SEMI-BROUWERIAN ALGEBRAS

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Introduction

Ever since David Ellis has shown that a Boolean algebra has a natural structure of an autometrized space, the interest in such spaces has led several authors to study various autometrized algebras like Brouwerian algebras [9], Newman algebras [4], Lattice ordered groups [6], Dually residuated lattice ordered semigroups [7] etc. However all these spaces are lattices (with the exception of Newman algebra which is not even a partially ordered set); and a natural question would be whether there are semilattices with a natural structure of an autometrized space. In the present paper we observe that the dual of an implicative semilattice [8] is a generalization of Brouwerian algebra and it has a natural structure of an autometrized space.

In §1 we define a semi-Brouwerian algebra and show that a semi-Brouwerian algebra is a semilattice with 0 satisfying (F) (see Theorem 1) which readily shows that a semi-Brouwerian algebra is the dual of an implicative semilattice. We also prove that a semi-Brouwerian algebra is a Boolean ring if and only if the symmetric difference is a group operation. In §2 we observe that a semi-Brouwerian algebra is an autometrized space and show that the entire Brouwerian geometry of E. A. Nordhaus and Leo Lapidus can be extended to these spaces. We also prove that a semi-Brouwerian algebra is a Boolean ring if and only if it admits a metric group operation. We further prove that a semi-Brouwerian geometry (see definition 2) is a Boolean geometry if and only if semilattice betweenness coincides with metric betweenness.

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DEFINITION 1. *An algebra* $L = (L; +, -, 0)$ *with two binary operations* $+$, $-$ *and a nullary operation* 0 *is called a semi-Brouwerian algebra if and only if* (1.1) $a + a = a$, (1.2) $a + b = b + a$, (1.3) $a - a = 0$, (1.4) $(a-b) + b = a + b$, (1.5) $(a-b) - c = a - (c + b)$ *for all* a, b, c *in* L.

We now show that these axioms are independent.

EXAMPLE 1. Let S be any non-empty set with more than one element and let 0 be an element of S. Define $a + b = 0$ and $a - b = 0$ for all a, b in S. Then S satisfies all the axioms except (1.1).

EXAMPLE 2. Let S be any set with more than one element and let $0 \in S$. Define $a + b = b$ and $a - b = 0$ for all a, b in S. Then S satisfies all the axioms except (1.2).

EXAMPLE 3. Let $(S, +)$ be the two element join semilattice $\{0, 1\}$ and define $1 - 0 = 1, 0 - 0 = 0 - 1 = 0, 1 - 1 = 1$ on S. Clearly S satisfies all the axioms except (1.3).

EXAMPLE 4. Let $(S, +)$ be the two element join semilattice $\{0, 1\}$ and define $a - b = 0$ for all a, b in S. Obviously S satisfies all the axioms except (1.4). Axiom (1.4) does not hold in S for $(1 - 0) + 0 = 0 + 0 = 0 \neq 1 = 1 + 0$.

EXAMPLE 5. Consider $S = \{0, a, b\}$. Define $a + b = b + a = b$, $a + 0 = 0 + a = a$, $b + 0 = 0 + b = b$, $a + a = a$, $b + b = b$, $0 + 0 = 0$ and $a - a = b - b = 0 - 0 = 0 - a = 0 - b = 0, a - 0 = a, b - 0 = b, a - b = a$, $b - a = b$ on S. Obviously S satisfies all the axioms except (1.5). Axiom (1.5) does not hold in S for $(a - b) - a = a - a = 0 \neq a = a - b = a - (a + b)$.

REMARK 1. Example 2 shows that associativity of + may be valid even without commutativity. However it is not known whether any significant results can be obtained by replacing commutativity in the definition 1 by associativity.

THEOREM 1. *Let* $L = (L; +, -, 0)$ *be a semi-Brouwerian algebra. If we write* $a \leqq b$ *to mean* $a + b = b$, *then* $(L, \leqq)$ *is a semilattice with* 0 *as the least element satisfying* (*F*) $a - b \leqq c$ *if and only if* $a \leqq c + b$. *Conversely if* $(L, +, 0)$ *is a semilattice with* 0 *and* $-$ *is a binary operation in L-with* (*F*), *then* $(L, +, -, 0)$ *is a semi-Brouwerian algebra.*

Obviously $(L, \leqq)$ is reflexive and antisymmetric; and to prove this theorem we need the following four lemmas in which we assume that L is a semi-Brouwerian algebra and $a, b, c, \cdots, \in L$.

LEMMA 1. (i) $0 + a = a$, (ii) $0 - a = 0$ *and* (iii) $a - (b + a) = 0$.

PROOF. (i): $(a - a) + a = a + a$ (by (1.4)) so that $0 + a = a$ (by (1.3) and (1.1)).

(ii) $(0 - a) = (a - a) - a$ (by (1.3)) $= a - (a + a)$ (by (1.5)) $= a - a = 0$,
(iii) $a - (b + a) = (a - a) - b$ (by (1.5)) $= 0 - b = 0$ (by (ii) above).

LEMMA 2. $a - b = 0$ *if and only if* $a \leqq b$.

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PROOF. If $a - b = 0$ then $b = 0 + b = (a - b) + b = a + b$ so that $a \leqq b$. Conversely suppose that $a \leqq b$. Then $0 = a - (b + a) = a - (a + b) = a - b$.

LEMMA 3. (i) $(a + b) - b = a - b$, (ii) $(a + b) - (a + c) = b - (a + c)$.

PROOF. (i) $[(a + b) - b] - (a - b) = (a + b) - [(a - b) + b]$ (by (1.5)) $= (a + b) - (a + b)$ (by (1.4)) $= 0$ and $(a - b) - [(a + b) - b] = a - \{[(a + b) - b] + b\}$ (by (1.5)) $= a - [(a + b) + b]$ (by (1.4)) $= (a - b) - (a + b)$ (by 1.5)) $= (a - b) - (b + a)$ (by (1.2)) $= [(a - b) - a] - b = [a - (a + b)] - b = [a - (b + a)] - b = 0 - b = 0$ so that by Lemma 2 we have $(a + b) - b = a - b$.

(ii) $[(a + b) - (a + c)] - [b - (a + c)] = (a + b) - \{[b - (a + c)] + (a + c)\}$ (by (1.5)) $= (a + b) - [b + (a + c)]$ (by (1.4)) $= (a + b) - [(a + c) + b]$ (by (1.2)) $= [(a + b) - b] - (a + c)$ (by 1.5)) $= (a - b) - (a + c)$ (by (i) above) $[(a - b) - a] - c$ (by (1.2) and (1.5)) $= [a - (a + b)] - c = 0 - c =$ (by (iii) of Lemma 1) $= 0$ and $[b - (a + c)] - [(a + b) - (a + c)] = b - \{[(a + b) - (a + c)] + (a + c)\}$ (by (1.5)) $= b - [(a + b) + (a + c)]$ (by (1.4)) $= b - [(a + c) + (a + b)]$ (by (1.2)) $= [b - (a + b)] - (a + c)$ (by (1.5)) $= 0 - (a + c) = 0$ so that by Lemma 2 we have $(a + b) - (a + c) = b - (a + c)$.

LEMMA 4. $a + (b + c) = (a + b) + c$.

PROOF. $[a + (b + c)] - [(a + b) + c] = \{[a + (b + c)] - (a + b)\} - c$ (by (1.2) and (1.5)) $= [(b + c) - (a + b)] - c$ (by (ii) of Lemma 3) $= [c - (b + a)] - c$ (by (1.2) and (ii) of Lemma 3) $= c - [c + (b + a)]$ (by (1.5)) $= 0$ so that by Lemma 2 we have $a + (b + c) \leqq (a + b) + c$. Now $(a + b) + c = c + (b + a) \leqq (c + b) + a = (b + c) + a = a + (b + c)$ so that $a + (b + c) = (a + b) + c$.

PROOF OF THEOREM 1. Lemma 1.(i) and Lemma 4 readily imply that $(L, \leqq)$ is a semilattice with 0 as the least element. Now for all a, b, c in L, $a \leqq c + b \Leftrightarrow a - (c + b) = 0 \Leftrightarrow (a - b) - c = 0 \Leftrightarrow a - b \leqq c$. For the proof of the converse see Nemitz [8].

REMARK 2. *If a semi-Brouwerian algebra is a lattice, then it is a Brouwerian algebra.*

For an example of a semi-Brouwerian algebra which is not a lattice see page 139 in [8].

Throughout this article L denotes a semi-Brouwerian algebra. The following theorem is an immediate consequence of Theorem 1 (see Nemitz [8]).

THEOREM 2. *For all a, b, c in L the following are valid.*

(2.1) $a = a - 0$.

(2.2) $a - b \leqq a$.

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(2.3) *If* $a \leqq b$, *then* $a - c \leqq b - c$ *and* $c - b \leqq c - a$.

(2.4) $a \leqq b$ *if and only if* $a - b = 0$.

(2.5) $(a - b) - b = a - b$.

(2.6) $(a + b) - c = (a - c) + (b - c)$.

(2.7) *If* L *is a lattice with greatest lower bound* $\cap$, *then* L *is distributive and* $a - (b \cap c) = (a - b) + (a - c)$.

THEOREM 3. *Let* $(L; +, \leqq, -)$ *be a system in which* $(L, +, \leqq)$ *is a semi-lattice with* 0, $+$ *denotes the least upper bound with respect to* $\leqq$ *and* $(L, -)$ *is a binary algebra. Then the following statements are equivalent.*

1. $a - b \leqq c$ *if and only if* $a \leqq c + b$.
2. (i) $(a-b) + b = a + b$, (ii) $a - a = 0$ *and* (iii) $(a-b) - c = a - (c + b)$.
3. (i) $(a-b) + b = a + b$, (ii) $a - (a + b) = 0$ *and* (iii) $(a-b) - c = a - (c + b)$.
4. (i) $(a-b) + b = a + b$, (ii) $(a + b) - c = (a-c) + (b-c)$, (iii) $a - a = 0$ *and* (iv) $(a - b) + a = a$.

PROOF. That 1 implies 2 follows from Theorem 1.

Assume 2. From 2(ii) and 2(iii) we have $0 - a = 0$ so that by 2(ii) $0 = (a-a) - b = a - (b + a) = a - (a + b)$. Hence 2 implies 3.

Assume 3. From 3(ii) and 3(iii) we have $a - a = 0 - a = 0$ and by 3 (i) $(a-b) + a = [(a-b) - a] + a = [a - (a + b)] + a = 0 + a = a$. From 3(i) and 3(ii) it is clear that $a \leqq b$ if and only if $a - b = 0$. We also observe that $a \leqq b$ implies $a - c \leqq b - c$. If $a \leqq b$, then $a + b = b$ so that $a + b + c = b + c$ and hence $a - (b + c) = a - (a + b + c) = 0$ (by 3(ii)). Now $(a-c) - (b-c) = a - [(b-c) + c]$ (by 3(iii)) $= a - (b + c)$ (by 3(i)) $= 0$ so that $a - c \leqq b - c$. $[(a + b) - c] - [(a-c) + (b-c)] = (a + b) - [(a-c) + (b-c) + c]$ (by 3(iii)) $= (a + b) - [(a-c) + b + c]$ (by 3(i)) $= (a + b) - [(a-c) + c + b] = (a + b) - (a + c + b)$ (by 3(i)) $= 0$ (by 3(ii)) so that $(a + b) - c \leqq (a-c) + (b-c)$. Also $a \leqq a + b$ and $b \leqq a + b$ imply $(a-c) + (b-c) \leqq (a + b) - c$ so that $(a + b) - c = (a-c) + (b-c)$. Thus 3 implies 4.

Assume 4 and let $a \leqq c + b$. From 4(ii) $a \leqq b$ implies $a - c \leqq b - c$; hence $a - b \leqq (c + b) - b = (c-b) + (b-b) = c - b \leqq c$ by 4(iii) and 4(iv). If $a - b \leqq c$ then $a \leqq a + b = (a-b) + b \leqq c + b$. Hence 4 implies 1.

We shall write $a * b = (a-b) + (b-a)$ for $a, b \in L$ and call $a * b$ the symmetric difference of a and b. It is known that in a Brouwerian algebra (with 1) the symmetric difference is a group operation if and only if it is a Boolean algebra and we show now that the same is true even in the case of a semi-Brouwerian algebra. Put $ab = (a + b) - (a * b)$.

LEMMA 5. $ab = a - (a * b) = b - (a * b)$.

PROOF. First observe that $ab = ba$. Now by (2.6) and (1.3) we have $a-(a*b) = [a+(a*b)]-(a*b) = [a+(a-b) + (b-a)]-(a*b) = (a+b)-(a*b)$ (and hence by symmetry) $= b-(a*b)$.

THEOREM 4. *If $a-(b-a) = a$ for all a, b in L, then L is a Boolean ring.*

To prove this theorem we require the next three lemmas in which we assume $a-(b-a) = a$ for all a, b in L.

LEMMA 6. *$a(bc) = (ab)c$ for all a, b, c in L.*

PROOF. By hypothesis $a-(a-b) = [a-(b-a)]-(a-b) = a-[(a-b)+(b-a)]$ (by (1.5)) $= a-(a*b)$ so that by the above lemma $ab = a-(a-b) = b-(b-a)$ (by symmetry). Now $(ab)c = ab-(ab-c) = [b-(b-a)] - \{[b-(b-a)]-c\} = [b-(b-a)]-\{b-[c+(b-a)]\}$ (by (1.5)) $= [b-(b-a)] - \{b-[(b-a)+c]\} = [b-(b-a)] - [(b-c)-(b-a)]$ (by (1.5)) $= b-\{[(b-c)-(b-a)]+(b-a)\}$ (by (1.5)) $= b-[(b-c)+(b-a)]$ (by (1.4)) $= b-[(b-a)+(b-c)] = (cb)a = a(bc)$.

LEMMA 7. *ab is the greatest lower bound of a and b in L.*

PROOF. Firstly if $ab = a$ then by Lemma 5, $b-(a*b) = a$ so that $a \leqq b$ and if $a \leqq b$ and $a-(b-a) = a$ then $ab = a-(a-b) = a-0 = a$. Obviously ab is a lower bound of a and b by Lemma 5 and (2.2); and now let t be a lower bound of a and b. Then $ta = t$ and $tb = t$ so that $t(ab) = (ta)b = tb = t$. Therefore $t \leqq ab$ so that ab is the greatest lower bound of a and b in L.

LEMMA 8. *L is a relatively complemented distributive lattice.*

PROOF. From lemmas 6 and 7 it follows that L is a lattice with greatest lower bound of a and b as ab so that by (2.7) it follows that L is a distributive lattice. Now let $a \in L$ and $0 \leqq x \leqq a$. Put $y = a-x$. Clearly by 2.2, $0 \leqq y \leqq a$. Further $y+x = (a-x)+x = a+x = a$. Also $xy = y-(x*y) = y-[(x-y)+(y-x)] = y-[(y-x)+(x-y)] = [y-(x-y)]-(y-x) = y-(y-x) = y-[(a-x)-x] = y-(a-x) = 0$. Thus $[0, a]$ is complemented for every $a \in L$ so that L is relatively complemented.

PROOF OF THEOREM 4. From lemmas 6, 7 and 8 it follows that L is a relatively complemented distributive lattice with 0 and hence L has the structure of a Boolean ring.

Now the following theorem shows that if $*$ is a group operation in L, then L has the structure of a Boolean ring.

THEOREM 5. *In L the following statements are equivalent.*

(1) *$*$ is a group operation.*

(2) *$*$ is associative.*

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(3) $*$ *is cancellative.*
(4) $a-(b-a)=a$ *for all* a,b *in* L.
(5) $(L,*,\cdot)$ *is a Boolean ring.*
(6) $(L,*)$ *is a loop.*
(7) ab *is the greatest lower bound of* a *and* b *in* L.
(8) $a+bc=(a+b)(a+c)$ *for all* a,b,c *in* L.
(9) $a-bc=(a-b)+(a-c)$ *for all* a,b,c *in* L.
(10) $x\leqq y$ *implies* $ax\leqq ay$ *for all* a *in* L.

PROOF. The order of demonstration is (1) $\Rightarrow$ (2), (3) and (6); (2) $\Rightarrow$ (1); (3) $\Rightarrow$ (4); (4) $\Rightarrow$ (5); (5) $\Rightarrow$ (1) and (7); (6) $\Rightarrow$ (3); (7) $\Rightarrow$ (4), (8), (9) and (10); (8) $\Rightarrow$ (4); (9) $\Rightarrow$ (7); (10) $\Rightarrow$ (7).

Now (1) $\Rightarrow$ (2), (3) and (6); (4) $\Rightarrow$ (5); (5) $\Rightarrow$ (1); (6) $\Rightarrow$ (3); (7) $\Rightarrow$ (8), (9) and (10) are all obvious.

Assume (2). Since $a*a=0$ and $a*0=0*a=a$ for every a in L it follows that $*$ is a group operation. Hence (2) $\Rightarrow$ (1).

Assume (3). Then $a*(b-a)=[a-(b-a)]+[(b-a)-a]=[a-(b-a)]+(b-a)=a+(b-a)=a+b$ and $[a-(b-a)]*(b-a)=(b-a)*[a-(b-a)]=(b-a)+a=a+b$ so that $a-(b-a)=a$. Hence (3) $\Rightarrow$ (4).

Assume (5). Let aXb be the greatest lower bound of a and b in L. Then since (5) implies (1) we have $ab=a-(a*b)=a*[aX(a*b)]$ (since in a Boolean ring $(B,+,\cdot)a-b=a+ab)=a*[(aXa)*(aXb)]=a*[a*(aXb)]=(a*a)*(aXb)=aXb$. Hence (5) $\Rightarrow$ (7).

Assume (7). Let $a,b\in L$. Then $a=(a+b)a=a-[a*(a+b)]=a-[(a+b)-a]=a-(b-a)$ so that (7) $\Rightarrow$ (4).

Assume (8). Now $a=a+ab=(a+a)(a+b)=a(a+b)=a-(b-a)$ so that (8) $\Rightarrow$ (4).

Assume (9). Let $a,b\in L$. We already know that ab is a lower bound of a and b. Let t be a lower bound of a and b. Then $0=(t-a)+(t-b)=t-ab$ so that $t\leqq ab$. Hence (9) $\Rightarrow$ (7).

Assume (10). Let $a,b\in L$ and let t be a lower bound of a and b. Now $t\leqq a$ and $t\leqq b$ implies $t=t^2\leqq tb\leqq ab$ so that $t\leqq ab$. Hence (10) $\Rightarrow$ (7).

2

It is well known that a relatively complemented distributive lattice with 0 is an autometrized space. We refer to this space as Boolean geometry (see [9]).

THEOREM 6. *The symmetric difference in a semi-Brouwerian algebra is a metric operation.*

PROOF. Obviously $a*a=0$ and suppose $a*b=0$; then $a-b=0$ and $b-a=0$ so that $a=b$ by (2.4). Now let $(a*b)+(b*c)=t$. Then each of

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$a-b$, $b-a$, $b-c$ and $c-b$ is $\leqq t$ so that $a \leqq t+b$, $b \leqq t+a$, $b \leqq t+c$ and $c \leqq t+b$. Hence $a \leqq t+c$ and $c \leqq t+a$ so that $a*c = (a-c)+(c-a) \leqq t$.

COROLLARY 1. *Every semi-Brouwerian algebra is an autometrized space (see definition* 1 *in* [5]).

DEFINITION 2. *A semi-Brouwerian algebra autometrized via the symmetric difference is called a semi-Brouwerian geometry.*

In the rest of this article L denotes a semi-Brouwerian geometry. We will regard a triple of elements a, b, c as the vertices of a triangle denoted by $\Delta(a, b, c)$ and call $a*b$, $b*c$, $c*a$ the sides of this triangle.

THEOREM 7. *L is a Boolean geometry if and only if it is free of isosceles triangles.*

PROOF. The necessity is obvious. Conversely suppose that $a*b = a*c$ and $b \neq c$; then it follows that a, b, c are all distinct and hence $\Delta(a, b, c)$ is an isosceles triangle. Hence if L is free of isosceles triangles, then $*$ is cancellative and therefore (by Theorem 5) L is a Boolean geometry.

The proofs of the following Theorems 8 to 14 are the same as for Brouwerian geometry (see [9]).

THEOREM 8. *In L the relation $(a, b, c)T$ is equivalent to each of the relations*
(i) $a+b = b+c = c+a = a+b+c$.
(ii) $a-b \leqq c$, $b-a \leqq c$, $c-a \leqq b$.
(iii) $a*b \leqq c \leqq a+b$.
(iv) $b-a = c-a$, $a-b = c-b$, $a-c = b-c$.

COROLLARY 2. *For $a, b \in L$ we have $(a, b, a*b)T$.*

THEOREM 9. *L is a chain if and only if all triangles are isosceles.*

THEOREM 10. *In L each side of a first distance triangle is under the opposite vertex.*

THEOREM 11. *In L every second distance triangle has fixity.*

THEOREM 12. *L is a Boolean geometry if and only if every first distance triangle has fixity.*

THEOREM 13. *L contains no equilateral triangles.*

THEOREM 14. *L contains no equilateral n-circuit for n-odd.*

Nordhaus and Lapidus [9] proved that a Brouwerian algebra with 1 is a Boolean algebra if and only if it admits a metric group operation. However we now show that even an improved result with much less hypothesis is valid.

THEOREM 15. *A semi-Brouwerian algebra is a Boolean ring if and only if it admits a metric group operation.*

To prove this theorem we need the following two lemmas.

LEMMA 9. *Let θ be a metric group operation in a semi-Brouwerian algebra L. Then for all a, b in L, $a \theta b \leqq a + b$.*

PROOF. Since $0 \theta 0 = 0$ the zero element of the group is zero. Thus $a \theta b \leqq (a \theta 0) + (0 \theta b) = a + b$.

LEMMA 10. *For any $a, b \in L$, $(a + b) * \{a - [a - (b - a)]\} = a + b$.*

PROOF. First we observe that $(a + b) - a = (a + b) - [a - (b - a)]$ for all $a, b \in L$. Since $a - (b - a) \leqq a$ we have $(a + b) - a \leqq (a + b) - [a - (b - a)]$. Also $(a + b) - [a - (b - a)] \leqq b - a = (a + b) - a$ since $(b - a) + [a - (b - a)] = a + (b - a) = a + b$. Thus $(a + b) - a = (a + b) - [a - (b - a)]$. Now putting $s = a - (b - a)$ and $t = (a + b) - (a - s)$ we have $t + (a - s) = (a + b) + (a - s) = a + b$ so that $a + b = t + (a - s) \leqq t + [(a + b) - s] \leqq a + b$. Hence $a + b = t + [(a + b) - s] = t + [(a + b) - a]$ (by the observation made above) $= t + \{[t + (a - s)] - a\} = t + (t - a) = t = (a + b) * \{a - [a - (b - a)]\}$.

PROOF OF THEOREM 15. Suppose that θ is a metric group operation in a semi-Brouwerian algebra L, and $c, d \in L$. Then $c = c \theta 0 \leqq (c \theta d) + (d \theta 0) = (c \theta d) + d$ so that $c - d \leqq c \theta d$; and similarly $d - c \leqq c \theta d$ so that $c * d \leqq c \theta d$. Now let $a, b \in L$ and put $a + b = x$. By Lemma 10 we have $x = x * \{a - [a - (b - a]\} \leqq x \theta \{a - [a - (b - a)]\}$ and on applying Lemma 9 we get $x \theta \{a - [a - (b - a)]\} = x = x \theta 0$ so that $a - [a - (b - a)] = 0$. Hence $a = a - (b - a)$ so that by Theorem 5, L is a Boolean ring. The converse is clear.

THEOREM 16. *The subgeometry (see definition 2.7 in [9]) generated by two elements of L contains atmost nine elements.*

PROOF. The same proof (with the same notation) as in Theorem 2.13 of [9] shows that $(a, b, c)T$ and $(a, d, c)T$ so that by (iv) of Theorem 8 it follows that $c = (a - b) + (b - a) = (c - b) + (c - a) \leqq (c - d) + (c - a)$ (since $d \leqq b$) $= (a - d) + (d - a) = a * d \leqq c$; hence $c = a * d$. The rest of the proof is the same as in [9].

COROLLARY 3. *The subgeometry generated by any two comparable elements of L contains atmost six elements.*

PROOF. See [9].

Theorems 3.5 and 3.6 of [9] are valid even if L is a semilattice.

REMARK 3. *The concept of semilattice betweenness (and symmetry) are as in [9] where ab is interpreted as $(a + b) - (a * b)$.*

THEOREM 17. *$(a, b, c)L$ implies $ac \leqq b \leqq a + c$.*

PROOF. If $ab + bc = b = (a + b)(b + c)$, then $b = ab + bc \leqq a + c$; also $a - \{[(a-c) - b] + [(c-a) - b]\} = a - \{[a - (b + c)] + [c - (a + b)]\} \leqq (a + b) - \{[a - (b + c)] + [c - (a + b)]\} = (a + b) - \{[(a + b) - (b + c)] + [(b + c) - (a + b)]\} = (a + b) - [(a + b) * (b + c)] = (a + b)(b + c) = b$ so that $a \leqq b + [(a-c) - b] + [(c-a) - b] \leqq b + (a - c) + (c - a) = b + (a * c)$ and hence $ac = a - (a * c) \leqq b$.

THEOREM 18. *L is a Boolean geometry if and only if $ac \leqq b \leqq a + c$ implies $(a, b, c)L$.*

PROOF. The necessity follows from the fact that in a distributive lattice $(L, +, \cdot)$ $ac \leqq b \leqq a + c$ if and only if $(a, b, c)L$. Conversely suppose that $ac \leqq b \leqq a + c$ implies $(a, b, c)L$. For $a, b \in L$, $ab \leqq b \leqq a + b$ and hence by hypothesis we have $(a, b, b)L$ so that $b = (a + b)(b + b) = (a + b)b = b - [(a + b) - b] = b - (a - b)$. Therefore by Theorem 5, L is a Boolean geometry.

THEOREM 19. *If $(a - b) + ab = a$ for all a, b in L, then* (i) *L is symmetric and* (ii) *semilattice betweenness implies metric betweenness (see definition* 3.2 *in* [9]).

PROOF. (i) since $a = (a-b) + ab$ we have $a - ab = [(a-b) + ab] - ab \leqq a - b \leqq a - ab$ so that $a - b = a - ab$. Hence $(a + b) * ab = (a+b) - ab = (a - ab) + (b - ab) = (a-b) + (b-a) = a * b$. (ii) Assuming $(a, b, c)L$ we have by Theorem 17, $ac \leqq b \leqq a + c$. Now $ac \leqq b$ implies $a - b \leqq a - ac = a - c$ and $c - b \leqq c - ac = c - a$. Also $b \leqq a + c$ implies $b - c \leqq a$ and $b - a \leqq c$ so that $b - c = (b - c) - c \leqq a - c$ and $b - a = (b - a) - a \leqq c - a$. Now we have $a - b \leqq a - c, b - a \leqq c - a, c - b \leqq c - a$ and $b - c \leqq a - c$ so that $a * b \leqq a * c$ and $b * c \leqq a * c$. Thus $(a * b) + (b * c) \leqq a * c \leqq (a * b) + (b * c)$ so that $(a, b, c)M$.

THEOREM 20. *L is a Boolean geometry if and only if semilattice betweenness coincides with metric betweenness.*

PROOF. Suppose L is a Boolean geometry. Then by Theorem 19 semilattice betweenness implies metric betweenness and a straightforward verification shows that metric betweenness implies semilattice betweenness. Conversely, suppose that semilattice betweenness coincides with metric betweenness. In view of Theorem 5 it is enough to show that $*$ is cancellative. Now let $a, b, c \in L$ with $a * b = a * c$. Then it follows that $(a, b, c)M$ and $(a, c, b)M$ from which we have $(a, b, c)L$ and $(a, c, b)L$. Hence $ab + bc = b$ and $ac + cb = c$ so that $ab = a - (a * b) = a - (a * c) = ac$. Therefore $b = ab + bc = ac + bc = c$.

THEOREM 21. *L is a Boolean geometry if and only if metric betweenness has transitivity t_1 (see definition* 3.3 *in* [9]).

PROOF. The necessity is obvious. Conversely suppose that the metric betweenness in L has transitivity t_1. Let $a, b, c \in L$ with $a * b = a * c$. It follows that $(a, b, c)M$ and $(a, c, b)M$ so that by transitivity t_1 we have $(c, b, c)M$. Therefore $b*c = (c*b)+(b*c) = c*c = 0$ so that $b = c$ and by Theorem 5, L is a Boolean geometry.

THEOREM 22. *A semi-Brouwerian geometry is a Boolean geometry if and only if it has congruence order three relative to the class of L-metrized spaces. (see definition* 1.7 *in* [5]).

PROOF. The necessity follows from Theorem 14 in [5]. We need only show that a semi-Brouwerian geometry L with congruence order three is a Boolean geometry. Now by supposing $a-[a-(b-a)] \neq 0$ we arrive at a contradiction just in the same way as in [9], where $a-[a-(b-a)]$ is in the place of $x \cdot \neg x$ and $a+b$ is in the place of 1. Therefore L is a Boolean geometry.

THEOREM 23. *A semi-Brouwerian geometry is a Boolean geometry if and only if its group of motions is simply transitive (see definitions* 1.4 *and* 1.5 *in* [5]).

PROOF. The necessity is obvious. The converse follows from Theorem 13 of [5] and Theorem 5 part (4) of the present paper.

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