha$ the condition $g*a=b\in V$ is satisfied.

Axiom B states that $a\in V$ is mapped into V by the convolution product $g*a$, as long as $g_n=O(|n|^{-\alpha})$.

Here $\alpha=\alpha(a)$ still may depend on the element a considered. However for many spaces V of interest there exists a universal $\alpha_0=\alpha_0(V)$ for all $a\in V$, i.e. all $g\in\mathcal{K}_{\alpha_0}$ act as convolution operators mapping V into itself.

We summarise some simple properties of the space V.

(i) For each $a=(a_n)_0^\infty\in V$ there exists $\beta=\beta(a)>0$ such that $a_n=O(n^\beta)$.

This follows from the fact that for $\alpha=\alpha(a)$ the sum $b_0=\sum_{k=0}^{\infty}(1+k)^{-\alpha}a_k$ exists (take $\beta(a)=\alpha(a)$).

(ii) There exists $\alpha_1=\alpha_1(V)>0$ such that for

$$a=(a_n)_0^\infty$$
$$a_n=O(n^{-\alpha_1}) \quad \text{implies} \quad (a_n)_0^\infty\in V.$$

Take $\alpha_1=\alpha(e_0)$ and for $a_n=O(n^{-\alpha_1})$ let

$$g_n=\begin{cases} a_n & n\geqq 0\\ 0 & n<0\end{cases}; \quad \text{then} \quad a=g*e_0\in V.$$

Let us denote by $\mathcal{F}$ the set

$$\{g=(g_n)_{-\infty}^{\infty} \mid g_n\neq 0 \quad \text{for only finitely many } n\in\boldsymbol{Z}\}$$

Since $g\in\mathcal{F}$ implies $g_n=0(|n|^{-\alpha})$ for all $\alpha>0$ we get

(iii) $g*a\in V$ for $g\in\mathcal{F}$,

i.e. for $g\in\mathcal{F}$ the convolution operation $g*a$ maps V into itself.

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A particular case is given by
(iv) The shift operators

$$\Gamma^{(k)}: a=(a_n)_0^\infty \to \Gamma^{(k)}a=(0,0,\ldots,0,a_0,a_1,\ldots)$$
$$(a_0 \text{ at } k\text{-th position})$$

$$\Gamma^{(-k)}: a=(a_n)_0^\infty \to \Gamma^{(-k)}a=(a_k,a_{k+1},\ldots)$$

($k \in \mathbf{N}$ in both cases) map V into itself.
If we consider the axioms

$$A_k: e_k \in V$$

then we get, from (iv),
(v) If B is assumed, then A_0 and A_k, $k=1,2,\ldots$ are equivalent.

3.

We now are able to state our main result

Theorem 1. *Let V be a linear sequence space over the field $\mathbf{C}$, satisfying the axioms A_0 and B. For $a=(a_n)_0^\infty$ consider the sequence $s=(s_n)_0^\infty$, $s_n=\sum_{k=0}^{n} a_k$, and the power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$.*
If the condition

$$(F-R): \quad f \text{ is analytic for } z=1, f(1)=0$$

is satisfied, then $(a_n)_0^\infty \in V$ implies $(s_n)_0^\infty \in V$.
We need the following lemma which is essentially known.

Lemma. *Given a function f analytic on the arc*

$$\{z=e^{i\phi} \mid \phi_1 \leqq \phi \leqq \phi_2\} \quad (\phi_2<2\pi+\phi_1)$$

having zeros of order $\geqq \gamma \in \mathbf{N}$ at $z_1=e^{i\phi_1}$ and $z_2=e^{i\phi_2}$. Then

$$\int_{\varphi_1}^{\varphi_2} f(e^{i\phi})e^{in\phi}d\phi = O(|n|^{-\gamma}) \quad \text{for} \quad |n|\to\infty.$$

Proof. There exists $R>1$ such that f is analytic in the closed domain

$$\{z=re^{i\phi} \mid 1/R \leqq r \leqq R, \phi_1 \leqq \phi \leqq \phi_2\}.$$

For $n<0$ we write the integral in the form

$$\frac{1}{i}\int_{e^{i\phi_1}}^{e^{i\phi_2}} f(z)z^{n-1}dz = \frac{1}{i}\int_1^R f(re^{i\phi_1})e^{in\phi_1}r^{n-1}dr$$
$$+\int_{\phi^1}^{\phi_2} f(Re^{i\phi})R^n e^{in\phi}d\phi + \frac{1}{i}\int_R^1 f(re^{i\phi_2})e^{in\phi_2}r^{n-1}dr: \;= I+II+III.$$

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The first integral may be estimated by

$$|I| \leqq M \int_1^R (r-1)^\gamma r^{-|n|} dr = O(|n|^{-\gamma})$$

the same estimate being valid for the third integral. The second integral is estimated by

$$|II| \leqq MR^{-|n|} = O(|n|^{-\gamma}).$$

For $n > 0$ we proceed in a similar way replacing R by $1/R$.

Proof of Theorem 1. Let

$$s(z) = \sum_{n=0}^{\infty} s_n z^n = \frac{1}{1-z} f(z)$$

then

$$s_n = \frac{1}{2\pi i} \oint_{|z|=1/2} \frac{s(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \oint_{|z|=1/2} \frac{f(z)}{(1-z)} \cdot \frac{dz}{z^{n+1}}.$$

Since f is analytic for $z = 1$ and $f(1) = 0$, s is analytic for $z = 1$, and there exists $0 < \phi_0 < \pi$, such that s is analytic on the arc $\{z = e^{i\phi} \mid -\phi_0 \leqq \phi \leqq \phi_0\}$.

Now there exists a polynomial $P(z) = \sum_{k=0}^{q} p_k z^k$ such that

$$g(z) = \frac{1}{1-z} - P(z)$$

at $z_1 = e^{-i\phi_0}$, $z_2 = e^{+i\phi_0}$ has zeros of order $\gamma \geqq \alpha + \alpha_1$ where $\alpha = \alpha(a)$ is the exponent from axiom B corresponding to the sequence $a = (a_n)_0^\infty$ and $\alpha_1 = \alpha(e_0)$ from property (ii).

We obtain

$$s_n = \frac{1}{2\pi i} \oint_{|z|=1/2} \frac{f(z)P(z)dz}{z^{n+1}} + \frac{1}{2\pi i} \oint_{|z|=1/2} \frac{f(z)g(z)dz}{z^{n+1}} := s_n^{(1)} + s_n^{(2)}.$$

If we let $p = (\ldots 0, 0, p_0, p_1, \ldots, p_q, 0, o, \ldots)$($p_0$ at 0-th position) we get $(s_n^{(1)})_0^\infty = s^{(1)} = p * a$ which implies $(s_n^{(1)})_0^\infty \in V$ by property (iii). In view of property (i) f has distributional boundary values

$$f(e)^{i\phi}) = \sum_{n=0}^{\infty} a_n e^{in\phi}$$

the sum converging in the distributional sense.

We can therefore write

$$s_n^{(2)} = \frac{1}{2\pi} \int_{-\phi_0}^{\phi_0} f(e^{i\phi}) g(e^{i\phi}) e^{-in\phi} d\phi + \frac{1}{2\pi} \int_{\phi_0}^{2\pi-\phi_0} f(e^{i\phi}) g(e^{i\phi}) e^{-in\phi} d\phi := s_n^{(3)} + s_n^{(4)}.$$

Since $f(z)g(z)$ is analytic on $\{z = e^{i\phi} \mid -\phi_0 \leqq \phi \leqq \phi_0\}$ having zeros of order $\geqq \gamma$ at $e^{\pm i\phi_0}$ we obtain from the lemma

$$s_n^{(3)} = O(n^{-\gamma}) \qquad n \to \infty$$

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and $\gamma \geqq \alpha_1$ (property (ii)) implies $(s_n^{(3)})_0^\infty \in V$. In order to estimate $s_n^{(4)}$ let be

$$h(e^{i\phi}) = \begin{cases} g(e^{i\phi}), & \phi_0 \leqq \phi \leqq 2\pi - \phi_0 \\ 0 & -\phi_0 < \phi < \phi_0 \end{cases}$$

Then h may be represented by a Fourier series

$$h(e^{i\phi}) = \sum_{n=-\infty}^{\infty} g_n e^{in\phi}.$$

Since $g(z)$ is analytic on the arc $\{z = e^{i\phi} \mid \phi_0 \leq \phi \leqq 2\pi - \phi_0\}$ having zeros of order $\geqq \gamma$ at $e^{\pm i\phi_0}$ we obtain from the lemma

$$g_n = O(|n|^{-\gamma}) \qquad |n| \to \infty.$$

Since $f(e^{i\phi})$ is analytic for $-\phi_0 \leqq \phi \leqq \phi_0$ and $g(e^{i\phi})$ is analytic for $\phi_0 \leqq \phi \leqq 2\pi - \phi_0$, the product is the well-defined distribution

$$f(e^{i\phi})h(e^{i\phi}) = \sum_{n=-\infty}^{\infty} b_n e^{in\phi}$$

where

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\phi})h(e^{i\phi})e^{-in\phi}d\phi = \sum_{k=0}^{\infty} g_{n-k}a_k.$$

So we obtain $b_n = s_n^{(4)}$ for $n \geqq 0$ and

$$(s_n^{(4)})_0^\infty = (g_n)_{-\infty}^\infty * (a_n)_0^\infty$$

which implies $(s_n^{(4)})_0^\infty \in V$ from axiom B and $\gamma \geqq \alpha(a)$. From

$$s_n = s_n^{(1)} + s_n^{(3)} + s_n^{(4)}$$

the statement of the theorem now follows.

4.

We give some examples of sequence spaces for which the Fatou-Riesz-theorem is valid.

Axiom A_0 is satisfied in nearly all sequence spaces of interest, although there exist trivial counter-examples where it does not hold (take the space spanned by the sequence $(1, 1, 1, \ldots)$). So the essential point is to show the validity of axiom B. In certain cases there exists a universal $\alpha_0 = \alpha_0(V)$ not depending on a such that B is satisfied. This might be checked in the following way. The convolution $b = g * a$ may be formally written as

$$b = S \cdot a$$

where

$$S = \sum_{k=-\infty}^{\infty} g_k \Gamma^{(k)}$$

is an infinite linear combination in the shift operators $\Gamma^{(k)}$.

In order to show that S is a well-defined map from V into itself if $g_n = O(|n|^{\alpha_0})$, one has to show that the operator sum $S = \sum_{k=-\infty}^{\infty} g_k \Gamma^{(k)}$ converges in a suitable sense. Usually

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this requires continuity arguments and consequently a topological structure on V.

We give a precise statement in the case of a Banach space.

Theorem 2. *Let V be a Banach sequence space. Assume that the shift operators $\Gamma^{(k)}$, $k \in \mathbf{Z}$, are bounded linear maps from V into itself with norms*

$$\|\Gamma^{(k)}\| = O(|k|^{\beta}) \quad as \quad |k| \to \infty$$

for some $\beta > 0$.

Then axiom B is satisfied in V. If in addition axiom A holds then the Fatou-Riesz-theorem is valid in V.

Proof. Choose $\alpha_0 > \beta + 1$, then $|g_n| = O(|n|^{-\alpha_0})$ implies the convergence of S in the operator norm topology.

In the spaces c_0 and b_v we have $\|\Gamma^k\| = 1$ and we obtain the known results. We now give some new examples.

(1) Take $V = l_p$ the (unweighted) l_p-space, $1 \leqq p \leqq \infty$, then $\|\Gamma^{(k)}\| = 1$, so Theorem 2 shows the validity of the Fatou-Riesz-theorem. (The case $p = \infty$ was mentioned already by Fatou (**1**).)

More generally we may take certain weighted l_p-spaces:

(2) For $1 \leqq p \leqq \infty$ let be

$$V = l_w^p = \left\{ a = (a_n)_0^\infty, \|a\|_{p,w} = \left\| \left(\frac{a_n}{w_n} \right)_0^\infty \right\|_p < \infty \right\}$$

where $\|(a_n)_0^\infty\|_p$ denotes the usual l_p-norm, and $w = (w_n)_0^\infty$ is a fixed positive sequence. Then axiom B is satisfied if the following regularity condition for w holds

$$\sup_{n=0,1,2,\ldots} \frac{w_{n+k}}{w_n} < M \cdot (|k|+1)^{\beta}$$

for some $M > 0$, $\beta > 0$ (where $w_{-n} = w_0$, $n = 1, 2, \ldots$). For we get

$$\|\Gamma^{(-k)} a\|_{p,w} = \left\| \left(\frac{w_{n+k}}{w_n} \frac{a_{n+k}}{w_{n+k}} \right) \right\|_p < \sup_{n=0,1,2,\ldots} \frac{w_{n+k}}{w_n} \left\| \frac{a_n}{w_n} \right\|_p < M(|k|+1)^{\beta} \|a\|_{p,w}.$$

(3) Take the space of C_1-summable series

$$C_1 = \left\{ (a_n)_0^\infty, \lim_{n \to \infty} \sigma_n \text{ exists} \right\} \quad \text{where} \quad \sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k.$$

This is a Banach space with norm $\|a\| = \sup_n |\sigma_n|$.

Then $\|\Gamma^{(k)}\| = 0(|k|)$, as seen by direct calculation.

5.

Finally we list some open problems.

1. Conditions for the validity of the Fatou-Riesz-theorem are given by the purely algebraic axioms A_0 and B. Do these axioms imply that the space V can be topologised in a suitable way?

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2. Theorem 1 gives only sufficient conditions for the validity of the Fatou–Riesz-theorem, and it remains open whether the theorem could be proved in a much wider class of spaces. Consider

$$V_1=\left\{a=(a_n)_0^\infty,\quad \limsup_{n\to\infty}|a_n|^{1/n}\leqq 1\right\}$$

$$V_2=\left\{a=(a_n)_0^\infty,\quad \limsup_{n\to\infty}\frac{\log|a_n|}{\log n}<\infty\right\}$$

$$V_3=\left\{a=(a_n)_0^\infty\quad f(z)=\sum_0^\infty a_n z^n\in H^p\right\}$$

(where H^p denote the Hardy space of order p), then $a\in V_i$ and $(F-R)$ implies $s\in V_i$ as seen directly without using Theorem 1, in V_1 and V_2 it holds even without the $(F-R)$ condition.

3. The classical Fatou-Riesz-theorem has the stronger statement: $\lim_{n\to\infty} a_n=0$ implies the uniform convergence of $s_n(z)$ on any closed arc of $|z|=1$ on which $f(z)=\sum_{n=0}^\infty a_n z^n$ is analytic. Does there exist a suitable generalisation in general sequence spaces?

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