



Images of multilinear graded polynomials on upper triangular matrix algebras

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Abstract. In this paper, we study the images of multilinear graded polynomials on the graded algebra of upper triangular matrices UT_n . For positive integers $q \leq n$, we classify these images on UT_n endowed with a particular elementary \mathbb{Z}_q -grading. As a consequence, we obtain the images of multilinear graded polynomials on UT_n with the natural \mathbb{Z}_n -grading. We apply this classification in order to give a new condition for a multilinear polynomial in terms of graded identities so that to obtain the traceless matrices in its image on the full matrix algebra. We also describe the images of multilinear polynomials on the graded algebras UT_2 and UT_3 , for arbitrary gradings. We finish the paper by proving a similar result for the graded Jordan algebra UJ_2 , and also for UJ_3 endowed with the natural elementary \mathbb{Z}_3 -grading.

1 Introduction

Let \mathcal{A} be an associative algebra over a field F , and let $f \in F\langle X \rangle$ be a multilinear polynomial from the free associative algebra $F\langle X \rangle$. Lvov posed the question to determine whether the image of a multilinear f when evaluated on $\mathcal{A} = M_n(F)$, is always a vector subspace of $M_n(F)$ (see [17, Problem 1.93]). The original question is attributed to Kaplansky and asks the determination of the image of a polynomial f on \mathcal{A} . It is well known that the above question is equivalent to that of determining whether the image of a multilinear f on $M_n(F)$ is 0, the scalar matrices, $sl_n(F)$, or $M_n(F)$. Clearly, the first possibility corresponds to f being a polynomial identity on $M_n(F)$, and the second gives the central polynomials.

Recall here that the description of all polynomial identities on $M_n(F)$ is known only for $n \leq 2$ (see [12, 36] for the case when F is of characteristic 0 and [28] for the case of F infinite of characteristic $p > 2$). The same holds for the central polynomials [9, 35]. The theorem of Amitsur and Levitzki gives the least degree polynomial identity for $M_n(F)$, the standard polynomial s_{2n} (see [2]). Recall also that one of the major breakthroughs in polynomial identities (PI) theory was achieved by Formanek and by Razmyslov [18, 37], who proved the existence of nontrivial (that is not identities) central polynomials for the matrix algebras. As for $sl_n(F)$, a theorem of Shoda [38]

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gives that every $n \times n$ matrix over a field of zero characteristic is the commutator of two matrices; later on, Albert and Muckenhoupt [1] generalized this to arbitrary fields. Hence, all four conjectured possibilities for the image of a multilinear polynomial on $M_n(F)$ can be achieved.

The study of images of polynomials on the full matrix algebra is of considerable interest for obvious reasons, among these the relation to polynomial identities. In [24], the authors settled the conjecture due to Lvov in the case of 2×2 matrices over a quadratically closed field F (that is, if f is a given polynomial in several variables, then every polynomial in one variable of degree $\leq 2 \deg f$ over F has a root in F). They proved that for every multilinear polynomial f and for every field F that is quadratically closed with respect to f , the image of f on $M_2(F)$ is 0 , F , $sl_2(F)$, or $M_2(F)$. It should be noted that the authors in [24] proved a stronger result. Namely, they considered a so-called semi-homogeneous polynomial f : a polynomial in m variables x_1, \dots, x_m of weights d_1, \dots, d_m , respectively, such that every monomial of f is of (weighted) degree d for a fixed d . They proved that the image of such a polynomial on $M_2(F)$ must be 0 , F , $sl_2(F)$, the set of all non-nilpotent traceless matrices, or a dense subset of $M_2(F)$. Here, the density is according to the Zariski topology. Later on, in [33], the author gave the solution to the problem for 2×2 matrices for the case when F is the field of the real numbers. In the case of 3×3 matrices, the known results can be found in [25]. The images of polynomials on $n \times n$ matrices for $n > 3$ are hard to describe, and there are only partial results (see, for example, [26]). Hence, in the case of $n \times n$ matrices, one is led to study images of polynomials of low degree. Interesting results in this direction are due to Špenko [40], who proved the conjecture raised by Lvov in case F is an algebraically closed field of characteristic 0, and f is a multilinear Lie polynomial of degree at most 4. Further advances in the field were made in [5–7]. In [5], the author proved that if A is an algebra over an infinite field F and $A = [A, A]$, then the image of an arbitrary polynomial which is neither an identity nor a central polynomial equals A . Recently, Malev [34] described completely the images of multilinear polynomials on the real quaternion algebra; he also described the images of semi-homogeneous polynomials on the same algebra.

If the base field F is finite, a theorem of Chuang [8] states that the image of a polynomial without constant term can be every subset of $M_n(F)$ that contains 0 and is closed under conjugation by invertible matrices. In the same paper, it was also shown that such a statement fails when F is infinite.

Therefore, it seems likely it should be very difficult to describe satisfactorily the images of multilinear polynomials on $M_n(F)$. That is why people started studying images of polynomials on “easier” algebras, and also on algebras with an additional structure. The upper triangular matrix algebras are quite important in Linear Algebra because of their applications to different branches of Mathematics and Physics. They are also very important in PI theory: they describe, in a sense, the subvarieties of the variety of algebras generated by $M_2(F)$ in characteristic 0. Block-triangular matrices appear in the description of the so-called minimal varieties of algebras. The images of polynomials on the upper triangular matrices have been studied rather extensively. In [43], the author described the images of multilinear polynomials on $UT_2(F)$, the 2×2 upper triangular matrices over a field F . The images of multilinear polynomials of degree up to 4 on $UT_n = UT_n(F)$ for every n were classified in [16], and in [15] the

first-named author of the present paper described the images of arbitrary multilinear polynomials on the strictly upper triangular matrices. It turned out that if f is a multilinear polynomial of degree m , then its image on the strictly upper triangular matrix algebra J is either 0 or J^m . The following conjecture was raised in [16]: Is the image of a multilinear polynomial on $UT_n(F)$ always a vector subspace of $UT_n(F)$? This conjecture was solved independently in [32], for infinite fields (or finite fields with sufficiently many elements), and in [19]. Further results concerning images of polynomials on the upper triangular matrix algebra can be found in [10, 44, 45].

Gradings on algebras appeared long ago; the polynomial ring in one or several variables is naturally graded by the infinite cyclic group \mathbb{Z} by the degree. Gradings on algebras by finite groups are important in Linear Algebra and also in Theoretical Physics: the Grassmann (or exterior) algebra is naturally graded by the cyclic group of order 2, \mathbb{Z}_2 . In fact, the Grassmann algebra is the most well-known example of a superalgebra. It should be noted that while in the associative case, the term “superalgebra” is synonymous to “ \mathbb{Z}_2 -graded algebra,” if one considers nonassociative algebras, these notions are very different: a Lie or a Jordan superalgebra seldom is a Lie or a Jordan algebra. We are not going to discuss further such topics because these are not relevant for our exposition.

In [42], Wall classified the finite-dimensional \mathbb{Z}_2 -graded algebras that are graded simple. Later on, the description of all gradings on matrix algebras was obtained as well as on simple Lie and Jordan algebras. We refer the readers to the monograph [14] for the state-of-the-art and for further references. In PI theory, gradings appeared in the works of Kemer (see [27]), and constituted one of the main tools in the classification of the ideals of identities of associative algebras, which in turn led him to the positive solution of the long-standing Specht Problem. It turns out that the graded identities are easier to describe than the ordinary ones; still they provide a lot of information on the latter. It is somewhat surprising that the images of polynomials have not been studied extensively in the graded setting. In [31], Kulyamin described the images of graded polynomials on matrix algebras over the group algebra of a finite group over a finite field.

The upper triangular matrix algebra admits various gradings, and these were shown to be isomorphic to elementary ones (see [41]). A grading on a subalgebra A of $M_n(F)$ is elementary if all matrix units $e_{ij} \in A$ are homogeneous in the grading. All elementary gradings on UT_n were classified in [11]; in the same paper, the authors described the graded identities for all these gradings. In this paper, we fix an arbitrary field F and the upper triangular matrix algebra UT_n .

In Section 3, we prove that for an arbitrary group grading on UT_n , $n > 1$, there are no nontrivial graded central polynomials. (Hence, the image of a graded polynomial on UT_n cannot be equal to the scalar matrices whenever $n > 1$.) In Section 4, we consider a specific grading on UT_n , and describe all possible images of multilinear graded polynomials for that grading. It turns out that the images are always homogeneous vector subspaces. We impose a mild restriction on the cardinality of the base field. As a by-product of the proof, we obtain a precise description of the graded identities for this grading.

In Section 5, we give a sufficient condition for the traceless matrices to be contained in the image of a multilinear graded polynomial. Once again, we require a mild

condition on the cardinality of the field. Section 6 studies the graded algebras UT_2 and UT_3 . We prove that the image of a multilinear graded polynomial on UT_2 , for every group grading, is a homogeneous subspace. In the case of UT_3 , the image of such a polynomial is also a homogeneous subspace provided that the grading is nontrivial. In the case of the trivial grading, if the field contains at least three elements, then the image of every multilinear polynomial is a vector subspace. Then we proceed with the Jordan algebra structure UJ_n obtained from UT_n by the Jordan (symmetric) product $a \circ b = ab + ba$ provided that the characteristic of the base field is different from 2. The description of all group gradings on UJ_n is more complicated than that on UT_n (see [30]), there appear gradings that are not isomorphic to elementary ones. The gradings on UJ_2 were described in [29]. We consider each one of these gradings, and prove that the image of a multilinear graded polynomial is always a homogeneous subspace. No restrictions on the base field are imposed (apart from the characteristic being different from 2). An analogous result is deduced for the Lie algebra $UT_2^{(-)}$ obtained from UT_n by substituting the associative product by the Lie bracket $[a, b] = ab - ba$. Finally, we consider UJ_3 equipped with the natural \mathbb{Z}_3 -grading: $\deg e_{ij} = j - i \pmod{3}$ for every $i \leq j$, assuming the base field infinite and of characteristic different from 2. We prove that the image of a multilinear graded polynomial is always a homogeneous subspace.

We hope that this paper will initiate a more detailed study of images of polynomials on algebras with additional structures.

2 Preliminaries

Unless otherwise stated, we denote by F an arbitrary field and \mathcal{A} an associative algebra over F . Given a group G , a G -grading on \mathcal{A} is a decomposition of \mathcal{A} in a direct sum of subspaces $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ such that $\mathcal{A}_g \mathcal{A}_h \subset \mathcal{A}_{gh}$, for all $g, h \in G$. We define the support of a G -grading on \mathcal{A} as the subset $\text{supp}(\mathcal{A}) = \{g \in G \mid \mathcal{A}_g \neq 0\}$. A subspace U of \mathcal{A} is called homogeneous if $U = \bigoplus_{g \in G} (U \cap \mathcal{A}_g)$. A graded homomorphism between two graded algebras $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ is defined as an algebra homomorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ such that $\varphi(\mathcal{A}_g) \subset \mathcal{B}_g$ for every $g \in G$. We denote by $F\langle X \rangle^{gr}$ the free G -graded associative algebra generated by a set of noncommuting variables $X = \{x_i^{(g)} \mid i \in \mathbb{N}, g \in G\}$. We also denote the neutral (that is of degree $1 \in G$) variables by y and call them even variables, and the nonneutral ones by z , and we call them odd variables. We draw the reader's attention that odd variables may have different degrees in the G -grading.

We define the image of a graded polynomial on an algebra as in [31].

Definition 2.1 Let $f \in F\langle X \rangle^{gr}$ be a G -graded polynomial. The image of f on the G -graded algebra \mathcal{A} is the set

$$\text{Im}(f) = \{a \in \mathcal{A} \mid a = \varphi(f) \text{ for some graded homomorphism } \varphi: F\langle X \rangle^{gr} \rightarrow \mathcal{A}\}.$$

Equivalently, if $f(x_1^{(g_1)}, \dots, x_n^{(g_n)}) \in F\langle X \rangle^{gr}$, then the image of f on the algebra \mathcal{A} is the set $\text{Im}(f) = \{f(a_1^{(g_1)}, \dots, a_n^{(g_n)}) \mid a_i^{(g_i)} \in \mathcal{A}_{g_i}\}$. We will also denote the image of f on \mathcal{A} by $f(\mathcal{A})$.

We now recall some basic properties of images of graded polynomials on algebras that will be used throughout the paper.

Proposition 2.2 *Let $f \in F\langle X \rangle^{gr}$ be a polynomial, and let \mathcal{A} be a G -graded algebra.*

- (1) *Let U be one-dimensional subspace of \mathcal{A} such that $\text{Im}(f) \subset U$ and assume that $\lambda \text{Im}(f) \subset \text{Im}(f)$ for every $\lambda \in F$. Then either $\text{Im}(f) = \{0\}$ or $\text{Im}(f) = U$.*
- (2) *If $1 \in \mathcal{A}$ and $f \in F\langle X \rangle^{gr}$ is a multilinear polynomial in neutral variables such that the sum of its coefficients is nonzero, then $\text{Im}(f) = \mathcal{A}_1$.*
- (3) *$\text{Im}(f)$ is invariant under graded endomorphisms of $F\langle X \rangle^{gr}$.*
- (4) *If $\text{supp}(\mathcal{A})$ is abelian and $f \in F\langle X \rangle^{gr}$ is multilinear, then $\text{Im}(f)$ is a homogeneous subset.*

Proof The proofs of the first and third items are straightforward. For the second item, it is enough to recall that if \mathcal{A} is a graded algebra with 1, then $1 \in \mathcal{A}_1$. Hence, given $a \in \mathcal{A}_1$, we have $a = f(\alpha^{-1}a, 1, \dots, 1)$, where $\alpha \neq 0$ is the sum of the coefficients of f , and then $\text{Im}(f) = \mathcal{A}_1$. For the last item, let g_1, \dots, g_m be the homogeneous degree of the variables that occur in f . If some $g_i \notin \text{supp}(\mathcal{A})$, then $\text{Im}(f) = \{0\}$ is a homogeneous subspace. Otherwise, since $\text{supp}(\mathcal{A})$ is abelian, we have that each monomial of f is of homogeneous degree $g_1 \cdots g_m$, and hence the same holds for f . ■

We say that a nonzero polynomial $f \in F\langle X \rangle^{gr}$ is a graded polynomial identity for a G -graded algebra \mathcal{A} if its image on \mathcal{A} is zero. The set of all graded polynomial identities of \mathcal{A} will be denoted by $\text{Id}^{gr}(\mathcal{A})$. It is easy to check that $\text{Id}^{gr}(\mathcal{A})$ is actually an ideal of $F\langle X \rangle^{gr}$ invariant under graded endomorphisms of $F\langle X \rangle^{gr}$. It is called the T_G -ideal of \mathcal{A} . Given a nonempty subset S of $F\langle X \rangle^{gr}$, we denote by $\langle S \rangle^{T_G}$ the T_G -ideal generated by S , that is the least T_G -ideal that contains the set S . The linearization process also holds for graded polynomials, and as in the ordinary case, we have the following statement.

Proposition 2.3 *If \mathcal{A} satisfies a graded polynomial identity, then \mathcal{A} also satisfies a multilinear one. Moreover, if $\text{char}(F) = 0$, then $\text{Id}^{gr}(\mathcal{A})$ is generated by its multilinear polynomials.*

Let now $\mathcal{A} = UT_n$ be the algebra of $n \times n$ upper triangular matrices over the field F . A G -grading on \mathcal{A} is said to be elementary if all elementary matrices are homogeneous in this grading, or equivalently, if there exists an n -tuple $(g_1, \dots, g_n) \in G^n$ such that $\text{deg}(e_{ij}) = g_i^{-1}g_j$. A theorem of Valenti and Zaicev states that every grading on UT_n is essentially elementary.

Theorem 2.4 [41] *Let G be a group, and let F be a field. Assume that $UT_n = \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ is G -graded. Then \mathcal{A} is G -graded isomorphic to UT_n endowed with some elementary G -grading.*

By Proposition 1.6 of [11], we have that an elementary grading on UT_n is completely determined by the sequence $(\text{deg}(e_{12}), \text{deg}(e_{23}), \dots, \text{deg}(e_{n-1,n})) \in G^{n-1}$.

We recall now some recent results about the description of images of multilinear polynomials on the algebra of upper triangular matrices. We start with the definition of the so-called commutator degree of an associative polynomial.

Definition 2.5 Let $f \in F\langle X \rangle$ be a polynomial. We say that f has commutator degree r if

$$f \in \langle [x_1, x_2] \cdots [x_{2r-1}, x_{2r}] \rangle^T \text{ and } f \notin \langle [x_1, x_2] \cdots [x_{2r+1}, x_{2r+2}] \rangle^T.$$

In [19], Gargate and de Mello used the above definition to give a complete description of images of multilinear polynomials on UT_n over infinite fields. Denoting by J the Jacobson radical of UT_n and $J^0 = UT_n$, they proved the following theorem.

Theorem 2.6 *Let F be an infinite field, and let $f \in F\langle X \rangle$ be a multilinear polynomial. Then $\text{Im}(f)$ on UT_n is J^r if and only if f has commutator degree r .*

One of the main steps in the proof of Theorem 2.6 was the characterization of the polynomials of commutator degree r in terms of their coefficients. An instance of such characterization has already been known (see [19, Lemma 3.3(2)]).

Lemma 2.7 [19] *Let F be an arbitrary field, and let $f \in F\langle X \rangle$ be a multilinear polynomial. Then $f \in \langle [x_1, x_2] \rangle^T$ if and only if the sum of its coefficients is zero.*

It is worth mentioning that the above theorem has been extended for a larger class of fields by Luo and Wang in [32].

Theorem 2.8 [32] *Let $n \geq 2$ be an integer, let F be a field with at least $n(n-1)/2$ elements, and let $f \in F\langle X \rangle$ be a multilinear polynomial. If f has commutator degree r , then $\text{Im}(f)$ on UT_n is J^r .*

In the next corollary, we denote by $UT_n^{(-)}$ the Lie algebra defined on UT_n by means of the Lie bracket $[a, b] = ab - ba$.

Corollary 2.9 *Let F be a field with at least $n(n-1)/2$ elements, and let $f \in L(X)$ be a multilinear Lie polynomial. Then $\text{Im}(f)$ on $UT_n^{(-)}$ is J^r , for some $0 \leq r \leq n$.*

Proof We use the Poincaré–Birkhoff–Witt theorem (and more precisely the Witt theorem) to consider the free Lie algebra $L(X)$ as the subalgebra of $F\langle X \rangle^{(-)}$ generated by the set X . Since $F\langle X \rangle$ is the universal enveloping algebra of $L(X)$, given a multilinear Lie polynomial $f \in L(X)$, there exists an associative polynomial $\tilde{f} \in F\langle X \rangle$ such that $\text{Im}(f)$ on $UT_n^{(-)}$ is equal to $\text{Im}(\tilde{f})$ on UT_n . Now, it is enough to apply Theorem 2.8. ■

Let $UT_n(d_1, \dots, d_k)$ be the upper block-triangular matrix algebra, that is, the subalgebra of $M_n(F)$ consisting of all block-triangular matrices of the form

$$\begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_k \end{pmatrix},$$

where $n = d_1 + \dots + d_k$ and A_i is a $d_i \times d_i$ matrix. We will denote by T the subalgebra of $UT_n(d_1, \dots, d_k)$, which consists of only triangular blocks of sizes d_i on the main diagonal and zero elsewhere. That is,

$$T = \begin{pmatrix} UT_{d_1} & & 0 \\ & \ddots & \\ 0 & & UT_{d_k} \end{pmatrix}.$$

As a consequence of the above theorem, we obtain the following lemma.

Lemma 2.10 *Let F be a field with at least $n(n-1)/2$ elements, and let $f \in F\langle X \rangle$ be a multilinear polynomial of commutator degree r . Then the image $\text{Im}(f)$ on T is J^r , where $J = \text{Jac}(T)$ is the Jacobson radical of T .*

Proof We note that $T \cong UT_{d_1} \times \cdots \times UT_{d_k}$. Hence, by [4, Proposition 5.60],

$$J = \begin{pmatrix} J_{d_1} & & \\ & \ddots & \\ & & J_{d_k} \end{pmatrix},$$

where $J_{d_i} = \text{Jac}(UT_{d_i})$. Therefore, by Theorem 2.8, we have

$$f(T) = \begin{pmatrix} f(UT_{d_1}) & & \\ & \ddots & \\ & & f(UT_{d_k}) \end{pmatrix} = \begin{pmatrix} J_{d_1}^r & & \\ & \ddots & \\ & & J_{d_k}^r \end{pmatrix} = J^r.$$

■

Throughout this paper, we use the letters w_i and $w_i^{(j)}$ to denote commuting variables. We recall the following well-known result about commutative polynomials.

Lemma 2.11 *Let F be an infinite field, and let $f_1(w_1, \dots, w_m), \dots, f_n(w_1, \dots, w_m)$ be nonzero commutative polynomials. Then there exist $a_1, \dots, a_m \in F$ such that*

$$f_1(a_1, \dots, a_m) \neq 0, \quad \dots, \quad f_n(a_1, \dots, a_m) \neq 0.$$

A similar result also holds for finite fields, as long as some boundedness on the degrees of the variables is given (see [13, Proposition 4.2.3]).

Lemma 2.12 *Let F be a finite field with n elements, and let $f_1(w_1, \dots, w_m), \dots, f_{n-1}(w_1, \dots, w_m)$ be nonzero polynomials in commuting variables. If $\deg_{w_i}(f_j) \leq 1$ for all i and j , then there exist $a_1, \dots, a_m \in F$ such that*

$$f_1(a_1, \dots, a_m) \neq 0, \quad \dots, \quad f_{n-1}(a_1, \dots, a_m) \neq 0.$$

3 Graded central polynomials for UT_n

Our goal in this section is to prove the nonexistence of graded central polynomials for the graded algebra of upper triangular matrices with entries in an arbitrary field. It is well known that the algebra of upper block triangular matrices has no central polynomials (see [20, Lemma 1]).

We will denote by $Z(\mathcal{A})$ the center of the algebra \mathcal{A} .

Definition 3.1 Let $f \in F\langle X \rangle^{gr}$. We say that f is a graded central polynomial for the algebra \mathcal{A} if $\text{Im}(f) \subset Z(\mathcal{A})$ and $f \notin \text{Id}^{gr}(\mathcal{A})$.

We recall the following fact from [11, Lemma 1.4].

Lemma 3.2 *Let UT_n be endowed with some elementary grading. Then the subspace of all diagonal matrices is homogeneous of neutral degree.*

Theorem 3.3 *Let $UT_n = \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be a G -grading on the algebra of upper triangular matrices over an arbitrary field. If $n > 1$, then there exist no graded central polynomials for \mathcal{A} .*

Proof By Theorem 2.4, we have that \mathcal{A} is graded isomorphic to some elementary grading on UT_n . Hence, we may reduce our problem to elementary gradings. Now, we assume that $f \in F\langle X \rangle^{gr}$ is a polynomial with zero constant term, such that $\text{Im}(f)$ on \mathcal{A}

is contained in $F = Z(\mathcal{A})$. We write f as $f = f_1 + f_2$ where f_1 contains neutral variables only and f_2 has at least one nonneutral variable in each of its monomials. Consider $a_1, \dots, a_m \in \mathcal{A}_1$, and b_1, \dots, b_l are nonneutral variables (of homogeneous degree $\neq 1$) that occur in f . Hence, $f(a_1, \dots, a_m, b_1, \dots, b_l) = f_1(\bar{a}_1, \dots, \bar{a}_m) + j$, where $j \in J$, the Jacobson radical of \mathcal{A} , and \bar{a}_i is the diagonal part of a_i . Since $\text{Im}(f) \subset F$, then $j = 0$ and hence $\text{Im}(f) = \text{Im}(f_1)$, where the image of f_1 is taken on diagonal matrices only. Now, note that if $\lambda_1, \dots, \lambda_m \in F$ are arbitrary, then

$$f_1(\lambda_1 e_{11}, \dots, \lambda_m e_{11}) = f_1(\lambda_1, \dots, \lambda_m) e_{11}.$$

Since $\text{Im}(f_1) \subset F$, we must have $f_1(\lambda_1, \dots, \lambda_m) = 0$. Hence, for diagonal matrices $D_i = \sum_{k=1}^n \lambda_k^{(i)} e_{kk}$, we have

$$f_1(D_1, \dots, D_m) = \sum_{k=1}^n f_1(\lambda_1^{(k)}, \dots, \lambda_m^{(k)}) e_{kk} = 0,$$

and thus $\text{Im}(f) = \{0\}$. We conclude the nonexistence of graded central polynomials for UT_n . ■

4 Certain \mathbb{Z}_q -gradings on UT_n

Throughout this section, we denote $UT_n = \mathcal{A}$, endowed with the elementary \mathbb{Z}_q -grading given by the following sequence in \mathbb{Z}_q^n :

$$(\bar{0}, \bar{1}, \dots, \overbrace{\bar{q}-2, \bar{q}-1, \bar{q}-1, \dots, \bar{q}-1}^{n-q+1 \text{ times}}).$$

Given $q \leq n$ an integer, we study the images of multilinear graded polynomials on \mathcal{A} .

One can see that, for $q = n$, we have the natural \mathbb{Z}_n -grading on UT_n given by $\text{deg } e_{ij} = j - i \pmod{n}$ for every $i \leq j$.

We note that the neutral component of UT_n is given by a block triangular matrix with $q - 1$ triangular blocks of size one each and a triangular block of size $n - q + 1$ in the bottom right corner:

$$A_0 = \begin{pmatrix} * & & & 0 \\ & \ddots & & \\ & & * & \\ 0 & & & UT_{n-q+1} \end{pmatrix}.$$

For $l \in \{1, \dots, q - 1\}$, we have that the homogeneous component of degree \bar{l} is given by

$$A_{\bar{l}} = \text{span}\{e_{i,i+l}, e_{q-l,j} \mid i = 1, \dots, q - l, j = q + 1, \dots, n\}.$$

For $1 \leq r \leq n - q$, we also define the following homogeneous subspaces of $A_{\bar{l}}$:

$$B_{\bar{l},r} = \text{span}\{e_{q-l,j} \mid j = q + r, \dots, n\}.$$

An immediate computation shows that the following are graded identities for \mathcal{A} :

- (1) $[y_1, y_2]z \equiv 0,$
- (2) $z_1z_2 \equiv 0,$
- (3) $[y_1, y_2] \cdots [y_{2(n-q+1)-1}, y_{2(n-q+1)}] \equiv 0,$

where the variables y_i are neutral ones, z, z_1, z_2 are nonneutral variables, and $\deg(z_1) + \deg(z_2) = \bar{0}$. A complete description of the graded polynomial identities for elementary gradings on UT_n was given in [11] for infinite fields and in [21] for finite fields.

We state several lemmas concerning the description of some graded polynomials on \mathcal{A} . In the upcoming lemmas, unless otherwise stated, we assume that the field F has at least $n(n - 1)/2$ elements and $f \in F\langle X \rangle^{gr}$ is a multilinear polynomial.

Lemma 4.1 *If $f = f(y_1, \dots, y_m)$, then $\text{Im}(f)$ on \mathcal{A} is a homogeneous vector subspace.*

Proof It is enough to apply Lemma 2.10. ■

In the next two lemmas, we will assume that $f = f(z_1, \dots, z_l, y_{l+1}, \dots, y_m)$, where $\deg(z_i) = \bar{1}, 1 \leq i \leq l$. It is obvious that, in this case, one must have $\text{Im}(f)$ on \mathcal{A} as a subset of $\mathcal{A}_{\bar{1}}$. Modulo the identity (1), we rewrite the polynomial f as

$$f = \sum_{i_1, \dots, i_l} y_{i_1}z_1y_{i_2}z_2 \cdots y_{i_l}z_lg_{i_1, \dots, i_l} + h,$$

where $y_{i_j} = y_{i_{j_1}} \cdots y_{i_{j_{k_j}}}$ is such that $i_{j_1} < \cdots < i_{j_{k_j}}$. Moreover g_{i_1, \dots, i_l} is the polynomial obtained by permuting the neutral variables whose indices are different from either of i_1, \dots, i_l , and forming a linear combination of such monomials. Furthermore, h is the sum of polynomial that differ from the first summand of f by nontrivial permutations of the odd variables.

Among all polynomials g_{i_1, \dots, i_l} (including those in h), we choose one of least commutator degree, say g , of commutator degree r . Up to permuting the odd variables, we can assume that the polynomial g occurs in the first summand of f .

Hence, in case $1 \leq r \leq n - q$, we can improve the inclusion $\text{Im}(f) \subset \mathcal{A}_{\bar{1}}$ to $\text{Im}(f) \subset \mathcal{B}_{\bar{1}, r}$. Our goal is to prove that $\text{Im}(f) = \mathcal{A}_{\bar{1}}$ in case $r = 0$ and $\text{Im}(f) = \mathcal{B}_{\bar{1}, r}$ otherwise.

Lemma 4.2 *If $1 \leq r \leq n - q$, then $\text{Im}(f) = \mathcal{B}_{\bar{1}, r}$.*

Proof Let $\bar{g} = y_{i_1}z_1y_{i_2}z_2 \cdots y_{i_l}z_lg$ be a nonzero summand of f written as above, where the commutator degree of g is r . We consider the following evaluations: the variables in y_{i_1} by $e_{q-l, q-l}$, the ones in y_{i_2} by $e_{q-l+1, q-l+1}, \dots$, and all variables in y_{i_l} by $e_{q-1, q-1}$. We also put $z_1 = e_{q-l, q-l+1}, z_2 = e_{q-l+1, q-l+2}, \dots, z_{l-1} = e_{q-2, q-1}$, and $z_l = \sum_{k=q}^n w_k e_{q-1, k}$. Since g is of commutator degree r , Theorem 2.8 enables us to evaluate the even variables in g by matrices from

$$\begin{pmatrix} 0 & 0 \\ 0 & UT_{n-q+1} \end{pmatrix}$$

in order to obtain the matrix $e_{q, q+r} + e_{q+1, q+r+1} + \cdots + e_{n-r, n}$.

Note that the evaluations that we have considered allow us to reduce the study of the image of f to the polynomial \bar{g} . Under these evaluations, we have

$$\begin{aligned} \bar{g} &= (w_q e_{q-l,q} + w_{q+1} e_{q-l,q+1} + \dots + w_n e_{q-l,n})(e_{q,q+r} + e_{q+1,q+r+1} + \dots + e_{n-r,n}) \\ &= w_q e_{q-l,q+r} + w_{q+1} e_{q-l,q+r+1} + \dots + w_{n-r} e_{q-l,n}. \end{aligned}$$

Taking a matrix $B \in \mathcal{B}_{\bar{l},r}$, say $B = b_q e_{q-l,q+r} + \dots + b_{n-r} e_{q-l,n}$, we can easily realize B as image of \bar{g} by choosing $w_j = b_j, j = q, \dots, n - r$.

Hence, $f(\mathcal{A}) = \mathcal{B}_{\bar{l},r}$. ■

Lemma 4.3 *If F is a field with at least n elements and $r = 0$, then $\text{Im}(f) = \mathcal{A}_{\bar{l}}$.*

Proof Denoting by \mathcal{D} the homogeneous subspace of diagonal matrices of \mathcal{A} , we consider the following homogeneous subalgebra of \mathcal{A} :

$$\mathcal{S} = \mathcal{D} \oplus \bigoplus_{1 \leq l \leq q-1} \mathcal{A}_{\bar{l}}.$$

We will show that $\text{Im}(f)$ on \mathcal{S} is $\mathcal{A}_{\bar{l}}$, which is enough to conclude the lemma. Note that \mathcal{S} still satisfies the identity (2) and it also satisfies $[y_1, y_2] \equiv 0$.

By the identity $[y_1, y_2] \equiv 0$, we may write the polynomial \bar{g} as

$$\beta y_{i_1} z_1 y_{i_2} z_2 \dots y_{i_l} z_l y_{i_{l+1}},$$

where β is the sum of all coefficients of the polynomial g and $y_{i_{l+1}}$ is the product of the variables of g in increasing order of the indices. Since $r = 0$, we get from Definition 2.5 that $\beta \neq 0$.

Now, we write $f = f(z_1, \dots, z_l, y_{l+1}, \dots, y_m)$ as

$$f = \sum_{j=1}^l f_j,$$

where f_j is the sum of all monomials of f such that the variable z_l is in the j th position in relation to the odd variables.

For each $j = 1, \dots, l$, we write

$$f_j = \sum_{\sigma \in S_l^{(j)}} f_{j,\sigma},$$

where $S_l^{(j)} = \{\sigma \in S_l \mid \sigma(l) = j\}$, and $f_{j,\sigma}$ is the sum of all monomials of f_j where the order of the odd variables is given by the permutation σ .

Taking $z_i = \sum_{k=1}^{q-1} w_k^{(i)} e_{k,k+1} + w_q^{(i)} e_{q-1,q+1} + \dots + w_{n-1}^{(i)} e_{q-1,n}$ and $y_j = \sum_{k=1}^n w_k^{(j)} e_{kk}$,

we have

$$\begin{aligned} f_{l,id}(z_1, \dots, z_l, y_{l+1}, \dots, y_m) &= \sum_{k=1}^{q-1} p_k w_k^{(1)} w_{k+1}^{(2)} \dots w_{k+l-1}^{(l)} e_{k,k+l} \\ &+ p_{q-l+1} w_{q-l}^{(1)} \dots w_{q-2}^{(l-1)} w_q^{(l)} e_{q-l,q+1} + \dots + p_{n-l} w_{q-l}^{(1)} \dots w_{q-2}^{(l-1)} w_{n-1}^{(l)} e_{q-l,n}, \end{aligned}$$

where $p_k, k = 1, \dots, n - l$, are polynomials in the variables $w^{(l+1)}, \dots, w^{(m)}$. We note that all polynomials $p_k, k = 1, \dots, n - l$, are nonzero ones. Indeed, we just have to check

that different monomials in $f_{l,id}$ give different monomials in p_k . To this end, note that if m_1 and m_2 are different monomials in $f_{l,id}$, then there exists some even variable y_j such that the quantity of preceding odd variables in relation to y_j is distinct in m_1 and m_2 . This gives us variables $w^{(j)}$ with different lower indices in the two monomials in p_k given by m_1 and m_2 , which proves our claim. Moreover, we note that every variable in each monomial of the polynomial p_k appears exactly once.

Since we have at most $n - 1$ polynomials p_k , by Lemma 2.12, there exist evaluations of the even variables y_j by diagonal matrices D_j such that p_k take nonzero values simultaneously for all k . In case F is infinite, the same conclusion holds by applying Lemma 2.11. Hence,

$$\begin{aligned}
 f_l &= \sum_{k=1}^{q-l} \left(\sum_{\sigma \in S_l^{(l)}} \alpha_\sigma w_k^{(\sigma(1))} \dots w_{k+l-2}^{(\sigma(l-1))} \right) w_{k+l-1}^{(l)} e_{k,k+l} \\
 &\quad + \left(\sum_{\sigma \in S_l^{(l)}} \alpha_\sigma w_{q-l}^{(\sigma(1))} \dots w_{q-2}^{(\sigma(l-1))} \right) w_q^{(l)} e_{q-l,q+1} \\
 &\quad + \dots + \left(\sum_{\sigma \in S_l^{(l)}} \alpha_\sigma w_{q-l}^{(\sigma(1))} \dots w_{q-2}^{(\sigma(l-1))} \right) w_{n-1}^{(l)} e_{q-l,n},
 \end{aligned}$$

with $\alpha_{id} \neq 0$. So the polynomials inside the brackets above are nonzero ones, and each of their monomials has variables of degree one. Applying Lemma 2.12 once again, we may evaluate the variables z_1, \dots, z_{l-1} by matrices in $C_1, \dots, C_{l-1} \in A_{\bar{l}}$ such that all these polynomials take nonzero values on F (in case F is infinite, we apply Lemma 2.11). Denote by $\alpha_{l,k} \in F \setminus \{0\}$, $k = 1, \dots, q - l$, the values of the polynomials inside the brackets after such evaluations.

Therefore,

$$\begin{aligned}
 &f(C_1, \dots, C_{l-1}, z_l, D_{l+1}, \dots, D_m) \\
 &= \sum_{k=1}^{q-l} \left(\alpha_{1,k} w_k^{(l)} + \dots + \alpha_{l-1,k} w_{k+l-2}^{(l)} + \alpha_{l,k} w_{k+l-1}^{(l)} \right) e_{k,k+l} \\
 &\quad + \left(\alpha_{1,q} w_{q-l}^{(l)} + \dots + \alpha_{l-1,q} w_{q-2}^{(l)} + \alpha_{l,q-l} w_q^{(l)} \right) e_{q-l,q+1} \\
 &\quad + \dots + \left(\alpha_{1,n-1} w_{q-l}^{(l)} + \dots + \alpha_{l-1,n-1} w_{q-2}^{(l)} + \alpha_{l,q-l} w_{n-1}^{(l)} \right) e_{q-l,n},
 \end{aligned}$$

where $\alpha_{l,k} \neq 0$ for every $k = 1, \dots, q - l$.

Then, given a matrix $B = \sum_{k=1}^{q-l} b_k e_{k,k+l} + b_{q-l+1} e_{q-l,q+1} + \dots + b_{n-l} e_{q-l,n} \in A_{\bar{l}}$, we take

$$f(C_1, \dots, C_{l-1}, z_l, D_{l+1}, \dots, D_m) = B,$$

and we obtain a linear system in the variables $w^{(l)}$ whose solution (not necessarily unique) can be found recursively. ■

Theorem 4.4 Let F be a field with at least $n(n-1)/2$ elements, let $UT_n = \bigoplus_{k \in \mathbb{Z}_q} \mathcal{A}_k$ be endowed with the elementary \mathbb{Z}_q -grading given by the sequence $(\bar{0}, \bar{1}, \dots, \overline{q-2}, \overline{q-1}, \dots, \overline{q-1})$, and let $f \in F\langle X \rangle^{gr}$ be a multilinear polynomial. Then $\text{Im}(f)$ on UT_n is $\{0\}$, J^r , $\mathcal{B}_{\bar{1},r}$, or $\mathcal{A}_{\bar{1}}$, where $J = \text{Jac}(\mathcal{A}_0)$. In particular, the image is always a homogeneous vector subspace.

Proof By Lemmas 4.1–4.3, we only need to analyze the case where f is a polynomial in neutral variables and nonneutral ones (that is of degree different from $\bar{1}$). However, we can reduce this case to the aforementioned lemmas. Indeed, modulo the graded identities (1)–(3), let f and g be as in the comments before Lemma 4.2 and let r be the commutator degree of g . Hence, $\text{Im}(f) \subset \mathcal{B}_{\bar{1},r}$ if $r \neq 0$ and $\text{Im}(f) \subset \mathcal{A}_{\bar{1}}$ otherwise. By Proposition 2.2(3), the image of the polynomial \tilde{f} obtained from f by evaluating every nonneutral variable z_i of homogeneous degree \bar{k} by a product of k variables of homogeneous degree $\bar{1}$, is contained in $\text{Im}(f)$. However, the polynomial g defined for \tilde{f} (see the comments before Lemma 4.2) is the same as the one defined for f . This allows us to get $\mathcal{B}_{\bar{1},r} \subseteq \text{Im}(\tilde{f})$ in case $r \neq 0$ and $\mathcal{A}_{\bar{1}} \subseteq \text{Im}(\tilde{f})$ otherwise. ■

Remark 4.5 Considering similar computations, one can easily see that the result is also valid for the elementary grading defined by the identities $z[y_1, y_2] \equiv 0$, (2), and (3).

In the next corollary, we are assuming that F is a field of characteristic zero and $\mathcal{A} = UT_n$ is endowed with the elementary \mathbb{Z}_q -grading given by the sequence $(\bar{0}, \bar{1}, \dots, \overline{q-2}, \overline{q-1}, \dots, \overline{q-1})$.

Corollary 4.6 The T_G -ideal $\text{Id}^{gr}(\mathcal{A})$ is generated by the graded identities (1)–(3).

Proof Let $f \in \text{Id}^{gr}(\mathcal{A})$. By Proposition 2.3, we may assume that f is multilinear. Note that in the proof of Theorem 4.4 and in the lemmas that precede it, we have shown that if f is not a consequence of the identities (1)–(3), then $\text{Im}(f) \neq \{0\}$. In other words, if $f \in \text{Id}^{gr}(\mathcal{A})$, then f is a consequence of the aforementioned identities. ■

We recall that in case $q = n$, we have the natural \mathbb{Z}_n -grading on UT_n .

Corollary 4.7 Let F be a field with at least n elements, let UT_n be endowed with the natural \mathbb{Z}_n -grading, and let $f \in F\langle X \rangle^{gr}$ be a multilinear polynomial. Then the image of f on UT_n is either zero or some homogeneous component.

Proof It follows from Proposition 2.2(2), Lemma 2.7, and the proof of Lemma 4.3. ■

Remark 4.8 Note that the same result also holds if we consider the natural \mathbb{Z} -grading on UT_n . Analogous results hold for the lower triangular matrix algebra LT_n as well (we will use this remark in the next section).

5 Graded identities and traceless matrices

In this section, we give a sufficient condition for the subspace of the traceless matrices to be contained in the image of a multilinear polynomial on the full matrix algebra.

We start by recalling the following result from [3].

Theorem 5.1 [3] *Let D be a division ring, $n \geq 2$ an integer, and $A \in M_n(D)$ a noncentral matrix. Then A is similar (conjugate) to a matrix in $M_n(D)$ with at most one nonzero entry on the main diagonal. In particular, if A has trace zero, then it is similar to a matrix in $M_n(D)$ with only zeros on the main diagonal.*

Consider the natural \mathbb{Z} -grading on $M_n(F) = \bigoplus_{r \in \mathbb{Z}} (M_n(F))_r$ given by

$$(M_n(F))_r = \begin{cases} \text{span}\{e_{k,k+r} \mid k = 1, \dots, n-r\}, & \text{if } 0 \leq r \leq n-1, \\ \text{span}\{e_{k-r,k} \mid k = 1, \dots, n+r\}, & \text{if } -n+1 \leq r \leq -1, \\ \{0\}, & \text{elsewhere.} \end{cases}$$

In this section, we denote $F\langle X \rangle^{gr}$ the free \mathbb{Z} -graded algebra. We also keep the notation of using variables y 's for neutral variables and z 's for nonneutral ones.

Theorem 5.2 *Let $n \geq 2$ be an integer, let F be a field with at least $(n-1)n+1$ elements where $\text{char}(F)$ does not divide n , and let $f \in F\langle X \rangle$ be a multilinear polynomial. If $f(y_1, \dots, y_{m-1}, z) \notin \langle [y_1, y_2] \rangle^{Tz}$ for every nonneutral variable z , then $\text{Im}(f)$ on $M_n(F)$ contains $sl_n(F)$.*

Proof Since $\text{char}(F)$ does not divide n , then we have that any nonzero traceless matrix is noncentral. Using further that $\text{Im}(f)$ is invariant under automorphisms, by Theorem 5.1, it is enough to show that $\text{Im}(f)$ contains all matrices with zero diagonal. Let A be a zero diagonal matrix, and write A as the sum of its homogeneous components

$$A = \sum_{i=-n+1}^{-1} A_i + \sum_{i=1}^{n-1} A_i,$$

where $A_i = \sum_{k=1}^{n+i} a_{k-i,k} e_{k-i,k}$, for $i = -n+1, \dots, -1$, and $A_i = \sum_{k=1}^{n-i} a_{k,k+i} e_{k,k+i}$, for $i = 1, \dots, n-1$.

By hypothesis, $f(y_1, \dots, y_{m-1}, z^{(i)})$ is not a graded polynomial identity for UT_n with the natural \mathbb{Z} -grading, for every variable $z^{(i)}$ of homogeneous degree i where $1 \leq i \leq n-1$.

We now consider the following evaluations on generic matrices: $y_j = \sum_{k=1}^n w_k^{(j)} e_{kk}$,

for all $j = 1, \dots, m-1$, and $z^{(i)} = \sum_{k=1}^{n-i} w_k^{(m,i)} e_{k,k+i}$.

Hence,

$$f(y_1, \dots, y_{m-1}, z^{(i)}) = \sum_{k=1}^{n-i} p_{k,i} w_k^{(m,i)} e_{k,k+i},$$

where $p_{k,i}$ is a polynomial in the variables $w_k^{(j)}$. Since $f \notin \text{Id}^{gr}(UT_n)$, Corollary 4.7 gives us that the image of $f(y_1, \dots, y_{m-1}, z^{(i)})$ on UT_n is exactly $(UT_n)_i$. Hence, all $p_{k,i}$ are nonzero polynomials. Moreover, note that $p_{k,i}$ is such that all its monomials are multilinear ones.

Analogously, we also have that $f(y_1, \dots, y_{m-1}, z^{(i)})$ is not a graded polynomial identity for the lower triangular matrix algebra LT_n endowed with the natural \mathbb{Z} -grading, for $i = -n + 1, \dots, -1$. Therefore,

$$f(y_1, \dots, y_{m-1}, z^{(i)}) = \sum_{k=1}^{n-i} q_{k,i} w_k^{(m,-i)} e_{k+i,k},$$

where $z^{(i)} = \sum_{k=1}^{n+i} w_k^{(m,-i)} e_{k-i,k}$ and $q_{k,i}$ are nonzero commutative polynomials with multilinear monomials.

The number of polynomials $p_{k,i}$ and $q_{k,i}$ is exactly $(n - 1)n$. We now apply Lemma 2.12 to get an evaluation of all variables $w_k^{(j)}$ such that the polynomials $p_{k,i}$ and $q_{k,i}$ assume simultaneously nonzero values in F . Such evaluations give us diagonal matrices D_1, \dots, D_{m-1} such that

$$f(D_1, \dots, D_{m-1}, z^{(i)}) = \sum_{k=1}^{n-i} \alpha_{i,k} w_k^{(m,i)} e_{k,k+i},$$

where $\alpha_{i,k}$ are nonzero scalars. Therefore, each matrix A_i can be realized as $f(D_1, \dots, D_{m-1}, B_i)$ for a suitable matrix $B_i \in (UT_n)_i$, for every $i = 1, \dots, n - 1$. Similarly, we also have that each matrix A_i can be realized as $f(D_1, \dots, D_{m-1}, C_i)$ for a suitable matrix $C_i \in (LT_n)_i$, for all $i = -n + 1, \dots, -1$. Hence,

$$A = \sum_{i=-n+1}^{-1} A_i + \sum_{i=1}^{n-1} A_i = \sum_{i=-n+1}^{-1} f(D_1, \dots, D_{m-1}, C_i) + \sum_{i=1}^{n-1} f(D_1, \dots, D_{m-1}, B_i),$$

and it is enough to use the linearity of f in one variable to get $A \in \text{Im}(f)$. ■

6 The low-dimension cases

6.1 Arbitrary gradings on UT_2 and UT_3

We start this section with the following proposition.

Proposition 6.1 *Let G be a group, and let \mathcal{A} and \mathcal{B} be two G -graded algebras such that \mathcal{B} is a graded homomorphic image of \mathcal{A} . Let $f \in F\langle X \rangle^{gr}$ be a graded polynomial and assume that $f(\mathcal{A})$ is a homogeneous subspace of \mathcal{A} . Then $f(\mathcal{B})$ is also a homogeneous subspace of \mathcal{B} .*

Proof Let $\phi: \mathcal{A} \rightarrow \mathcal{B}$ be a graded epimorphism. We start by noting that $\phi(f(\mathcal{A})) = f(\phi(\mathcal{A}))$. Then taking $b_i^{(1)}, b_i^{(2)} \in \mathcal{B}_{g_i}$, $i = 1, \dots, m$, we have $b_i^{(j)} = \phi(a_i^{(j)})$ for some $a_i^{(j)} \in \mathcal{A}_{g_i}$, since ϕ is surjective. This leads us to

$$\begin{aligned} & \alpha f(b_1^{(1)}, \dots, b_m^{(1)}) + f(b_1^{(2)}, \dots, b_m^{(2)}) \\ &= \alpha f(\phi(a_1^{(1)}), \dots, \phi(a_m^{(1)})) + f(\phi(a_1^{(2)}), \dots, \phi(a_m^{(2)})) \\ &= \phi(\alpha f(a_1^{(1)}, \dots, a_m^{(1)}) + f(a_1^{(2)}, \dots, a_m^{(2)})), \end{aligned}$$

which is an element from $\phi(f(\mathcal{A})) = f(\phi(\mathcal{A}))$ since we are assuming that $f(\mathcal{A})$ is a subspace. Then $f(\mathcal{B})$ is also a subspace.

Now, we assume that $f(\mathcal{A})$ is a homogeneous subspace and let $b = b_{h_1} + \dots + b_{h_k}$ be an element in $f(\mathcal{B})$ written as the sum of its homogeneous components. Since $b \in f(\mathcal{B})$, let

$$b = f(b_1, \dots, b_m) = f(\phi(a_1), \dots, \phi(a_m)) = \phi(f(a_1, \dots, a_m))$$

for some $b_i = \phi(a_i)$, and let

$$f(a_1, \dots, a_m) = a_{g_1} + \dots + a_{g_l}$$

be the sum of its homogeneous components. It follows that

$$b = \phi(a_{g_1}) + \dots + \phi(a_{g_l}),$$

and that since ϕ is a graded homomorphism, we must have $k = l$ and every g_t must be equal to some h_s . Without loss of generality, we assume that $b_{g_t} = \phi(a_{g_t})$. Now, it is enough to use that $f(\mathcal{A})$ is homogeneous and $\phi(f(\mathcal{A})) = f(\phi(\mathcal{A}))$. ■

Remark 6.2 In the proof of Proposition 6.1, we have not used the associativity of \mathcal{A} . Therefore, it also holds for arbitrary algebras, in particular an analogous proposition holds for graded Jordan algebras.

As a consequence of Proposition 6.1, we have the following theorem.

Theorem 6.3 *Let $UT_2 = \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be some grading on \mathcal{A} , and let $f \in F\langle X \rangle^{sr}$ be a multilinear graded polynomial. Then $f(\mathcal{A})$ is a homogeneous subspace of \mathcal{A} .*

Proof By Theorem 2.4 and Proposition 6.1, it is enough to consider images of multilinear graded polynomials on elementary gradings only. We note that just two elementary G -gradings can be defined on $\mathcal{A} = UT_2$. Indeed, an elementary grading on UT_2 is completely determined by the homogeneous degree of e_{12} . If $\deg(e_{12}) = 1$, then we have the trivial grading, and we apply Theorem 2.8. Hence, we assume that $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}\}$ and $\mathcal{A}_g = \text{span}\{e_{12}\}$, where $g \neq 1$. In this grading, the images of multilinear polynomials in neutral variables are handled by Lemma 2.7 and Proposition 2.2(2). Since $\mathcal{A}_g^2 = \{0\}$, it is enough to consider multilinear polynomials in one variable of homogeneous degree g and all remaining variables of neutral degree. In this case, the image is contained in \mathcal{A}_g , and by Proposition 2.2(1), we are done. ■

Now, we prove an analogous fact to Theorem 6.3 with $\mathcal{A} = UT_3$ instead of UT_2 . From now on, in this subsection, we assume that \mathcal{A} is endowed with some elementary G -grading given by a tuple $(g_1, g_2) \in G^2$. Hence, $g_1 = \deg(e_{12})$, $g_2 = \deg(e_{23})$, and $g_3 := g_1g_2 = \deg(e_{13})$.

Hence, the elementary gradings on UT_3 are exactly the ones given by following relations.

- (I) $\{1\} \cap \{g_1, g_2, g_3\} \neq \emptyset$.
 - (a) $g_1 = g_2 = 1$, which implies $g_3 = 1$;
 - (b) $g_1 = 1$, which implies $g_2 = g_3$;
 - (c) $g_2 = 1$, which implies $g_1 = g_3$;

- (d) $g_3 = 1$ and $g_1 = g_2$;
- (e) $g_3 = 1$ and $g_1 \neq g_2$.
- (II) $\{1\} \cap \{g_1, g_2, g_3\} = \emptyset$.
 - (a) $1, g_1, g_2, g_3$ are pairwise distinct elements;
 - (b) $g_1 = g_2 \neq g_3$.

In the following lemmas, we discuss the grading on UT_3 determined by each relation above and the respective image of a multilinear graded polynomial on such a graded algebra.

Lemma 6.4 *Let UT_3 be endowed with the grading (I)(b). Then $\text{Im}(f)$ on UT_3 is a homogeneous subspace.*

Proof We denote $g_2 = g$, then we have $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}, e_{33}, e_{12}\}$ and $\mathcal{A}_g = \text{span}\{e_{13}, e_{23}\}$. Note that $\mathcal{A}_g^2 = \{0\}$ and hence we only need to analyze multilinear polynomials in at most one variable of homogeneous degree g .

The case when f is a multilinear polynomial in neutral variables is settled by Lemma 2.7 and Proposition 2.2(2).

Now, we consider f as a multilinear polynomial in one nonneutral variable and $m - 1$ neutral ones. Since \mathcal{A} satisfies the graded identity $z[y_1, y_2] \equiv 0$, then modulo this identity, we write f as

$$\sum_{1 \leq i_1 < \dots < i_k \leq m-1} h_{i_1, \dots, i_k} z_m y_{i_1} \dots y_{i_k}.$$

If all polynomials h_{i_1, \dots, i_k} have commutator degree different from 0, then $\text{Im}(f) \subset \text{span}\{e_{13}\}$ and then we apply Proposition 2.2(1). Otherwise, we may assume, without loss of generality, that $h_{1, \dots, k}$ has commutator degree 0. Then we perform the following evaluations: $y_1 = \dots = y_k = e_{33}$, $y_j = e_{11} + e_{22}$ for every $j \notin \{1, \dots, k\}$, and $z_m = \alpha^{-1}(a_1 e_{13} + a_2 e_{23})$, where α is the sum of the coefficients of $h_{1, \dots, k}$. Note that, under such an evaluation, we have $a_1 e_{13} + a_2 e_{23} \in \text{Im}(f)$, which proves that $\text{Im}(f) = \mathcal{A}_g$. ■

Lemma 6.5 *Let UT_3 be endowed with the grading (I)(c). Then $\text{Im}(f)$ on UT_3 is a homogeneous subspace.*

Proof Note that $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}, e_{33}, e_{23}\}$, $\mathcal{A}_{g_1} = \text{span}\{e_{12}, e_{13}\}$, and \mathcal{A} satisfies the identities $[y_1, y_2]z \equiv 0$ and $z_1 z_2 \equiv 0$. Thus, the proof is similar to the one for (I)(b). ■

Lemma 6.6 *Let UT_3 be endowed with the grading (I)(e). Then $\text{Im}(f)$ on UT_3 is a homogeneous subspace.*

Proof Here, we must have $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}, e_{33}, e_{13}\}$, $\mathcal{A}_{g_1} = \text{span}\{e_{12}\}$, and $\mathcal{A}_{g_2} = \text{span}\{e_{23}\}$. Note that $\mathcal{A}_{g_1}^2 = \mathcal{A}_{g_2}^2 = \{0\}$, $\mathcal{A}_{g_2} \mathcal{A}_{g_1} = \{0\}$, and $\mathcal{A}_{g_1} \mathcal{A}_{g_2} \subset \text{span}\{e_{13}\}$.

The case when f is a multilinear polynomial in neutral variables can be treated as in the grading (I)(b). Hence, we may consider f is a multilinear polynomial in one variable of degree g_1 (resp. g_2) and $m - 1$ neutral variables, or in one variable of degree g_1 , one of degree g_2 , and $m - 2$ neutral ones. In each of these situations, we have that $\text{Im}(f)$ is contained in a one-dimensional space and we apply Proposition 2.2(1). ■

Lemma 6.7 *Let UT_3 be endowed with the grading (II)(a). Then $\text{Im}(f)$ on UT_3 is a homogeneous subspace.*

Proof We have $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}, e_{33}\}$, $\mathcal{A}_{g_1} = \text{span}\{e_{12}\}$, $\mathcal{A}_{g_2} = \text{span}\{e_{23}\}$, and $\mathcal{A}_{g_3} = \text{span}\{e_{13}\}$. The only nontrivial relation among the nonneutral homogeneous components is given by $\mathcal{A}_{g_1}\mathcal{A}_{g_3} = \mathcal{A}_{g_2}$.

The case of f in neutral variables is the same as for the grading (I)(b).

Since the nonneutral components are one-dimensional, then the image of a multilinear polynomial in one nonneutral variable and $m - 1$ neutral ones is always zero or the respective homogeneous component.

In case f has one variable of homogeneous degree g_1 , one of degree g_3 , and $m - 2$ neutral ones, then the image is contained in \mathcal{A}_{g_2} , and we are done. ■

Lemma 6.8 *Let UT_3 be endowed with the grading (II)(b). Then $\text{Im}(f)$ on UT_3 is a homogeneous subspace.*

Proof Note that $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}, e_{33}\}$, $\mathcal{A}_{g_1} = \text{span}\{e_{12}, e_{23}\}$, and $\mathcal{A}_{g_3} = \text{span}\{e_{13}\}$. We only need to consider the case when f is a multilinear polynomial in $m - 1$ neutral variables and one of homogeneous degree g_1 , since the remaining cases can be treated as above. We write $f = \sum_{j=1}^m f_j$ where f_j is the sum of all monomials from f which contain the variable z_m in the j th position. Hence, modulo $[y_1, y_2] \equiv 0$, we have

$$f_j = \sum_{1 \leq i_1 < \dots < i_{j-1} \leq m-1} \alpha_{i_1, \dots, i_{j-1}} y_{i_1} \dots y_{i_{j-1}} z_m y_{k_1} \dots y_{k_{m-j}},$$

where $k_1 < \dots < k_{m-j}$. We evaluate $y_i = w_1^{(i)} e_{11} + e_{22} + w_3^{(i)} e_{33}$ and $z_m = w_1^{(m)} e_{12} + w_2^{(m)} e_{23}$. Thus, $f(y_1, \dots, y_{m-1}, z_m)$ is given by

$$\begin{pmatrix} 0 & p_1(w_1^{(1)}, \dots, w_1^{(m-1)})w_1^{(m)} & 0 \\ 0 & 0 & p_2(w_3^{(1)}, \dots, w_3^{(m-1)})w_2^{(m)} \\ 0 & 0 & 0 \end{pmatrix},$$

where $p_1(w_1^{(1)}, \dots, w_1^{(m-1)}) = \sum_{j=1}^m \sum_{1 \leq i_1 < \dots < i_{j-1} \leq m-1} \alpha_{i_1, \dots, i_{j-1}} w_1^{(i_1)} \dots w_1^{(i_{j-1})}$ and p_2 is given analogously.

We claim that p_1 takes nonzero values on F . Indeed, assume that p_1 is a polynomial identity for F and denote $e_j = \sum_{1 \leq i_1 < \dots < i_{j-1} \leq m-1} \alpha_{i_1, \dots, i_{j-1}} w_1^{(i_1)} \dots w_1^{(i_{j-1})}$. Note that $e_1 \in$

F and taking $w_1^{(1)} = \dots = w_1^{(m-1)} = 0$ we have $e_1 = 0$. Taking $w_1^{(1)} = 1$ and zero for the remaining values of w_1 , we have $\alpha_l = 0$ for all $l \in \{1, \dots, m - 1\}$ and hence $e_2 = 0$. Now, assume $e_l = 0$ for all $l < k$, and we shall prove that $e_k = 0$. For each chosen i_1, \dots, i_{k-1} , we take $w_1^{(r)} = 0$ for all $r \notin \{i_1, \dots, i_{k-1}\}$, then $e_l = 0$ for all $l > k$ and $e_k = \alpha_{i_1, \dots, i_{k-1}} w_1^{(i_1)} \dots w_1^{(i_{k-1})}$. Then we take $w_1^{(i_1)} = \dots = w_1^{(i_{k-1})} = 1$, and we conclude that $\alpha_{i_1, \dots, i_{k-1}} = 0$. Hence, $p_1 = 0$, which is a contradiction. Analogous claim holds for p_2 . Therefore, it is enough to use the variables $w_1^{(m)}$ and $w_2^{(m)}$ to realize any matrix in \mathcal{A}_{g_1} in the image of f . ■

Lemma 6.9 *Let UT_3 be endowed with the grading (I)(d). Then $\text{Im}(f)$ on UT_3 is a homogeneous subspace.*

Proof We denote $g = g_1$ and note that $\mathcal{A}_1 = \text{span}\{e_{11}, e_{22}, e_{33}, e_{13}\}$ and $\mathcal{A}_g = \text{span}\{e_{12}, e_{23}\}$. Then $\mathcal{A}_g^2 \subset \text{span}\{e_{13}\}$ and \mathcal{A} satisfies the identities $z[y_1, y_2] \equiv 0$ and $[y_1, y_2]z \equiv 0$. The case when f has one variable of homogeneous degree g and $m - 1$ neutral variables can be treated as in the previous lemma. The remaining cases are considered as above. ■

Hence, we have the following theorem.

Theorem 6.10 *Let F be an arbitrary field, let $UT_3 = \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be some nontrivial grading on \mathcal{A} , and let $f \in F\langle X \rangle^{gr}$ be a multilinear graded polynomial. Then $\text{Im}(f)$ on \mathcal{A} is a homogeneous subspace of \mathcal{A} . If $|F| \geq 3$ and \mathcal{A} is equipped with the trivial grading, then the image is also a subspace.*

Proof The proof is clear from the previous lemmas and Proposition 6.1. ■

6.2 The graded Jordan algebra UJ_2

Throughout this subsection, we assume that F is a field of characteristic different from 2 and we denote by UJ_n the Jordan algebra of the upper triangular matrices with product $a \circ b = ab + ba$. Unlike the associative setting, gradings on UJ_n are not only elementary ones. Actually, a second kind of grading also occurs on UJ_n , the so-called mirror-type gradings, and we define these below. First of all, let us introduce the following notation.

Let i, m be nonnegative integers and set

$$E_{i:m}^+ = e_{i,i+m} + e_{n-i-m+1, n-i+1} \text{ and } E_{i:m}^- = e_{i,i+m} - e_{n-i-m+1, n-i+1}.$$

Definition 6.11 A G -grading on UJ_n is called of mirror type if the matrices $E_{i:m}^+$ and $E_{i:m}^-$ are homogeneous, and $\text{deg}(E_{i:m}^+) \neq \text{deg}(E_{i:m}^-)$.

We recall the following theorem from [30].

Theorem 6.12 [30] *The G -gradings on the Jordan algebra UJ_n are, up to a graded isomorphism, elementary or of mirror type.*

In particular, we have the following classification of the gradings on UJ_2 .

Proposition 6.13 *Up to a graded isomorphism, the gradings on UJ_2 are given by $UJ_2 = \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ where:*

- (I) elementary ones;
 - (a) trivial grading;
 - (b) $\mathcal{A}_1 = Fe_{11} + Fe_{22}, \mathcal{A}_g = Fe_{12}$;
- (II) mirror type ones;
 - (a) $\mathcal{A}_1 = F(e_{11} + e_{22}), \mathcal{A}_g = F(e_{11} - e_{22}) + Fe_{12}$;
 - (b) $\mathcal{A}_1 = F(e_{11} + e_{22}) + Fe_{12}, \mathcal{A}_g = F(e_{11} - e_{22})$;
 - (c) $\mathcal{A}_1 = F(e_{11} + e_{22}), \mathcal{A}_g = F(e_{11} - e_{22}), \mathcal{A}_h = Fe_{12}$;

where $g, h \in G$ are elements of order 2.

In [30], it was also proved that the support of a grading on UJ_n is always abelian (see [30, Theorem 24]). Hence, by Proposition 2.2(4), we have that $\text{Im}(f)$ on UJ_n is a homogeneous subset for any multilinear graded polynomial $f \in J(X)$.

Next, we analyze the images of a multilinear graded Jordan polynomial f on the gradings considered above.

Lemma 6.14 *Let UJ_2 be endowed with the grading (I)(b). Then $\text{Im}(f)$ on UJ_2 is a homogeneous subspace.*

Proof We start with a multilinear polynomial f in m neutral variables. We evaluate each variable y_i to an arbitrary diagonal matrix D_i . Therefore, each monomial \mathbf{m} in f is evaluated to $2^{m-1}\beta D_1 \cdots D_m$, where $\beta \in F$ is the coefficient of \mathbf{m} . Hence,

$$f(D_1, \dots, D_m) = 2^{m-1}\alpha D_1 \cdots D_m,$$

where $\alpha \in F$ is the sum of all coefficients of f . In case $\alpha = 0$, then $f = 0$ is a graded polynomial identity for UJ_2 ; otherwise, we can take $D_2 = \cdots = D_m = I_2$ and use D_1 in order to obtain every diagonal matrix in the image of f .

Since UJ_2 satisfies the graded identity $z_1 \circ z_2 = 0$ such that $\text{deg}(z_1) = \text{deg}(z_2) = g$, then we only need to analyze the case where f is a multilinear polynomial in $m - 1$ neutral variables and one of homogeneous degree g . Obviously, we must have $\text{Im}(f) \subset \mathcal{A}_g$ and this homogeneous component is one-dimensional, then we are done. ■

For the grading (II)(a), we recall a lemma from [22] applied to multilinear polynomials. In order to make the notation more compact, we omit the symbol \circ for the Jordan product, and we write ab instead of $a \circ b$. If no brackets are given in a product, we assume that these are left-normed, that is, $abc = (ab)c$.

Lemma 6.15 [22] *Let UJ_2 be endowed with the grading (II)(a), and let $f \in J(X)_g$ be a multilinear \mathbb{Z}_2 -graded polynomial. Then, modulo the graded identities of UJ_2 , we can write f as a linear combination of monomials of the type*

$$y_1 \cdots y_l z_{i_0} (z_{i_1} z_{i_2}) \cdots (z_{i_{2m-1}} z_{i_{2m}}), 1 < \cdots < l, i_1 < i_2 < i_3 < \cdots < i_m < i_{m+1}, i_0 > 0.$$

Lemma 6.16 *Let UJ_2 be endowed with the grading (II)(a). Then $\text{Im}(f)$ on UJ_2 is a homogeneous subspace.*

Proof Since $\dim \mathcal{A}_1 = 1$, it follows that if the image of a multilinear polynomial on UJ_2 is contained in \mathcal{A}_1 , then it must be either $\{0\}$ or \mathcal{A}_1 .

Now, we consider a multilinear polynomial f in homogeneous variables of degree 1 and g such that $\text{deg } f = g$. Let $\mathbf{m} = y_1 \cdots y_l z_{i_0} (z_{i_1} z_{i_2}) \cdots (z_{i_{2m-1}} z_{i_{2m}})$ be a monomial as in Lemma 6.15. We note that the main diagonal of a matrix in $m(UJ_2)$ is such that the entry (k, k) is given by $(-1)^{k+1} 2^{m+l+1} a$ where a is the product of the entries at position $(1, 1)$ of all matrices y and z . Hence, every matrix in $\text{Im}(f)$ is of the form

$$\begin{pmatrix} 2^{m+l+1}\alpha \cdot a & * \\ & -2^{m+l+1}\alpha \cdot a \end{pmatrix},$$

where α is the sum of all coefficients of f .

In case $\alpha = 0$, then $\text{Im}(f) \subset \text{span}\{e_{12}\}$ and then the image is completely determined.

We consider now $\alpha \neq 0$. Without loss of generality, we assume that the nonzero scalar occurs in the monomial $y_1 \cdots y_l z_0(z_1 z_2) \cdots (z_{2m-1} z_{2m})$. Then we make the following evaluation: $y_1 = \cdots = y_l = I_2, z_0 = w_1(e_{11} - e_{22}) + w_2 e_{12}$, and $z_i = e_{11} - e_{22}$, for every $i = 1, \dots, 2m$, where w_1 and w_2 are commutative variables. Therefore,

$$f(y_1, \dots, y_l, z_0, \dots, z_{2m}) = \begin{pmatrix} 2^{m+l+1} \alpha w_1 & 2^{m+l+1} w_2 \\ & -2^{m+l+1} \alpha w_1 \end{pmatrix}.$$

Since $\text{char}(F) \neq 2$ and $\alpha \neq 0$, it follows that $\text{Im}(f) = \mathcal{A}_g$. ■

Now, we consider the grading (II)(b) and we recall another lemma from [22].

Lemma 6.17 [22] *Let $f \in J(X)_1$ be a multilinear polynomial. Then, modulo the graded identities of UJ_2 , f can be written as a linear combination of monomials of the form:*

- (1) $(y_{i_1} \cdots y_{i_r})(z_{j_1} \cdots z_{j_l})$;
- (2) $((y_i z_{j_1}) z_{j_2}) y_{i_1} \cdots y_{i_r} z_{j_3} \cdots z_{j_l}$,

where $l \geq 0$ is even, $r \geq 0, i_1 < \cdots < i_r$, and $z_{j_1} < z_{j_2} < z_{j_3} < \cdots < z_{j_l}$.

Lemma 6.18 *Let UJ_2 be endowed with the grading (II)(b). Then $\text{Im}(f)$ on UJ_2 is a homogeneous subspace.*

Proof We start with a multilinear polynomial f in m neutral variables. Note that a multilinear monomial of degree m evaluated on \mathcal{A}_1 is a matrix whose main diagonal is given by $2^{m-1} a I_2$ where a is the product of the entries on the main diagonal of the matrices used in the evaluation. Hence, a matrix in $\text{Im}(f)$ must be of the form

$$\begin{pmatrix} 2^{m-1} \alpha \cdot a & * \\ & 2^{m-1} \alpha \cdot a \end{pmatrix},$$

where α is the sum of all coefficients of f . In case $\alpha = 0$, we have $\text{Im}(f) \subset \text{span}\{e_{12}\}$ and the image is completely determined. From now on, we assume $\alpha \neq 0$, and we evaluate $m - 1$ matrices by I_2 and one matrix, say y_1 , by $w_1(e_{11} + e_{22}) + w_2 e_{12}$, where w_1 and w_2 are commuting variables. Therefore,

$$f(y_1, \dots, y_m) = \begin{pmatrix} 2^{m-1} \alpha w_1 & 2^{m-1} \alpha w_2 \\ & 2^{m-1} \alpha w_1 \end{pmatrix},$$

and since $\text{char}(F) \neq 2$ and $\alpha \neq 0$, we have that $\text{Im}(f) = \mathcal{A}_1$.

Now, we consider a multilinear polynomial f which has at least one variable of homogeneous degree g . In case $\text{deg } f = g$, the image $\text{Im}(f)$ is completely determined, since $\dim \mathcal{A}_g = 1$. So we assume $\text{deg } f = 1$. In case f is a multilinear polynomial in variables of homogeneous degree g , then $\text{Im}(f)$ is contained in the vector space of the scalars matrices, and therefore the image is completely determined. Hence, we assume further that f has at least one variable of neutral degree and let f be a multilinear polynomial in l neutral variables y_1, \dots, y_l , and $m - l$ variables z_{l+1}, \dots, z_m of homogeneous degree g . Then, by Lemma 6.17, we write f as

$$f = \alpha_1 (y_1 \cdots y_l)(z_{l+1} \cdots z_m) + \sum_{i=1}^l \alpha_{i+1} (((y_i z_{l+1}) z_{l+2}) y_1 \cdots \widehat{y}_i \cdots y_l) z_{l+3} \cdots z_m.$$

Here, \widehat{y}_i means that the variable y_i does not appear in the product $y_1 \cdots \widehat{y}_i \cdots y_l$.

We replace $y_i = w_1^{(i)}(e_{11} + e_{22}) + w_2^{(i)}e_{12}$ and $z_j = w_1^{(j)}(e_{11} - e_{22})$, where the w 's are commuting variables. Note that the Jordan product of two matrices y_1 and y_2 is given by $2y_1 \cdot y_2$ where the dot \cdot stands for the usual product of matrices. On the other hand, the usual product of n matrices y_1, \dots, y_n is given by

$$\begin{pmatrix} w_1^{(1)} \cdots w_n^{(n)} & w \\ & w_1^{(1)} \cdots w_n^{(n)} \end{pmatrix}.$$

Here, $w = \sum_{\substack{1 \leq i_1 < \dots < i_{n-1} \leq n \\ i_n \in \{1, \dots, n\} \setminus \{i_1, \dots, i_{n-1}\}}} w_1^{(i_1)} \cdots w_1^{(i_{n-1})} w_2^{(i_n)}$, as one can see by induction on

n . Hence, the image of the monomial $\alpha_{i+1}(((y_i z_{l+1}) z_{l+2}) y_1 \cdots \widehat{y}_i \cdots y_l) z_{l+3} \cdots z_m$ is equal to

$$2^{m-1} \alpha_{i+1} \begin{pmatrix} w_1^{(l)} \cdots w_1^{(l)} w_1^{(l+1)} \cdots w_1^{(m)} & w_i w_1^{(l+1)} \cdots w_1^{(m)} \\ w_1^{(l)} \cdots w_1^{(l)} w_1^{(l+1)} \cdots w_1^{(m)} & \end{pmatrix},$$

where w_i is given as w above but $i_n \neq i$.

Therefore, the main diagonal of $f(y_1, \dots, y_l, z_{l+1}, \dots, z_m)$ is given by

$$2^{m-1} \alpha w_1^{(1)} \cdots w_1^{(m)},$$

where α is the sum of all coefficients in f . The entry at position $(1, 2)$ is

$$\sum_{\substack{1 \leq i_1 < \dots < i_{l-1} \leq l \\ k \in \{1, \dots, l\} \setminus \{i_1, \dots, i_{l-1}\}}} \beta_k w_1^{(i_1)} \cdots w_1^{(i_{l-1})} w_2^{(k)} w_1^{(l+1)} \cdots w_1^{(m)},$$

where $\beta_k = \alpha_1 + \sum_{\substack{j=1 \\ j \neq k}}^l \alpha_{j+1}$.

If all β_k are equal to zero, then $\text{Im}(f)$ is contained in the space of the scalar matrices and we are done. So we may assume that some of the β_k is nonzero, and without loss of generality, we suppose $\beta_l \neq 0$. In this case, we claim that the image will be the whole neutral component. Indeed, take $A = a_1(e_{11} + e_{22}) + a_2 e_{12} \in \mathcal{A}_1$. We evaluate the matrices of degree g by $e_{11} - e_{22}$, that is, we take $w_1^{(l+1)} = \dots = w_1^{(m)} = 1$. We also evaluate the neutral matrices y_1, \dots, y_{l-1} by the identity matrix, that is, we take $w_1^{(1)} = \dots = w_1^{(l-1)} = 1$ and $w_2^{(1)} = \dots = w_2^{(l-1)} = 0$. Hence, the equality

$$f(I_2, \dots, I_2, y_l, e_{11} - e_{22}, \dots, e_{11} - e_{22}) = A$$

leads us to the following linear system:

$$\begin{cases} 2^{m-1} \alpha w_1^{(l)} = a_1, \\ 2^{m-1} \beta_l w_2^{(l)} = a_2, \end{cases}$$

which has $w_1^{(1)} = (2^{m-1} \alpha)^{-1} a_1$ and $w_2^{(2)} = (2^{m-1} \beta_l)^{-1} a_2$ as the solution. ■

Theorem 6.19 *Let $UJ_2 = \bigoplus_{g \in G} \mathcal{A}_g$ be a nontrivial G -grading, and let $f \in J(X)$ be a multilinear graded Jordan polynomial. Then $\text{Im}(f)$ on UJ_2 is a homogeneous subspace. The same conclusion also holds for the trivial grading on UJ_2 for arbitrary fields.*

Proof We first consider a nontrivial grading on UJ_2 . By Remark 6.2, we may reduce the defined grading on UJ_2 to one of those described above. We note that the case of the grading (II)(c) follows from the fact that $\text{Im}(f)$ on UJ_2 is a homogeneous subset and all homogeneous components in this grading are one-dimensional. We use Lemmas 6.14, 6.16, and 6.18 for the remaining nontrivial gradings. Now, we consider the trivial grading on UJ_2 . Let $f \in J(X)$ be a multilinear polynomial. We may assume that $f \notin \text{Id}(UJ_2)$. By [39], the algebra $J(X)/\text{Id}(UJ_2)$ is a special Jordan algebra, and hence we may assume that f is an element in the free special Jordan algebra. Therefore, the image $\text{Im}(f)$ on UJ_2 is equal to the image of some associative polynomial on UT_2 . Hence, $\text{Im}(f) \in \{J, UJ_2\}$. ■

Remark 6.20 Consider the Lie algebra $UT_n^{(-)}$ with product given by the Lie bracket. Given a grading on $UT_n^{(-)}$, note that $J = [UT_n^{(-)}, UT_n^{(-)}]$ is always a homogeneous ideal. We also note that if $f \in L(X)^{gr}$ is a multilinear polynomial of degree ≥ 2 , then $\text{Im}(f)$ on $UT_n^{(-)}$ is contained in J . In particular, for $n = 2$, we must have that $\text{Im}(f)$ is contained in $\text{span}\{e_{12}\}$ which is a homogeneous subspace. Since the image of multilinear polynomials of degree 1 is trivial, we have that $\text{Im}(f)$ on the graded algebra $UT_2^{(-)}$ is always a homogeneous subspace, regardless of the grading defined on $UT_2^{(-)}$.

6.3 The natural elementary \mathbb{Z}_3 -grading in the Jordan algebra UJ_3

In this subsection, we study images of multilinear polynomials on the Jordan algebra $\mathcal{A} = UJ_3$ endowed with the elementary \mathbb{Z}_3 -grading given by the sequence $(\bar{0}, \bar{1}, \bar{2})$, that is, $\mathcal{A}_{\bar{0}} = \text{span}\{e_{11}, e_{22}, e_{33}\}$, $\mathcal{A}_{\bar{1}} = \text{span}\{e_{12}, e_{23}\}$, and $\mathcal{A}_{\bar{2}} = \text{span}\{e_{13}\}$.

We denote by $(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3)$ the associator of the elements x_1, x_2, x_3 .

We recall the following identity which holds in any Jordan algebra.

Lemma 6.21 Let \mathcal{J} be a Jordan algebra. Then

$$abcd + adcb + bdca = (ab)(cd) + (ac)(bd) + (ad)(bc)$$

for all $a, b, c, d \in \mathcal{J}$.

Proof See, for example, [23, p. 34]. ■

As an easy consequence of Lemma 6.21, we have

$$(4) \quad abcd + adcb + bdca = abdc + acdb + bcda$$

for every $a, b, c, d \in \mathcal{J}$.

The next lemma points out some graded identities for the algebra UJ_3 .

Lemma 6.22 The identities

$$(y_1, y_2, y_3) \equiv 0, (y_1, z, y_2) \equiv 0 \text{ and } z_1z_2 \equiv 0$$

hold for UJ_3 , where z is an odd variable and $\text{deg}(z_1) + \text{deg}(z_2) = \bar{0}$.

Proof A straightforward computation, hence omitted. ■

The next lemma has the same proof as [22, Lemma 5.3]. However, we will consider its proof here for the sake of completeness.

Lemma 6.23 *The polynomial*

$$g = y_1(y_2(y_3z)) - \frac{1}{2} \left(y_1(z(y_2y_3)) + y_2(z(y_1y_3)) + y_3(z(y_1y_2)) - z(y_1(y_2y_3)) \right)$$

is a consequence of (y_1, z, y_2) , where $\deg(z) \in \{\bar{1}, \bar{2}\}$.

Proof By identity (4), we have

$$-((y_2y_3)z)y_1 - ((y_1y_3)z)y_2 - ((y_1y_2)z)y_3 + ((y_2y_3)y_1)z = -((y_2z)y_1)y_3 - ((y_3z)y_1)y_2.$$

Hence, we can write $h = 2g$ as

$$\begin{aligned} h &= 2(y_1(y_2(y_3z)) - ((y_2z)y_1)y_3 - ((y_3z)y_1)y_2) \\ &= (y_3, z, y_2)y_1 + (y_2, zy_3, y_1) + (y_3, zy_2, y_1), \end{aligned}$$

which implies that g is a consequence of (y_1, z, y_2) . ■

Given two even variables y_i and y_j , we set $y_i < y_j$ if $i < j$. Hence, we define an order on words in even variables $Y_1 < Y_2$ considering the left lexicographic order in case Y_1 and Y_2 have the same length, and $Y_1 < Y_2$ in case Y_2 is longer than Y_1 . For the next lemma, we use ideas from [22, Lemma 5.6]. We denote by T the T -ideal generated by the identities from Lemma 6.22.

Lemma 6.24 *Let $f = f(y_1, \dots, y_{m-1}, z_m) \in J(X)$ be a multilinear polynomial, where $\deg(z_m) \in \{\bar{1}, \bar{2}\}$. Then, modulo T , f is a linear combination of monomials of the form $Y_1(zY_2)$, where each Y_i is an increasingly ordered product of even variables and $Y_1 < Y_2$.*

Proof It is enough to consider $f = f(y_1, \dots, y_{m-1}, z_m)$ as a monomial. We apply induction on m . If $m = 1$ or $m = 2$, then the conclusion is obvious. So we assume $m \geq 3$ and we write $f = gh$ where $g, h \in J(X)$. Without loss of generality, we may assume that the odd variable z_m occurs in g . Hence, $h = Y_1$, and by the induction hypothesis, we must have g as a linear combination of monomials of the form $Y_2(zY_3)$. On the other hand, we have

$$(Y_2(zY_3))Y_1 = \frac{1}{2} \left(Y_1(z(Y_2Y_3)) + Y_2(z(Y_1Y_3)) + Y_3(z(Y_1Y_2)) - z(Y_1(Y_2Y_3)) \right).$$

Now, it is enough to use the identities $(y_1, y_2, y_3) \equiv 0$, $(y_1, z, y_2) \equiv 0$, and the commutativity of the Jordan product to get the desired conclusion. ■

Now, we are ready to prove the main theorem of this subsection.

Theorem 6.25 *Let F be an infinite field of characteristic different from 2, and let $f \in J(X)$ be a multilinear graded polynomial. Then the image of f on the graded Jordan algebra UJ_3 endowed with the natural elementary \mathbb{Z}_3 -grading is either $\{0\}$ or some homogeneous component.*

Proof Since f is a homogeneous polynomial in the graded algebra $J(X)$ and $z_1z_2 \equiv 0$ holds on UJ_3 , for $\deg z_1 + \deg z_2 = \bar{0}$, we will consider the following three cases in our proof.

Case I: $\deg f = \bar{0}$. Here, we must have $f = f(y_1, \dots, y_m)$ and the proof is the same as the first paragraph of the proof of Lemma 6.14.

Case 2: $\deg f = \bar{1}$. Let $f = f(y_1, \dots, y_{m-1}, z_m)$ be such that $\deg z_m = \bar{1}$. By Lemma 6.24, modulo T , we may write f as a linear combination of monomials of the form $Y_1(z_m Y_2)$, where $Y_1 < Y_2$. On the other hand, given

$$(5) \quad y_i = \sum_{k=1}^3 w_k^{(i)} e_{kk} \text{ and } z_m = w_1^{(m)} e_{12} + w_2^{(m)} e_{23},$$

note that $y_i z_m = \begin{pmatrix} 0 & (w_1^{(i)} + w_2^{(i)})w_1^{(m)} & 0 \\ & 0 & (w_2^{(i)} + w_3^{(i)})w_2^{(m)} \\ & & 0 \end{pmatrix}$ and then

$$f(y_1, \dots, y_{m-1}, z_m) = \begin{pmatrix} 0 & p_1 w_1^{(m)} & 0 \\ & 0 & p_2 w_2^{(m)} \\ & & 0 \end{pmatrix},$$

where p_1 and p_2 are polynomials in the variables $w^{(i)}$, $i = 1, \dots, m - 1$. We claim that if $f \neq 0$ modulo T , then $p_1 \neq 0$ and $p_2 \neq 0$. Indeed, consider the monomial $\mathbf{m} = \alpha Y_1(z_m Y_2)$, where $Y_1 = y_{j_1} \cdots y_{j_r}$, $Y_2 = y_{l_1} \cdots y_{l_s}$, and $Y_1 < Y_2$. Note that the $(1, 2)$ entry of the image of \mathbf{m} under the evaluation (5) is given by

$$2^a \alpha (w_1^{(j_1)} \cdots w_1^{(j_r)} + w_2^{(j_1)} \cdots w_2^{(j_r)}) w_1^{(m)} (w_1^{(l_1)} \cdots w_1^{(l_s)} + w_2^{(l_1)} \cdots w_2^{(l_s)})$$

for some power of 2. Hence, p_1 contains the following monomials:

$$2^a \alpha w_1^{(j_1)} \cdots w_1^{(j_r)} w_2^{(l_1)} \cdots w_2^{(l_s)} \text{ and } 2^a \alpha w_2^{(j_1)} \cdots w_2^{(j_r)} w_1^{(l_1)} \cdots w_1^{(l_s)}.$$

Since $Y_1 < Y_2$, the two monomials above can only be obtained from the monomial \mathbf{m} . Hence, if $f \neq 0$ modulo T , then f contains some monomial \mathbf{m} as above for some nonzero α , which will imply in nonzero monomials in p_1 that are not multiple of any other one that comes from the remaining monomials of f . The same ideas also prove that $p_2 \neq 0$.

Now, we use the fact that F is infinite to get evaluations of the even variables for diagonal matrices such that p_1 and p_2 assume nonzero values on F , simultaneously. We finally use the variables $w_1^{(m)}$ and $w_2^{(m)}$ to get arbitrary odd matrices in $\text{Im}(f)$, that is, $\text{Im}(f) = (UJ_3)_{\bar{1}}$.

Case 3: $\deg f = \bar{2}$. This last case follows from the fact that the homogeneous component of degree $\bar{2}$ is one-dimensional. ■

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