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ON THE EMBEDDING INTO A RING OF AN ARCHIMEDEAN l-GROUP

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We shall prove the following about the "ringification" ρA of [**2**] and [**5**] of an archimedean l-group A: (a) Any "minimal ring" containing A is ρA; (b) $A \mapsto \rho A$ is a reflector; (c) ρA need not be laterally complete when A is. These constitute the solutions to the problems posed in [**2**] by Paul Conrad.

1. The embedding into a ring. Let $\mathscr{L}$ be the category which has objects archimedean l-groups A with distinguished positive weak unit e_A, and morphisms l-group homomorphisms $h: A \rightarrow B$ with $h(e_A) = e_B$. Let $\mathscr{R}$ be the category with objects archimedean f-rings R with identity 1_R which is a weak unit, and morphisms l-ring homomorphisms $h: R \rightarrow S$ with $h(1_R) = 1_S$.

1.1 THEOREM. *If $A \in \mathscr{L}$, then there is $\rho A \in \mathscr{R}$ and an $\mathscr{L}$-embedding $\rho_A: A \rightarrow \rho A$ with the universal mapping property: If $h: A \rightarrow R$ is an $\mathscr{L}$-morphism, with $R \in \mathscr{R}$, then there is a unique $\mathscr{R}$-morphism $\rho h: \rho A \rightarrow R$ with $(\rho h) \circ \rho_A = h$.*

It seems appropriate to credit this theorem to Conrad and the present authors: In [**2**], Conrad creates a "ringification" $c_A: A \rightarrow cA$ by embedding A into its essential closure $D(Q)$ (Q being the Stone space of the polar algebra of A) with $e_A \mapsto 1$; then cA is the subring generated by A. He then shows that any f-ring B (archimedean or not) in which A is large (see § 4) with e_A an identity for B, is essentially cA; this uniqueness is a weak version of the mapping property of 1.1. (Actually, Conrad proceeded more generally, with a fixed order basis rather than simply a weak unit; this will not concern us at present). In ignorance of [**2**], we proved Theorem 1.1 in [**5**], but assuming scaler multiplication on the groups and rings. The appendix to [**5**] discusses these things, and points out that the scalar multiplication is not needed.

We shall see in 3.2 below, that for each A, $(\rho_A, \rho A) \cong (c_A, cA)$. The consequences are discussed there.

Remark on reflections. There is the obvious forgetful functor $F: \mathscr{R} \rightarrow \mathscr{L}$, and 1.1 says exactly that F has a left adjoint ρ for which the adjunctions ρ_A are embeddings. In fact, F is one-to-one. (I.e., each $\mathscr{L}$-object admits at most one compatible multiplication making the weak unit the ring identity; this is proved in 2.2 of [**2**], and again, differently, in §4 of [**5**].) This permits inter-

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preting $\mathscr{R}$ as a subcategory of $\mathscr{L}$, and with this interpretation, 1.1 says that $\mathscr{R}$ is reflective in $\mathscr{L}$. (Terminology from, e.g., [**6**].)

While we shall use this terminology below—ρA will be called the *$\mathscr{R}$-reflection* of A, and ρ_A the *reflection embedding*—we would like to point out that 1.1 and (a) of the introduction ($=$ 3.1 below) permit another proof of the uniqueness of multiplication: Given $R \in \mathscr{R}$, the identity 1_R: $R \rightarrow R$ is a "minimal $\mathscr{L}$-embedding" of $R \in \mathscr{L}$ into $R \in \mathscr{R}$, hence by 3.1 has the universal mapping property of 1.1. Thus, if R' denotes the $\mathscr{L}$-object R with another multiplication, we have

$$\begin{tikzcd} R \arrow[r, "1_R"] \arrow[d, "1_R"'] & R \arrow[dl, dashed, "f"] \\ R' & \end{tikzcd}$$

for a unique $\mathscr{R}$-morphism f. Clearly $f = 1_R$, which says the multiplications of R' and R are the same.

2. Point-separating representation. Wide use has been made of various representations of archimedean l-groups and rings as structures of continuous extended-real-valued functions defined over spaces of ideals or something similar. In our proof [**5**] of 1.1 (sketched in § 3) we employed a version of the Yosida representation [**9**] (which, it seems to us, has been under-exploited), and we shall need it further here.

Given $A \in \mathscr{L}$, let X_A denote the space of ideals of A (also called *solid subgroups*) which are maximal with respect to the property of not containing the weak unit e_A, given the hull-kernel topology.

If X is any topological space, let

$$D(X) = \{f \colon X \rightarrow [-\infty, +\infty] \mid f \text{ continuous}, f^{-1}(-\infty, +\infty) \text{ dense}\}.$$

2.1 THEOREM. *If $A \in \mathscr{L}$, then X_A is non-void compact Hausdorff, and there is an $\mathscr{L}$-isomorphism $A \mapsto \hat{A}$ onto an "l-subgroup of $D(X_A)$" such that $\hat{e}_A$ is the function constantly* 1 *and $\hat{A}$ separates the points of X_A.*

If $A \mapsto \bar{A}$ is an $\mathscr{L}$-isomorphism onto an l-subgroup of $D(X)$, with X compact Hausdorff, such that $\bar{e}_A \equiv 1$ and $\bar{A}$ separates points, then there is a homeomorphism τ: $X \rightarrow X_A$ such that $\hat{a} \circ \tau = \bar{a}$ for each $a \in A$.

2.2 THEOREM. *The representation $A \mapsto \hat{A} \subset D(X_A)$ of* 2.1 *has the property: If $h : A \rightarrow B$ is an $\mathscr{L}$-morphism, then there is a unique continuous map $\tau : X_B \rightarrow X_A$ such that $h(a)\hat{} = \hat{a} \circ \tau$ for each $a \in A$.*

The proofs of 2.1 and 2.2 are in [**5**] for vector lattices; the necessary modifications (described in [**5**]) are easy.

2.3 PROPOSITION. *With h and τ as in* 2.2, *h is one-to-one if and only if τ is onto.*

Proof. We need to know that (for any A), $\hat{A}$ separates points from closed sets in X_A (equivalently, the sets coz $\hat{a} = \{p | \hat{a}(p) \neq 0\}$ form a base). This follows by a compactness argument from the point-separating. Then, $\tau(X_B) \neq X_A$ is equivalent to the existence of $a \in A$, $a \neq 0$, with (coz $\hat{a}$) $\cap\, \tau(X_B) = \emptyset$. For such a, we have $\hat{a}(\tau(x)) = 0$ for each x, hence $h(a) = 0$ and h is not one-to-one. The converse is now clear.

3. Strong uniqueness of ρA.

The principal result of this section (and the paper) is

3.1 THEOREM. *Let $A \in \mathscr{L}$, $R \in \mathscr{R}$, let $h : A \rightarrow R$ be an $\mathscr{L}$-embedding, and suppose there is no $\mathscr{R}$-object properly between $h(A)$ and R. Then there is an $\mathscr{R}$-isomorphism $i : \rho A \rightarrow R$ with $i \circ \rho_A = h$.*

This is the exact statement of (a) of the introduction. Before proceeding to the proof, we give some corollaries.

3.2 COROLLARY. *For each $A \in \mathscr{L}$, Conrad's ringification* (c_A, cA) *and* $(\rho_A, \rho A)$ (*of* 1.1) *are related by a commuting $\mathscr{R}$-isomorphism* (*per* 3.1).

Proof. Conrad's construct satisfies the condition in 3.1, by 1.1 of [**2**].

3.3 COROLLARY. *For each $A \in \mathscr{L}$, Conrad's* (c_A, cA) *has the universal mapping property of* 1.1.

Proof. By 3.2 and 1.1.

Thus referring to the questions at the end of [**2**]: 3.1 and 3.2 yield an an affirmative answer to (1); 3.1 and 3.3 yield an affirmative answer to (2).

We now present the proof to 3.1, in several steps.

3.4 Given $A \in \mathscr{L}$, consider a diagram

$$\begin{tikzcd} A \arrow[r, "\rho_A"] \arrow[d, "h"'] & \rho A \arrow[dl, dashed, "\rho h"] \\ R & \end{tikzcd}$$

with $A \xrightarrow{h} R$ as in the hypotheses of 3.1; the unique $\mathscr{R}$-morphism ρh with $(\rho h) \circ \rho_A = h$ exists by 1.1. Thus, the image $(\rho h)(\rho A) \in \mathscr{R}$, and by minimality, ρh is onto. We seek to prove that ρh is $1 - 1$. Then ρh will be the $\mathscr{R}$-isomorphism of 3.1.

Applying 2.2 to the above diagram yields the "dual" commuting diagram

$$\begin{tikzcd} X_A & X_{\rho A} \arrow[l, "\alpha"'] \\ X_R \arrow[u, "\beta"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

in the category of compact Hausdorff spaces. (Here, $\hat{a} \circ \alpha = \rho_A(a)\hat{}$ for each $a \in A$; $\hat{a} \circ \beta = h(a)\hat{}$ for each $a \in A$; $\hat{b} \circ \gamma = (\rho h)(b)$ for each $b \in \rho A$.) According to 2.3, α and β are onto because ρ_A and h are one-to-one; and, ρh will be one-to-one *if* γ is onto. We prove this in two steps:

3.5 LEMMA. *Consider a commuting diagram*

$$\begin{tikzcd} X & Y \arrow[l, "\alpha"'] \\ Z \arrow[u, "\beta"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

in compact Hausdorff spaces. If α is irreducible and β is onto, then γ is onto.

(A continuous function is called *irreducible* if it is onto, and maps *no* proper closed subspace of the domain onto the range.)

Proof. $\gamma(Z)$ is compact, hence closed. If γ were not onto, then $\gamma(Z)$ would be a proper closed subspace; hence $\alpha(\gamma(Z)) \neq X$, because α is irreducible. But $\alpha(\gamma(Z)) = \beta(Z) = X$, because β is onto.

3.6 PROPOSITION. *For each $A \in \mathscr{L}$, the continuous map $X_A \overset{\alpha}{\leftarrow} X_{\rho A}$ which "induces" the reflection embedding $A \xrightarrow{\rho A} \rho A$ per 2.2 is irreducible.*

Proof. We need to recall the construction of ρA from § 6 of [**5**]:

Let L be the set of principal ideals (= solid subgroups) which contain the weak unit e_A, directed by set inclusion. For each $I \in L$, ρI is constructed (below) with the universal mapping property of 1.1. Then, if $J \supset I$, the inclusion "lifts" to an $\mathscr{R}$-morphism $\varphi_I{}^J : \rho I \rightarrow \rho J$, which is shown to be one-to-one. This process provides a direct limit system in $\mathscr{R}$. ρA is defined as

$$\varinjlim \{\rho I | I \in L\} \quad \text{(the direct limit in } \mathscr{R}\text{)},$$

and the reflection embedding $\rho_A : A \rightarrow \rho A$ results from the fact that $A = \bigcup \{I | I \in L\}$.

For $I \in L$, the construction of ρI is this: For some $a \geqq e_A$, we have $I = \{b \in A | \, |b| \leqq na$ for some integer $n\}$. Letting $R_a = \hat{a}^{-1}(-\infty, +\infty)$, each

$b \in I$ is real-valued on R_a, and we may view $\hat{I}$ as a subset of $C(R_a)$ (the continuous real-valued functions on R_a). Since $C(R_a) \in \mathscr{R}$, there is a least "$\mathscr{R}$-subobject" of $C(R_a)$ which contains I; this is taken as ρI, with $I \to \hat{I} \subset \rho I$ the reflection embedding.

Now I has weak unit e_A and $\hat{I}$ separates the points of X_A. By 2.1, $X_I = X_A$. Since $\rho I \subset C(R_a)$, there is a compactification of R_a over which all functions in ρI extend (with values in $[-\infty, +\infty]$) and with the extensions separating points (take a quotient of the Stone-Čech compactification βR_a). By 2.1, this is $X_{\rho I}$. Evidently, there is a map $\alpha_I : X_{\rho I} \to X_I = X_A$ extending the inclusion $R_a \subset X_A$.

For $J \supset I$, $\varphi_I{}^J : \rho I \to \rho J$ is, by 2.2, "dually induced" by a map $\alpha_I{}^J : X_{\rho J} \to X_{\rho I}$. We now have an inverse limit system of compact Hausdorff spaces. Let

$$Z = \varprojlim \{X_{\rho I} | I \in L\}.$$

For $I \in L$, let $\prod_I : Z \to X_{\rho I}$ be the projection and let $\alpha_I \circ \prod_I \equiv \alpha : Z \to X_I = X_A$. This is easily seen to not depend on the choice of I. We now shall show that α is irreducible, and then that $Z = X_{\rho A}$.

Let F be a proper closed set in Z. We may as well suppose that $F = Z - U$, where U is nonvoid and basic, of the form

$$Z \cap \bigcap_{i=1} \textstyle\prod_{I_i}^{-1} (U_i),$$

U_i open in $X_{\rho I_i}$. For each i, let a_i be a generator for I_i with $a_i \geqq e_A$, and let I be generated by $a_1 \vee \cdots \vee a_n$. Since $I \supset I_i$ for each i, U collapses to $Z \cap \prod_I^{-1}(G)$, where

$$G = \bigcap_{i=1}^{n} (\alpha_{I_i})^{-1} (U_i).$$

Then

$$\alpha(F) = \alpha_I\left(\textstyle\prod_I (F)\right) = \alpha_I\left(\textstyle\prod_I (Z - U)\right) \doteq \alpha_I(X_{\rho I} - G).$$

Now $\alpha_I(X_{\rho I} - R_a) = X_A - R_a$ $(a = a_1 \vee \cdots \vee a_n)$ by 6.11 of [**3**]. Since $G \cap R_a \neq \emptyset$, $\alpha_I(X_{\rho I} - G)$ misses some points of R_a, hence is a proper subset of X_A. Thus, α is irreducible.

To see that $Z = X_{\rho A}$, we create a point-separating representation of ρA on Z and use 2.2. If $f \in \rho A$, then $f \in \rho I$ for some I and is viewed as a $[-\infty, +\infty]$-valued function on $X_{\rho I}$. Let $\hat{f} = f \circ \prod_I$; this is independent of I. We want to show that

$$\hat{f}^{-1}(-\infty, +\infty) = \textstyle\prod_I^{-1} (f^{-1}(-\infty, +\infty))$$

is dense, and it is easy to check that irreducible maps inversely preserve dense sets. So, it suffices that $\prod_I$ be irreducible. But $\alpha_I \circ \prod_I = \alpha$ is irreducible,

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and so is α_I (from the end of the last paragraph); thus Π_I must be. To see that the $\hat{f}$'s separate points, suppose $p \neq q$ in Z. For some coordinate I, $p_I \neq q_I$, and there is $f \in \rho I$ with $f(p_I) \neq f(q_I)$; then $\hat{f}(p) \neq \hat{f}(q)$.

The proof of 3.6 is complete.

4. On irreducibility and strong uniqueness. We make a few remarks on the material of the preceding section. The first gives an algebraic version of the irreducibility condition, and the second points out a generalization of 3.1.

4.1 PROPOSITION. *Let $h : A \to B$ be an $\mathscr{L}$-embedding. Then, $h(A)$ is large in B if and only if the map $\alpha : X_B \to X_A$ "dually inducing" h (per 2.2) is irreducible.*

("Large" means that each nonzero ideal of B meets $h(A)$ non-trivially, or that whenever $0 < b \in B$, then there are $a \in A$ and an integer n, with $0 < h(a) < nb$.)

Proof. First, irreducibility of α is easily seen to be equivalent to: For O open and nonvoid in X_B, there is an open nonvoid V in X_A with $\alpha^{-1}(V) \subset O$. The sets O and V may be taken to be basic; a convenient basis for an X_A is $\{\operatorname{coz} \hat{a} | a \in A^+\}$ (as in the proof of 2.2).

So, let $h(A)$ be large in B, let $0 = \operatorname{coz} \hat{b}$ be given and choose a and n with $0 < h(a) < nb$. Obviously, $\operatorname{coz} h(a)\hat{} \subset \operatorname{coz} (nb)\hat{} = \operatorname{coz} \hat{b}$. But $h(a)\hat{} = \hat{a} \circ \alpha$, and $\operatorname{coz} h(a)\hat{} = \alpha^{-1}(\operatorname{coz} \hat{a})$. Thus $V = \operatorname{coz} \hat{a}$ has $\alpha^{-1}(V) \subset O$.

Conversely, let α be irreducible and let $0 < b \in B$. Choose n and open O with $\hat{b}|O \geqq 1/n$. Now take $V = \operatorname{coz} \hat{a}_1$ ($a_1 \in A^+$) with $\alpha^{-1}(V) \subset O$, and let $a = a_1 \wedge e_A$ (so $\hat{a} = \hat{a}_1 \wedge 1$). Then $\operatorname{coz} h(a)\hat{} = \alpha^{-1} (\operatorname{coz} \hat{a}) = \alpha^{-1}(\operatorname{coz} \hat{a}_1) \subset O \subset \operatorname{coz} \hat{b} = \operatorname{coz} (nb)\hat{}$. Clearly, $0 < h(a) < nb$.

Combining 4.1 with the obvious generalizations of 3.4 and 3.5 (whose proofs work here), we have

4.2 PROPOSITION. *Let $\mathscr{S}$ be a subcategory of $\mathscr{L}$, and let $A \in \mathscr{L}$ have an $\mathscr{S}$-reflection $S_A : A \to sA$ with S_A an $\mathscr{L}$-embedding. If $S_A(A)$ is large in sA, then any $\mathscr{L}$-embedding $h : A \to S$ for which there is no $\mathscr{S}$-object properly between $h(A)$ and S is, up to $\mathscr{S}$-isomorphism (as in 3.1), the $\mathscr{S}$-reflection.*

Remarks. (a) We do not suppose $\mathscr{S}$ is reflective, only that there be $S_A : A \to sA$ ($sA \in \mathscr{S}$) with the required universal mapping property (as in 1.1).

(b) Conrad showed that for his (c_A, cA), $c_A(A)$ is large in cA. But without knowing reflectivity (one of Conrad's questions), we don't see how to get the conclusion of 4.2 (another of his questions).

The conclusion of 4.2 (or 3.1) seems a striking property. It is well known to fail for "almost reflections" like Dedekind or lateral completion, and is very uncommon for *topological* embedding — reflections. (The only such examples we know are in Hausdorff uniform spaces; those *epi*-reflective subcategories which contain all complete spaces.) On the other hand, we know of no embedding—reflection in $\mathscr{L}$ for which it fails.

4.3 *Question.* It it true that whenever $\mathscr{S}$ is an embedding-reflective subcategory of $\mathscr{L}$, then for each reflection $S_A : A \rightarrow sA$, $S_A(A)$ is large in sA?

5. Lateral completeness. We present two examples of laterally complete l-groups A for which ρA fails to be laterally complete; thus Conrad's question (3) is answered negatively. The first example below was contributed by the referee, who called it "the discrete version of the (second) example . . ."; this is quite simpler then the second example. The second example (our original one) is, however, a vector lattice, and since we don't see how to get a simpler vector lattice example, we include it here. (These examples are also orthocomplete with ρA failing to be (the first directly, the second by [**8**]); this was part of Conrad's question).

5.1. *Example.* Let T be the subgroup of the additive reals R generated by 1 and π. For each i, let T_i be a copy of T and let

$$A = \prod_{i=1}^{\infty} T_i.$$

Then A is laterally complete, but the subring A' of $\prod_{i=1}^{\infty} R_i$ generated by A is not laterally complete. And $A' = \rho A$ either by 3.1 or by 1.1 of [**2**] (since A is large in $\prod_{i=1}^{\infty} R_i$).

5.2. *Example.* Let A be the lateral completion of the vector lattice P of continuous piecewise linear functions on the unit interval. We shall see that A is not laterally complete. We require a description of A.

Let X be the projective cover of the unit interval [**4**]: X is compact, extremally disconnected, and there is an irreducible map $\alpha : X \rightarrow [0, 1]$. (Or, X is the Stone space of the Boolean algebra of polars of P.) A familiar [**7**; **1**] representation of P is created, with

$$\bar{P} = \{p \circ \alpha | p \in P\}.$$

(As in § 3, the sets $(P \circ \alpha)^{-1}(-\infty, +\infty)$ are dense, by irreducibility of α). Then the lateral completion A emerges as the set of all $f \in D(X)$ such that for some disjoint family $\mathscr{U}$ of clopen sets with $\bigcup \mathscr{U}$ dense, $f|U \in \bar{P}$ for each $U \in \mathscr{U}$ (Theorem 11 of [**8**]).

The description of these functions can be simplified a bit: First, such $\mathscr{U}$'s are at most countable because [0, 1] has the countable chain condition, and irreducible maps inversely preserve this property. Second, then, if $f \in A$ there is $\mathscr{U} = \{U_n\}$ countable, and for each n, $P_n \in P$ such that $f|U_n = \overline{P_n}|U_n$. Associated with P_n is a "partition" of [0, 1], say $0 = x_0 < x_1 < \cdots < x_{k+1} = 1$, such that on each $[x_i, x_{i+1}]$, p_n is linear. Then $C_i^n = \mathrm{cl}_X \alpha^{-1}(x_i, x_{i+1})$ is clopen in X (by extermal disconnectivity). Thus, for each i, there is *linear* q_i, such that $\overline{P_n}|C_i^n = q_i|C_i^n$. So, the system $\mathscr{V} = \{U_n \cap C_i^n\}_{n,i}$ consists of disjoint clopen sets, the union is still dense, and for each $V \in \mathscr{V}$ there is a linear q_V such that $f|V = \overline{q_V}|V$.

Changing the notation, we have the following characterizing description of those $f \in A$: There is a countable dense disjoint family $\{U_n\}$ of clopen sets, and for each n, linear $q_n \in P$, such that $f|U_n = \overline{q_n}|U_n$.

By extremal disconnectivity, $D(X) \in \mathscr{R}$, so by 3.1 or by § 1 of [**2**] ρA is the smallest $\mathscr{R}$-subobject (i.e., l-subring) of $D(X)$ which contains A. This, it is easy to see, consists of those $f \in D(X)$ for which there is a positive integer d and a countable, dense, disjoint family $\{U_n\}$ of clopen sets, and for each n a polynomial p_n of degree $\leqq d$, with $f|U_n = \overline{p_n}|U_n$.

Thus we see that ρA is not laterally complete: Take any such infinite system $\{U_n\}$, let $\mathscr{X}_n$ be the characteristic function of $\mathscr{U}_n$, and for each n choose a polynomial p_n of degree $\geqq n$. Then each $\mathscr{X}_n\overline{P_n} \in \rho A$ but $\vee_n\mathscr{X}_n\overline{P_n} \notin \rho A$.

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