

LEAVITT PATH ALGEBRAS OF WEIGHTED AND SEPARATED GRAPHS

PERE ARA

(Received 11 May 2022; accepted 17 July 2022; first published online 12 September 2022)

Communicated by Daniel Chan

Abstract

In this paper, we show that Leavitt path algebras of weighted graphs and Leavitt path algebras of separated graphs are intimately related. We prove that any Leavitt path algebra $L(E, \omega)$ of a row-finite vertex weighted graph (E, ω) is $*$ -isomorphic to the lower Leavitt path algebra of a certain bipartite separated graph $(E(\omega), C(\omega))$. For a general locally finite weighted graph (E, ω) , we show that a certain quotient $L_1(E, \omega)$ of $L(E, \omega)$ is $*$ -isomorphic to an upper Leavitt path algebra of another bipartite separated graph $(E(w)_1, C(w)^1)$. We furthermore introduce the algebra $L^{\text{ab}}(E, w)$, which is a universal tame $*$ -algebra generated by a set of partial isometries. We draw some consequences of our results for the structure of ideals of $L(E, \omega)$, and we study in detail two different maximal ideals of the Leavitt algebra $L(m, n)$.

2020 *Mathematics subject classification*: primary 16S88; secondary 16S10.

Keywords and phrases: weighted graph, separated graph, Leavitt path algebra, ideal.

1. Introduction

A weighted graph is a pair (E, ω) consisting of a directed graph $E = (E^0, E^1, r, s)$ and a weight function $\omega : E^1 \rightarrow \mathbb{N}$, where \mathbb{N} is the set of positive integers. Leavitt path algebras of weighted graphs were introduced in [13] to obtain a graph theoretical model of Leavitt algebras $L(m, n)$ for arbitrary values $1 \leq m \leq n$. Recall that Leavitt algebras were introduced by Leavitt in [14], who showed that the (Leavitt) type of $L(m, n)$ is $(m, n - m)$. Some years later, Bergman reported in [9] the precise structure of the monoid $\mathcal{V}(L(m, n))$ of isomorphism classes of finitely generated projective $L(m, n)$ -modules. Recently, an interesting connection between the K -theory of Leavitt path algebras of weighted graphs and the theory of abelian sandpile models has been

Partially supported by DGI-MINECO-FEDER grant PID2020-113047GB-I00, and the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M).

© The Author(s), 2022. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.



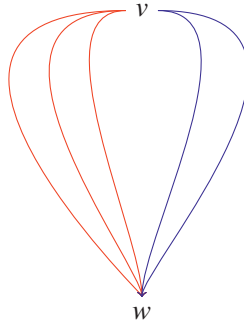


FIGURE 1. Separated graph $(E(2, 3), C(2, 3))$.

developed in [2]. We refer the reader to [16] for an excellent survey on Leavitt path algebras of weighted graphs.

The algebras $L(m, n)$ were also described by the author and Goodearl in [6] as full corners of the Leavitt path algebras of the separated graphs $(E(m, n), C(m, n))$, and this was one of the key motivations to introduce this new type of algebras. Recall that a separated graph [6] is a pair (E, C) consisting of a directed graph E and a partition C of the set of edges of E that refines the natural partition induced by the source function.

For all integers $1 \leq m \leq n$, define the separated graph $(E(m, n), C(m, n))$ as follows:

- (1) $E(m, n)^0 := \{v, w\}$;
- (2) $E(m, n)^1 := \{e_1, \dots, e_n, f_1, \dots, f_m\}$ ($n + m$ distinct edges);
- (3) $s(e_i) = s(f_j) = v$ and $r(e_i) = r(f_j) = w$ for all i, j ;
- (4) $C(m, n) = C(m, n)_v := \{X, Y\}$, where $X = \{e_1, \dots, e_n\}$ and $Y = \{f_1, \dots, f_m\}$.

By [6, Proposition 2.12], we have an isomorphism between $L(m, n)$ and the corner algebra $wL(E(m, n), C(m, n))w$, and hence $L(m, n)$ is isomorphic to a full corner of the Leavitt path algebra of the separated graph $(E(m, n), C(m, n))$. Since a full corner eRe of a ring R is Morita-equivalent to R , the rings eRe and R share many properties. For instance, they have the same module theory and the same lattice of (two-sided) ideals.

Observe that the graph $E(m, n)$ is a *bipartite graph*, that is, there is a partition of the set of vertices $E^0 = E^{0,0} \sqcup E^{0,1}$ such that $s(E^1) \subseteq E^{0,0}$ and $r(E^1) \subseteq E^{0,1}$. As in Figure 1, we represent a bipartite separated graph (E, C) by a diagram in which we draw the vertices in $E^{0,0}$ in the upper level and the vertices in $E^{0,1}$ in the lower level of the diagram. According to this representation, we introduce in this paper the notions of the *upper Leavitt path algebra* $LV(E, C)$ and the *lower Leavitt path algebra* $LW(E, C)$ of a bipartite separated graph (see Section 2 for the precise definitions). Under the mild hypothesis that $s(E^1) = E^{0,0}$ and $r(E^1) = E^{0,1}$, it follows readily from the definitions that the upper and the lower Leavitt path algebras of a bipartite finitely separated graph (E, C) are Morita-equivalent to the full Leavitt path algebra $L(E, C)$. It turns out that, in many examples, the significant algebra to consider is an upper or a lower Leavitt path algebra of a bipartite separated graph, see [4, Section 9].

The purpose of this paper is to show that Leavitt path algebras of weighted graphs and Leavitt path algebras of separated graphs are intimately related. A *vertex weighted graph* is a weighted graph (E, ω) such that $\omega(e) = \omega(f)$ for every pair of edges e, f such that $s(e) = s(f)$. We show in Section 2 that any Leavitt path algebra of a row-finite vertex weighted graph (E, ω) is $*$ -isomorphic to the lower Leavitt path algebra of a certain bipartite separated graph $(E(\omega), C(\omega))$. For a general row-finite weighted graph, we cannot construct a bipartite separated graph satisfying the property above, but we show in Section 3 that a certain quotient $*$ -algebra $L_1(E, w)$ of $L(E, w)$ is $*$ -isomorphic to an *upper* Leavitt path algebra of another bipartite separated graph $(E(w)_1, C(w)^1)$. We furthermore introduce in Section 4 the algebra $L^{\text{ab}}(E, w)$, called the *abelianized Leavitt path algebra* of (E, ω) , for any locally finite weighted graph (E, w) , and we show that it is $*$ -isomorphic to a full corner of the $*$ -algebra $L^{\text{ab}}(E(w)_1, C(w)^1)$ introduced in [4]. These abelianized algebras have a strong dynamical behaviour, being the crossed products of certain partial actions on totally disconnected Hausdorff topological spaces. Finally, we draw in Section 5 some consequences of our results for the structure of ideals of $L(E, \omega)$, shedding light on the second Open Problem in [16, Section 12]. We illustrate our results by studying the ideals of the Leavitt algebras $L(m, n)$. In particular, two specific examples of maximal ideals of $L(m, n)$ are described.

2. Bipartite separated graphs and weighted graphs

We start with the definition of a separated graph. Concerning directed graphs, we follow the conventions and notation in [1]. In particular, we use the following definition of a path. Let $E = (E^0, E^1, r, s)$ be a directed graph. Then a trivial path (or path of length 0) is just a vertex in E and a nontrivial path is a sequence $e_1 e_2 \cdots e_n$ of edges in E such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n - 1$. The extended (or double) graph of E , denoted by \hat{E} , is the graph obtained from E by adding a new edge e^* for each edge $e \in E^1$, with $r(e^*) = s(e)$ and $s(e^*) = r(e)$.

A *row-finite graph* is a graph E such that $|s^{-1}(v)| < \infty$ for all $v \in E^0$. The set E_{reg}^0 of *regular vertices* of a row-finite graph is the set of vertices v such that $s^{-1}(v) \neq \emptyset$. A *locally finite graph* is a graph E such that both $s^{-1}(v)$ and $r^{-1}(v)$ are finite, for all $v \in E^0$.

DEFINITION 2.1 ([6, Definition 2.1] and [4, Definition 4.1]). A *separated graph* is a pair (E, C) , where E is a (directed) graph and $C = \bigcup_{v \in E^0} C_v$, in which C_v is a partition of $s^{-1}(v)$ into pairwise disjoint nonempty subsets for each vertex v . If all the sets in C are finite, we say that (E, C) is a *finitely separated graph*. This is automatically true when E is row-finite.

A *bipartite separated graph* is a separated graph (E, C) such that $E^0 = E^{0,0} \sqcup E^{0,1}$, and $s(e) \in E^{0,0}$, $r(e) \in E^{0,1}$ for each $e \in E^1$.

Note that our notation concerning ranges and sources of edges is the same as that from [1, 6, 13, 16], but it is distinct from the one used in [4, 7] and other sources.

We can now define Leavitt path algebras of separated graphs, following [6]. We consider through the paper algebras and $*$ -algebras over an arbitrary but fixed coefficient field K , endowed with an involution $*$. Note that our definitions usually refer to *presentations of $*$ -algebras in the category of $*$ -algebras*.

DEFINITION 2.2. The *Leavitt path algebra* of a separated graph (E, C) with coefficients in K is the $*$ -algebra $L(E, C)$ with generators $\{v, e \mid v \in E^0, e \in E^1\}$, subject to the following relations:

- (V) $vv' = \delta_{v,v'}v$ and $v = v^*$ for all $v, v' \in E^0$;
- (E) $s(e)e = e = er(e)$ for all $e \in E^1$;
- (SCK1) $e^*f = \delta_{e,f}r(e)$ for all $e, f \in X, X \in C$ and
- (SCK2) $v = \sum_{e \in X} ee^*$ for every finite set $X \in C_v, v \in E^0$.

Note that the path algebra $P_K(\hat{E})$ of the extended graph \hat{E} of E , endowed with its canonical involution, is precisely the $*$ -algebra defined by relations (V) and (E), so that $L(E, C)$ is the quotient of $P_K(\hat{E})$ by the $*$ -ideal corresponding to the relations (SCK1) and (SCK2).

We need the normal form of elements of $L(E, C)$, which was obtained in [6].

DEFINITION 2.3. For two nontrivial paths $\mu, \nu \in \text{Path}(E)$ with $s(\mu) = s(\nu) = v$, we say that μ and ν are *C-separated* if the initial edges of μ and ν belong to different sets $X, Y \in C_v$.

DEFINITION 2.4. For each finite $X \in C$, we select an edge $e_X \in X$. Let $\mu, \nu \in \text{Path}(E)$ be two paths such that $r(\mu) = r(\nu)$, and let e and f be the terminal edges of μ and ν , respectively. The path $\mu\nu^*$ is said to be *reduced* if $(e, f) \neq (e_X, e_X)$ for every finite $X \in C$. In the case where either μ or ν has length zero, then $\mu\nu^*$ is automatically reduced.

THEOREM 2.5 [6, Corollary 2.8]. *Let (E, C) be a separated graph. Then the set of elements of the form*

$$\mu_1\nu_1^*\mu_2\nu_2^*\cdots\mu_n\nu_n^*, \quad \mu_i, \nu_i \in \text{Path}(E),$$

such that ν_i and μ_{i+1} are C-separated paths for all $i \in \{1, \dots, n-1\}$ and $\mu_i\nu_i^$ is reduced for all $i \in \{1, \dots, n\}$, forms a linear basis of $L(E, C)$. We call $\mu_1\nu_1^*\cdots\mu_n\nu_n^*$ a C-separated reduced path.*

For each edge e of a separated graph (E, C) , we denote by X_e the unique element of C such that $e \in X_e$.

We are now ready for our definitions of the upper and lower path algebras of a bipartite separated graph. We first introduce these algebras using a presentation, and we show below in Proposition 2.8 that these are precisely the corner algebras of $L(E, C)$ corresponding to the upper and lower subsets $E^{0,0}$ and $E^{0,1}$ of E^0 .

DEFINITION 2.6. Let (E, C) be a row-finite bipartite separated graph with $s(E^1) = E^{0,0}$, $r(E^1) = E^{0,1}$. Let $LV(E, C)$ be the universal $*$ -algebra with generators $P_V \sqcup T$,

where $P_V = \{p_v\}_{v \in E^{0,0}}$ and $T = \{\tau(e, f)\}_{\{e, f \in E^1 \mid r(e)=r(f)\}}$, and subject to the relations:

- (V') $p_v p_{v'} = \delta_{v, v'} p_v$ and $p_v^* = p_v$ for all $v, v' \in E^{0,0}$;
- (T) $\tau(e, f)^* = \tau(f, e)$;
- (E') $\tau(e, f) \cdot p_{s(f)} = p_{s(e)} \cdot \tau(e, f) = \tau(e, f)$;
- (SCK1') $\tau(e, f) \cdot \tau(g, h) = \delta_{f, g} \tau(e, h)$ for $f, g \in E^1$ belonging to the same set $Y \in C$;
- (SCK2') $p_v = \sum_{e \in Y} \tau(e, e)$ for all $v \in E^{0,0}$ and all $Y \in C_v$.

DEFINITION 2.7. Let (E, C) be a row-finite bipartite separated graph with $s(E^1) = E^{0,0}$, $r(E^1) = E^{0,1}$. Let $LW(E, C)$ be the universal $*$ -algebra with generators $P_W \sqcup R$, where $P_W = \{p_w\}_{w \in E^{0,1}}$ and $R = \{\rho(e, f)\}_{\{e, f \in E^1 \mid s(e)=s(f) \text{ and } X_e \neq X_f\}}$, and subject to the relations:

- (V'') $p_w p_{w'} = \delta_{w, w'} p_w$ and $p_w^* = p_w$ for all $w, w' \in E^{0,1}$;
- (R) $\rho(e, f)^* = \rho(f, e)$;
- (E'') $\rho(e, f) \cdot p_{r(f)} = p_{r(e)} \cdot \rho(e, f) = \rho(e, f)$;
- (SCK1'') Suppose that $e, h \in E^1$ with $v := s(e) = s(h)$ and $X \in C_v$ with $X \neq X_e$ and $X \neq X_h$. Then:

$$\sum_{f \in X} \rho(e, f) \cdot \rho(f, h) = \begin{cases} \delta_{e, h} p_{r(e)} & \text{if } X_e = X_h \\ \rho(e, h) & \text{if } X_e \neq X_h. \end{cases}$$

We make use of the multiplier algebra $M(A)$ of an algebra A , see for instance [8] and [10, Ch. 7], as a convenient way of defining our algebras. Note that $M(A) = A$ if A is unital.

Let (E, C) be a row-finite bipartite separated graph such that $s(E^1) = E^{0,0}$ and $r(E^1) = E^{0,1}$. Let $V = \sum_{v \in E^{0,0}} v \in M(L(E, C))$ and $W = \sum_{w \in E^{0,1}} w \in M(L(E, C))$. Since the finite sums of vertices give a family of local units of $L(E, C)$, one can easily show that V and W exist in $M(L(E, C))$ and that $V + W = 1$. It follows readily that $VL(E, C)V$ is linearly spanned by all the paths μ in \hat{E} such that $s(\mu) \in E^{0,0}$ and $r(\mu) \in E^{0,0}$. A similar description holds for $LW(E, C)$.

PROPOSITION 2.8. We have natural $*$ -isomorphisms $\varphi_V : LV(E, C) \rightarrow VL(E, C)V$ and $\varphi_W : LW(E, C) \rightarrow WL(E, C)W$ such that

$$\varphi_V(p_v) = v, \quad \varphi_V(\tau(e, f)) = ef^*, \quad \varphi_W(p_w) = w, \quad \varphi_W(\rho(e, f)) = e^*f.$$

PROOF. We only show the isomorphism $LW(E, C) \cong WL(E, C)W$. The other isomorphism is proved in the same way.

One can easily see that the assignments $\varphi_W(p_w) = w$ and $\varphi_W(\rho(e, f)) = e^*f$ give a well-defined $*$ -algebra homomorphism

$$\varphi_W : LW(E, C) \rightarrow WL(E, C)W.$$

Clearly φ_W is surjective. To show that φ_W is injective, we observe that using the defining relations of $LW(E, C)$, we can write each element of $LW(E, C)$ as a linear combination of terms of the form

$$\rho(e_1, f_1)\rho(e_2, f_2) \cdots \rho(e_n, f_n)$$

such that $s(e_i) = s(f_i)$, $X_{e_i} \neq X_{f_i}$ for $i = 1, \dots, n$, $r(f_i) = r(e_{i+1})$ and $f_i e_{i+1}^*$ is reduced for $i = 1, \dots, n - 1$.

Now suppose that α is a nonzero element of $LW(E, C)$ and that $\alpha = \sum \lambda_i \alpha_i$, where $\lambda_i \in K \setminus \{0\}$ and α_i are pairwise distinct terms as described in the above paragraph. Then $\varphi_W(\alpha) = \sum \lambda_i \varphi_W(\alpha_i)$ and $\varphi_W(\alpha_i)$ are pairwise distinct C -separated reduced paths in $L(E, C)$. Since these elements are linearly independent in $L(E, C)$, we conclude that $\varphi_W(\alpha) \neq 0$. □

In view of Proposition 2.8, we identify the algebras $LW(E, C)$ and $VL(E, C)V$, and also the algebras $LW(E, C)$ and $WL(E, C)W$.

We recall now the definitions of weighted graph and of Leavitt path algebra of a weighted graph, following [16].

An *isolated vertex* of a graph E is a vertex v such that $s^{-1}(v) = r^{-1}(v) = \emptyset$. To avoid trivialities, we restrict attention to graphs with no isolated vertices.

DEFINITION 2.9. A *weighted graph* is a pair (E, ω) , where E is a row-finite graph with no isolated vertices, and $\omega : E^1 \rightarrow \mathbb{N}$ is a map. For each $v \in E_{\text{reg}}^0$, set $\omega(v) := \max\{\omega(e) : e \in s^{-1}(v)\}$ and set $\omega(v) = 0$ if v is a sink. We say that (E, ω) is a *vertex weighted graph* if $w(e) = w(v)$ for all $e \in s^{-1}(v)$.

DEFINITION 2.10. Let (E, ω) be a weighted graph and K a field. Then we define the *weighted Leavitt path algebra* of (E, ω) to be the $*$ -algebra $L(E, \omega)$ with generating set $\{v, e_i, | v \in E^0, e \in E^1, 1 \leq i \leq \omega(e)\}$ subject to the relations:

- (1) $uv = \delta_{u,v}$ and $v = v^*$ for all $u, v \in E^0$;
- (2) $s(e)e_i = e_i = e_i r(e)$, where $e \in E^1, 1 \leq i \leq \omega(e)$;
- (3) $\sum_{e \in s^{-1}(v)} e_i e_j^* = \delta_{ij} v$, where $v \in E_{\text{reg}}^0, 1 \leq i, j \leq \omega(v)$;
- (4) $\sum_{1 \leq i \leq \omega(v)} e_i^* f_i = \delta_{e,f} r(e)$, where $v \in E_{\text{reg}}^0$ and $e, f \in s^{-1}(v)$.

In relations (3) and (4), we set e_i and e_i^* zero whenever $i > \omega(e)$.

Now we consider a vertex weighted graph (E, ω) and we build a bipartite separated graph $(E(\omega), C(\omega))$ such that $L(E, \omega) \cong LW(E(\omega), C(\omega))$.

DEFINITION 2.11. Let (E, ω) be a vertex weighted graph. Define a bipartite separated graph $(E(\omega), C(\omega))$ as follows. Let V_0 and V_1 be two copies of E^0 , with bijections $E^0 \rightarrow V_i$ given by $v \mapsto v_i$, for $i = 0, 1$. Set $E(\omega)^{0,0} = \{v_0 \mid v \in E_{\text{reg}}^0\}$, $E(\omega)^{0,1} = V_1$ and $E(\omega)^1 = E(\omega)^{0,0} \sqcup E(\omega)^{0,1}$. For each $e \in E^1$, we define an edge $\tilde{e} \in E(\omega)^1$ such that $s(\tilde{e}) = s(e)_0$ and $r(\tilde{e}) = r(e)_1$. We set $X_v = \{\tilde{e} \mid e \in s^{-1}(v)\}$ for $v \in E_{\text{reg}}^0$. In addition, we define another set of edges $Y_v = \{h(v, i) : 1 \leq i \leq \omega(v)\}$ for each $v \in E_{\text{reg}}^0$, where $s(h(v, i)) = v_0$ and $r(h(v, i)) = v_1$ for all $v \in E_{\text{reg}}^0$ and all $1 \leq i \leq \omega(v)$. Finally, we define $C(\omega)_{v_0} = \{X_v, Y_v\}$ for each $v \in E_{\text{reg}}^0$, and $E(\omega)^1 = \bigsqcup_{v \in E_{\text{reg}}^0} (X_v \sqcup Y_v)$.

We call $(E(\omega), C(\omega))$ the *separated graph* of the vertex weighted graph (E, ω) . Observe that, since E has no isolated vertices, we have $s(E^1) = E^{0,0}$ and $r(E^1) = E^{0,1}$.

THEOREM 2.12. *Let (E, ω) be a vertex weighted row-finite graph. Then there is a unique isomorphism of $*$ -algebras*

$$\Phi = \Phi_{(E, \omega)} : L(E, \omega) \longrightarrow LW(E(\omega), C(\omega))$$

such that $\Phi(v) = p_v = v_1$ for $v \in E^0$, $\Phi(e_i) = \rho(h(s(e), i), \tilde{e}) = h(s(e), i)^* \tilde{e}$ for $e \in E^1$ and $1 \leq i \leq \omega(s(e))$.

PROOF. Write $(F, D) = (E(\omega), C(\omega))$, $A = L(E, \omega)$ and $B = LW(E(\omega), C(\omega))$. To show that Φ gives a well-defined $*$ -algebra homomorphism, we need to check that the defining relations of $A = L(E, \omega)$ are satisfied in B . It is quite easy to show that relations (1) and (2) in Definition 2.10 are preserved by Φ . To show that (3) is also preserved, take $v \in E_{\text{reg}}^0$ and $1 \leq i, j \leq \omega(v)$. We then have

$$\begin{aligned} \sum_{e \in s^{-1}(v)} \Phi(e_i) \Phi(e_j)^* &= \sum_{e \in s^{-1}(v)} h(v, i)^* \tilde{e} \tilde{e}^* h(v, j) \\ &= h(v, i)^* \left(\sum_{e \in s^{-1}(v)} \tilde{e} \tilde{e}^* \right) h(v, j) \\ &= h(v, i)^* h(v, j) \\ &= \delta_{i,j} v_1 = \delta_{i,j} \Phi(v). \end{aligned}$$

For (4), let $e, f \in s^{-1}(v)$, with $v \in E_{\text{reg}}^0$. Then we have

$$\begin{aligned} \sum_{1 \leq i \leq \omega(v)} \Phi(e_i)^* \Phi(f_i) &= \sum_{1 \leq i \leq \omega(v)} \tilde{e}^* h(v, i) h(v, i)^* \tilde{f} \\ &= \tilde{e}^* \left(\sum_{1 \leq i \leq \omega(v)} h(v, i) h(v, i)^* \right) \tilde{f} \\ &= \tilde{e}^* \tilde{f} \\ &= \delta_{e,f} r(e)_1 = \delta_{e,f} \Phi(r(e)). \end{aligned}$$

Hence, we have a well-defined $*$ -homomorphism $\Phi : A \rightarrow B$. To build the inverse of Φ , consider the map $\Psi : B \rightarrow A$ defined by

$$\Psi(p_v) = v \quad (v \in E^0), \quad \Psi(\rho(h(v, i), \tilde{e})) = e_i \quad (e \in s^{-1}(v)).$$

Here we observe that for each $v \in E_{\text{reg}}^0$, we have $|D_{v_0}| = 2$. Therefore, it follows from Definition 2.7 that we only have generators of the form $\rho(h(v, i), \tilde{e})$ and $\rho(\tilde{e}, h(v, i)) = \rho(h(v, i), \tilde{e})^*$ for $e \in s^{-1}(v)$. Therefore, to define Ψ as a $*$ -homomorphism, it is enough to define it on the given generators. We need to show the preservation of the defining relations of $LW(F, D)$. We only deal with (SCK1''). Let $v \in E_{\text{reg}}^0$. There are two cases. Suppose first that $e = h(v, i)$ and $h = h(v, j)$ for $1 \leq i, j \leq \omega(v)$. Then the unique option for X in (SCK1'') is $X = X_v$, and so

$$\begin{aligned} \sum_{\tilde{f} \in X_v} \Psi(\rho(h(v, i), \tilde{f})) \cdot \Psi(\rho(\tilde{f}, h(v, j))) &= \sum_{\tilde{f} \in X_v} \Psi(\rho(h(v, i), \tilde{f})) \cdot \Psi(\rho(h(v, j), \tilde{f}))^* \\ &= \sum_{f \in s^{-1}(v)} f_i f_j^* = \delta_{i,j} v = \Psi(\delta_{i,j} p_{r(h(v,i))}). \end{aligned}$$

We now consider the second case. In this case, we have two edges $\tilde{e}, \tilde{h} \in X_v$, that is, $e, f \in E^1$, with $s(e) = s(h) = v \in E^0$, and the unique possible value for X in (SCK1'') is $X = Y_v$. We have

$$\begin{aligned} \sum_{1 \leq i \leq \omega(v)} \Psi(\rho(\tilde{e}, h(v, i))) \cdot \Psi(\rho(h(v, i), \tilde{h})) &= \sum_{1 \leq i \leq \omega(v)} \Psi(\rho(h(v, i), \tilde{e}))^* \cdot \Psi(\rho(h(v, i), \tilde{h})) \\ &= \sum_{1 \leq i \leq \omega(v)} e_i^* h_i = \delta_{e,h} r(e) = \Psi(\delta_{e,h} p_{r(\tilde{e})}). \end{aligned}$$

Hence, Ψ is a well-defined $*$ -homomorphism. It is clear that Ψ and Φ are mutually inverse. This concludes the proof. \square

3. The algebra $L_1(E, \omega)$

In this section, we introduce a new $*$ -algebra $L_1(E, \omega)$ associated to a row-finite weighted graph. This algebra is a certain quotient of $L(E, \omega)$ and has the property of being generated by partial isometries. This is not the case, in general, for the $*$ -algebra $L(E, \omega)$. We show that if the graph E is locally finite, then $L_1(E, \omega)$ is $*$ -isomorphic to the upper Leavitt path algebra of a finitely separated graph.

DEFINITION 3.1. Let (E, ω) be a weighted graph and K a field. The $*$ -algebra $L_1(E, \omega)$ is the free $*$ -algebra generated by $\{v, e_i \mid v \in E^0, e \in E^1, 1 \leq i \leq \omega(e)\}$ subject to the relations:

- (1) $uv = \delta_{u,v}$ and $v = v^*$ for all $u, v \in E^0$;
- (2) $s(e)e_i = e_i = e_i r(e)$, where $e \in E^1, 1 \leq i \leq \omega(e)$;
- (3) $e_i e_j^* = 0$ for all $e \in E^1$ and all $1 \leq i, j \leq \omega(e)$ with $i \neq j$;
- (4) $e_i^* f_i = 0$ for all $e, f \in s^{-1}(v)$ with $e \neq f, v \in E^0, 1 \leq i \leq \min\{\omega(e), \omega(f)\}$;
- (5) $\sum_{e \in s^{-1}(v), \omega(e) \geq i} e_i e_i^* = v$, where $v \in E_{\text{reg}}^0$, and $1 \leq i \leq \omega(v)$;
- (6) $\sum_{1 \leq i \leq \omega(e)} e_i^* e_i = r(e)$ for all $e \in E^1$.

REMARK 3.2

- (1) Observe that relations (3) with $i \neq j$ and (4) with $e \neq f$ in Definition 2.10 are automatically satisfied in $L_1(E, \omega)$ because of relations (3) and (4) above. Therefore, we have

$$L_1(E, \omega) \cong L(E, \omega) / I_0,$$

where I_0 is the $*$ -ideal of $L(E, \omega)$ generated by the elements $e_i^* e_j$ for $e \in E^1$ and $1 \leq i \neq j \leq \omega(e)$, and $e_i^* f_i$ for $e, f \in s^{-1}(v)$ with $e \neq f, v \in E^0, 1 \leq i \leq \min\{\omega(e), \omega(f)\}$.

- (2) Notice that the elements e_i , for $e \in E^1$ and $1 \leq i \leq \omega(e)$, are partial isometries in $L_1(E, \omega)$, that is, $e_i e_i^* e_i = e_i$. This follows by either right multiplying relation (5), or left multiplying relation (6), by e_i , and using relations (4) or (3) accordingly.

Let (E, ω) be a weighted graph and let (E, ω^M) be the unique vertex weighted graph such that $\omega^M(v) = \omega(v)$ for all $v \in E_{\text{reg}}^0$. Then $L(E, \omega)$ is the quotient $*$ -algebra of $L(E, \omega^M)$ by the $*$ -ideal generated by e_j , for $\omega(e) < j \leq \omega(v)$, for each $v \in E^0$ and $e \in s^{-1}(v)$. The corresponding quotient $*$ -algebra of $LW(E(\omega^M), C(\omega^M))$ is an object that cannot be exactly modelled with a separated graph.

However, through the consideration of a related graph, we show that the $*$ -algebra $L_1(E, \omega) = L(E, \omega)/I_0$ is an upper Leavitt path algebra of a bipartite separated graph.

We need a definition from [4] (see also [7]).

DEFINITION 3.3. Let (E, C) be any locally finite bipartite separated graph, and write

$$C_u = \{X_1^u, \dots, X_{k_u}^u\}$$

for all $u \in E^{0,0}$. Then the 1-step resolution of (E, C) is the locally finite bipartite separated graph denoted by (E_1, C^1) , and defined by:

- $E_1^{0,0} := E^{0,1}$ and $E_1^{0,1} := \{v(x_1, \dots, x_{k_u}) \mid u \in E^{0,0}, x_j \in X_j^u\}$;
- $E_1^1 := \{\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u}) \mid u \in E^{0,0}, i = 1, \dots, k_u, x_j \in X_j^u\}$;
- $s(\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u})) := r(x_i)$ and $r(\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u})) := v(x_1, \dots, x_{k_u})$;
- for $v \in E_1^{0,0} = E^{0,1}$, $C_v^1 := \{X(x) \mid x \in r^{-1}(v)\}$, where

$$X(x_i) := \{\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u}) \mid x_j \in X_j^u \text{ for } j \neq i\}.$$

A sequence of locally finite bipartite separated graphs $\{(E_n, C^n)\}_{n \geq 0}$ with $(E_0, C^0) := (E, C)$ is then defined inductively by letting (E_{n+1}, C^{n+1}) denote the 1-step resolution of (E_n, C^n) . Finally, set $(F_n, D^n) = \bigcup_{i=0}^n (E_i, C^i)$ and let (F_∞, D^∞) be the infinite layer graph

$$(F_\infty, D^\infty) := \bigcup_{n=0}^\infty (F_n, D^n) = \bigcup_{n=0}^\infty (E_n, C^n).$$

It is clear by construction that (F_∞, D^∞) is a separated Bratteli diagram in the sense of [7, Definition 2.8], called the separated Bratteli diagram of the locally finite bipartite graph (E, C) . (Note that only the case of a finite bipartite separated graph (E, C) was considered in [4, 7]. However the extension to locally finite bipartite separated graphs is straightforward.)

By [4, Theorem 5.1], there is a canonical surjective $*$ -homomorphism

$$\phi_0 : L(E, C) \twoheadrightarrow L(E_1, C^1),$$

which is defined by

$$\phi_0(u) = \sum_{(x_1, \dots, x_{k_u}) \in \prod_{i=1}^{k_u} X_i^u} v(x_1, \dots, x_{k_u}),$$

where $C_u = \{X_1^u, \dots, X_{k_u}^u\}$ for $u \in E^{0,0}$, $\phi_0(w) = w$ for all $w \in E^{0,1}$ and

$$\phi_0(x_i) = \sum_{x_j \in X_j^u, j \neq i} (\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u}))^*$$

for an arrow $x_i \in X_i^u$.

To state our next proposition, we need to extend the definition of the generators of $LW(E, C)$ for a bipartite separated graph (E, C) to the case where e, f belong to the same set of the partition C . Concretely, we set for $e, f \in X \in C$:

$$\rho(e, f) = \delta_{e,f} r(e), \quad (e, f \in X, X \in C). \tag{3-1}$$

PROPOSITION 3.4. *The kernel of the *-homomorphism $\phi_0 : LW(E, C) \rightarrow LV(E_1, C^1)$ is the *-ideal I of $LW(E, C)$ generated by the elements*

$$\rho(e, f)\rho(f, g)\rho(g, h) - \rho(e, g)\rho(g, f)\rho(f, h)$$

for all $e, f, g, h \in E^1$ such that $s(e) = s(f) = s(g) = s(h)$. Here, $\rho(e, f)$ are the canonical generators of $LW(E, C)$ as given in Definition 2.7 when $X_e \neq X_f$, and the elements given by (3-1) when $X_e = X_f$.

PROOF. Let I be the *-ideal of $LW(E, C)$ generated by all the elements of the form $\rho(e, f)\rho(f, g)\rho(g, h) - \rho(e, g)\rho(g, f)\rho(f, h)$, where $s(e) = s(f) = s(g) = s(h)$.

By [4, Theorem 5.1], the kernel of the map ϕ_0 is the *-ideal J of $L(E, C)$ generated by all the commutators $[ee^*, ff^*]$, for $e, f \in E^1$. Observe that

$$\rho(e, f)\rho(f, g)\rho(g, h) - \rho(e, g)\rho(g, f)\rho(f, h) = e^*[ff^*, gg^*]h \in J,$$

so we get $I \subseteq WJW = \text{Ker}(\phi_0)$. For the other inclusion, observe that the family $e^*[ff^*, gg^*]h$, where $s(e) = s(f) = s(g) = s(h)$, is a family of generators for the ideal WJW . Indeed, we have $[ff^*, gg^*] = 0$ except that $s(f) = s(g)$, so J is generated as an ideal by the set of elements $[ff^*, gg^*]$, where $s(f) = s(g)$. Hence, every element of WJW is a linear combination of terms of the form

$$\gamma[ff^*, gg^*]\lambda,$$

where $v := s(f) = s(g)$, γ, λ are paths in \widehat{E} , with $r(\gamma) = v = s(\lambda)$ and $s(\gamma), r(\lambda) \in E^{0,1}$, so $\gamma = \gamma'e^*, \lambda = h\lambda'$, with $e, h \in E^1$, $s(e) = s(h) = v$, and $\gamma', \lambda' \in LW(E, C)$, so that

$$\gamma[ff^*, gg^*]\lambda = \gamma'(e^*[ff^*, gg^*]h)\lambda' \in I.$$

This concludes the proof. □

We deduce from the above that $LW(E, C)/I \cong LV(E_1, C^1)$ through the *-homomorphism induced by ϕ_0 , where I is the ideal generated by elements

$\rho(e, f)\rho(f, g)\rho(g, h) - \rho(e, g)\rho(g, f)\rho(f, h)$, with $s(e) = s(f) = s(g) = s(h)$. With the help of Theorem 2.12, we can thus identify a suitable quotient of $L(E, \omega)$ for a vertex weighted graph (E, ω) , that is an upper Leavitt path algebra of a separated graph. We show next that this algebra is precisely the algebra $L_1(E, \omega)$ from Definition 3.1.

Recall from Definition 3.1 that $L_1(E, \omega) = L(E, \omega)/I_0$, where I_0 is the $*$ -ideal of $L(E, \omega)$ generated by the elements:

$$e_i e_j^* \quad \text{for } e \in E^1 \quad \text{and} \quad 1 \leq i, j \leq \omega(e) \text{ with } i \neq j, \tag{3-2}$$

$$e_i^* f_i \quad \text{for } e, f \in s^{-1}(v) \text{ with } e \neq f, v \in E^0 \quad \text{and} \quad 1 \leq i \leq \min\{\omega(e), \omega(f)\}. \tag{3-3}$$

THEOREM 3.5. *Let (E, ω) be a locally finite vertex weighted graph and let $L_1(E, \omega) = L(E, \omega)/I_0$ be the $*$ -algebra from Definition 3.1. Then we have a canonical $*$ -isomorphism*

$$\Phi_1 : L_1(E, \omega) \longrightarrow LV(E(\omega)_1, C(\omega)^1)$$

such that $\Phi_1(v) = v_1$ and $\Phi_1(e_i) = \tau(\alpha^{h(v,i)}(\tilde{e}), \alpha^{\tilde{e}}(h(v, i)))$ for $e \in E^1$, $1 \leq i \leq \omega(e)$ and $v = s(e)$. Hence, $L_1(E, \omega)$ is $*$ -isomorphic to the upper Leavitt path algebra of a separated graph.

PROOF. By Theorem 2.12, we have a $*$ -isomorphism

$$\Phi : L(E, \omega) \rightarrow LW(E(\omega), C(\omega)).$$

Composing with the surjective $*$ -homomorphism $\phi_0| : LW(E(\omega), C(\omega)) \rightarrow LV(E(\omega)_1, C(\omega)^1)$, we obtain a surjective $*$ -homomorphism

$$\tilde{\Phi}_1 : L(E, \omega) \rightarrow LV(E(\omega)_1, C(\omega)^1).$$

By Proposition 3.4, we only have to check that $\Phi(I_0) = I$, where I is the $*$ -ideal of $LW(E(\omega), C(\omega))$ generated by all the elements

$$\gamma(a, b, c, d) := \rho(a, b)\rho(b, c)\rho(c, d) - \rho(a, c)\rho(c, b)\rho(b, d),$$

for $a, b, c, d \in s^{-1}(v_0)$, $v \in E^0$. Given such an element, there are various possibilities to consider depending on which elements a, b, c, d belong to X_v or to Y_v . Observe that $\gamma(a, b, c, d) = 0$ if b and c belong to the same set, so by symmetry, we only need to consider the case where $b \in X_v$ and $c \in Y_v$. Assuming this, write $b = \tilde{f}$ and $c = h(v, i)$ for $f \in s^{-1}(v)$ and $1 \leq i \leq \omega(v)$. We have four cases to consider.

(1) $a = \tilde{e} \in X_v$ and $d = \tilde{g} \in X_v$. Then we have

$$\begin{aligned} \Phi^{-1}(\gamma(a, b, c, d)) &= \Phi^{-1}(\tilde{e}^* \tilde{f} \tilde{f}^* h(v, i) h(v, i)^* \tilde{g} - \tilde{e}^* h(v, i) h(v, i)^* \tilde{f} \tilde{f}^* \tilde{g}) \\ &= \delta_{e,f} f_i^* g_i - \delta_{f,g} e_i^* f_i. \end{aligned}$$

This element is nonzero only if $e = f$ and $f \neq g$, or $e \neq f$ and $f = g$. In both of these cases, we get an element of the form (3-3).

- (2) Similarly, if $a, d \in Y_v$, then we get that $\Phi^{-1}(\gamma(a, b, c, d))$ is an element of the form (3-2).
- (3) $a = h(v, j) \in Y_v$ and $d = \tilde{e} \in X_v$. In this case, we have

$$\Phi^{-1}(\gamma(a, b, c, d)) = f_j f_i^* e_i - \delta_{i,j} \delta_{e,f} f_i.$$

For $i = j$ and $e = f$, this gives the element $e_i e_i^* e_i - e_i$, which belongs to I_0 by Remark 3.2(2). When $i \neq j$ or $e \neq f$, this gives an element that also belongs to I_0 .

- (4) If $a \in X_v$ and $d \in Y_v$, then $\gamma(a, b, c, d) = -\gamma(d, b, c, a)^*$, and we reduce to case (3).

We thus obtain that $\Phi^{-1}(I) \subseteq I_0$, that is, $I \subseteq \Phi(I_0)$. The reverse inclusion $\Phi(I) \subseteq I_0$ follows easily from the above computations.

Hence, we obtain a *-isomorphism $\Phi_1 : L_1(E, \omega) \rightarrow LV(E(\omega)_1, C(\omega)^1)$, with $\Phi_1(v) = v_1$ and

$$\begin{aligned} \Phi_1(e_i) &= \phi_0(h(v, i))^* \phi_0(\tilde{e}) \\ &= \left(\sum_{f \in s^{-1}(v)} \alpha^{h(v,i)}(\tilde{f}) \right) \left(\sum_{j=1}^{\omega(v)} \alpha^{\tilde{e}}(h(v, j))^* \right) \\ &= \alpha^{h(v,i)}(\tilde{e}) \alpha^{\tilde{e}}(h(v, i))^*, \end{aligned}$$

where the last equality follows from $r(\alpha^{h(v,i)}(\tilde{f})) = v(\tilde{f}, i)$ and $r(\alpha^{\tilde{e}}(h(v, j))) = v(\tilde{e}, j)$, which imply that the only nonzero summand is the one corresponding to $f = e$ and $j = i$.

This concludes the proof of the theorem. □

It is time now to show that Theorem 3.5 generalizes to arbitrary weighted graphs. The point is that in $L(E, \omega^M)/I_0$, we can kill the projections $e_j e_j^*$ for $i + 1 \leq j \leq \omega(v)$, and this is equivalent to killing the projections $v(\tilde{e}, h(v, j))$ in $L(E(\omega^M)_1, C(\omega^M)^1)$.

Before proceeding to the statement of the next theorem, we introduce a useful simplified notation for the generators of $L(E(\omega)_1, C(\omega)^1)$. This notation is used in the rest of the paper, whenever there is no danger of confusion.

NOTATION 3.6. Let (E, ω) be a locally finite vertex weighted graph, and let $R = L(E(\omega)_1, C(\omega)^1)$ be the Leavitt path algebra of the separated graph $(E(\omega)_1, C(\omega)^1)$. The elements of $E_1^{0,0}$ are simply denoted by v , where $v \in E^0$. (These elements are denoted v_1 in the previous, general notation.) The elements of $E_1^{0,1}$ are denoted by $v(e, i)$, where $e \in E^1$ and $1 \leq i \leq \omega(e)$. (These elements are denoted $v(\tilde{e}, h(v, i))$ in the previous, general notation.) The elements in $X(e)$ for $e \in E^1$ are denoted by $\alpha^e(i)$, where $e \in E^1$ and $1 \leq i \leq \omega(e)$. (These elements are denoted $\alpha^{\tilde{e}}(h(v, i))$ in the previous, general notation.) The elements in $X(v, i) := X(h(v, i))$ are denoted by $\alpha^i(e)$, where $e \in E^1$ and $1 \leq i \leq \omega(e)$. (These elements are denoted $\alpha^{h(v,i)}(\tilde{e})$ in the previous, general notation.)

With the new notation, we have, by Theorem 3.5,

$$\Phi_1(e_i) = \alpha^i(e) \alpha^e(i)^* = \tau(\alpha^i(e), \alpha^e(i))$$

for $e \in E^1$ and $1 \leq i \leq \omega(v)$.

We can now extend the definition of the separated graph $(E(\omega)_1, C(\omega)^1)$ to any locally finite weighted graph.

DEFINITION 3.7. Let (E, ω) be a locally finite weighted graph. We define $(E(\omega)_1, C(\omega)^1)$ as the bipartite locally finite separated graph with

$$E(\omega)_1^{0,0} = E^0, \quad E(\omega)_1^{0,1} = \{v(e, i) : e \in E^1, 1 \leq i \leq \omega(e)\},$$

and for each $v \in E^0$:

$$C(\omega)_v^1 = \{X(v, 1), \dots, X(v, \omega(v))\} \bigcup \{X(e) \mid e \in E^1, r(e) = v\},$$

where, for $1 \leq i \leq \omega(v)$,

$$X(v, i) = \{\alpha^i(e) : e \in s^{-1}(v), \omega(e) \geq i\}, \quad s(\alpha^i(e)) = s(e) = v, \quad r(\alpha^i(e)) = v(e, i),$$

and, for $e \in E^1$ with $r(e) = v$,

$$X(e) = \{\alpha^e(i) : 1 \leq i \leq \omega(e)\}, \quad s(\alpha^e(i)) = r(e) = v, \quad r(\alpha^e(i)) = v(e, i).$$

We now recall the definition of the quotient graph $(E/H, C/H)$, see [6, Construction 6.8]. We use the pre-order \leq on E^0 given by $v \leq w$ if and only if there is a path γ in E such that $s(\gamma) = w$ and $r(\gamma) = v$.

DEFINITION 3.8. Let (E, C) be a finitely separated graph. A subset H of E^0 is *hereditary* if $v \leq w$ and $w \in H$ imply that $v \in H$, and H is *C-saturated* if whenever we have $v \in E^0$ such that $r(x) \in H$ for all $x \in X$, for some $X \in C_v$, then necessarily $v \in H$. Given a hereditary C-saturated subset H of E^0 , the quotient separated graph $(E/H, C/H)$ has $(E/H)^0 = E^0 \setminus H$, and $(C/H)_v = \{X/C \mid X \in C_v\}$, where for $v \in E^0 \setminus H$ and $X \in C_v$, $X/C := \{x \in X \mid r(x) \notin H\} \neq \emptyset$. We denote by $\mathcal{H}(E, C)$ the lattice of hereditary C-saturated subsets of E^0 .

THEOREM 3.9. Let (E, ω) be a locally finite weighted graph, let $L_1(E, \omega)$ be the L_1 -algebra of (E, ω) (Definition 3.1) and let $(E(\omega)_1, C(\omega)^1)$ be the bipartite separated graph from Definition 3.7. Then there exists a canonical *-isomorphism

$$\Phi_1 : L_1(E, \omega) \xrightarrow{\cong} LV(E(\omega)_1, C(\omega)^1)$$

such that $\Phi_1(v) = v$ for $v \in E^0$ and $\Phi_1(e_i) = \tau(\alpha^i(e), \alpha^e(i))$ for $e \in E^1$ and $1 \leq i \leq \omega(e)$. In particular, $L_1(E, \omega)$ is isomorphic to the upper Leavitt path algebra of a separated graph.

PROOF. By Theorem 3.5, we have a *-isomorphism

$$\Phi_1 : L_1(E, \omega^M) \longrightarrow LV(E(\omega^M)_1, C(\omega^M)^1),$$

where (E, ω^M) is the unique vertex weighted graph such that $\omega^M(v) = \omega(v)$ for all $v \in E^0$.

We consider the following subset of $E(\omega^M)_1^{0,1}$:

$$H = \{v(e, j) \mid e \in E^1, \omega(e) + 1 \leq j \leq \omega(s(e))\}.$$

Then $H \subset E(\omega^M_1)^{0,1}$ is a hereditary subset of $E(\omega^M_1)^0$, because the vertices in $E(\omega^M_1)^{0,1}$ are sinks in $E(\omega^M_1)$. Let us check that H is also $C(\omega^M_1)$ -saturated. The sets in $C(\omega^M_1)$ are of one of the forms $X(v, i)$ for $1 \leq i \leq \omega(v)$, or $X(e)$ for $e \in E^1$. It is enough to show that $r(X) \not\subseteq H$ for X of these forms. We consider first the case $X = X(v, i)$ for $1 \leq i \leq \omega(v)$. The elements of $X(v, i)$ are of the form $\alpha^i(e)$, where $e \in s^{-1}(v)$. Since $\omega(v) = \max\{\omega(e) \mid e \in s^{-1}(v)\}$, there is some $e \in s^{-1}(v)$ such that $\omega(e) = \omega(v)$, and then

$$r(\alpha^i(e)) = v(e, i) \notin H.$$

Now consider $e \in E^1$ and set $v = s(e)$. Since $\omega(e) \geq 1$, we have that $r(\alpha^e(1)) = v(e, 1) \notin H$.

Hence, we obtain that H is a hereditary $C(\omega^M_1)$ -saturated subset of $E(\omega^M_1)^0$. The ideal $I(H)$ of $L(E(\omega^M_1), C(\omega^M_1))$ generated by H satisfies that

$$L(E(\omega^M_1), C(\omega^M_1))/I(H) \cong L(E(\omega^M_1)/H, C(\omega^M_1)/H)$$

(see the proof of [7, Theorem 5.5]).

We define $(E(\omega)_1, C(\omega)^1) := (E(\omega^M_1)/H, C(\omega^M_1)/H)$.

Observe that the $*$ -isomorphism $L(E(\omega^M_1), C(\omega^M_1))/I(H) \rightarrow L(E(\omega)_1, C(\omega)^1)$ induces a $*$ -isomorphism

$$\pi : LV(E(\omega^M_1), C(\omega^M_1))/IV(H) \longrightarrow LV(E(\omega)_1, C(\omega)^1),$$

where $IV(H) = I(H) \cap LV(E(\omega^M_1), C(\omega^M_1))$. Next, we let \mathfrak{I} be the $*$ -ideal of $LV(E(\omega^M_1), C(\omega^M_1))$ generated by the elements $\tau(\alpha^j(e), \alpha^e(j))$, for which $\omega(e) + 1 \leq j \leq \omega(s(e))$. We claim that $\mathfrak{I} = IV(H)$. Clearly, $\mathfrak{I} \subseteq IV(H)$. To show equality, observe first that \mathfrak{I} contains the elements

$$\tau(\alpha^j(e), \alpha^j(e)) = \tau(\alpha^j(e), \alpha^e(j))\tau(\alpha^j(e), \alpha^e(j))^*$$

and

$$\tau(\alpha^e(j), \alpha^e(j)) = \tau(\alpha^j(e), \alpha^e(j))^*\tau(\alpha^j(e), \alpha^e(j))$$

for $\omega(v) + 1 \leq j \leq \omega(s(e))$. We define a $*$ -homomorphism $\varphi : LV(E(\omega)_1, C(\omega)^1) \rightarrow LV(E(\omega^M_1), C(\omega^M_1))/\mathfrak{I}$ by $\varphi(p_v) = \bar{p}_v$ for $v \in E^0$, and $\varphi(\tau(x, y)) = \tau(x, y)$, for $x, y \in (E(\omega)_1)^1$ with $r(x) = r(y)$. Here we indicate by \bar{z} the class of an element z in the quotient $LV(E(\omega^M_1), C(\omega^M_1))/\mathfrak{I}$. To see that φ is well defined, we have to check that the relations in Definition 2.6 are preserved by φ . This is obvious for all relations except for (SCK2'). Let us check relation (SCK2') for $Y = X(e)$, where $e \in E^1$. Since we are working in the graph $(E(\omega)_1, C(\omega)^1)$, this relation reads

$$p_v = \sum_{1 \leq i \leq \omega(e)} \tau(\alpha^e(i), \alpha^e(i)).$$

Now, using that $\overline{\tau(\alpha^e(j), \alpha^e(j))} = 0$ for $\omega(e) + 1 \leq j \leq \omega(s(e))$, we have

$$\varphi\left(\sum_{i=1}^{\omega(e)} \tau(\alpha^e(i), \alpha^e(i))\right) = \sum_{i=1}^{\omega(s(e))} \overline{\tau(\alpha^e(i), \alpha^e(i))} = \overline{p_v} = \varphi(p_v).$$

Similarly, we can check that (SCK2') is preserved for $Y = X(v, i)$ for $1 \leq i \leq \omega(v)$. Let

$$\zeta : LV(E(\omega^M)_1, C(\omega^M)^1)/\mathfrak{S} \longrightarrow LV(E(\omega^M)_1, C(\omega^M)^1)/IV(H)$$

be the canonical quotient map. We clearly have the equality $\varphi \circ \pi \circ \zeta = \text{Id}_{LV(E(\omega^M)_1, C(\omega^M)^1)/\mathfrak{S}}$, and hence ζ is injective, which implies that $\mathfrak{S} = IV(H)$, as desired.

Now the generators $\tau(\alpha^j(e), \alpha^e(j))$, for $\omega(e) + 1 \leq j \leq \omega(s(e))$, of the $*$ -ideal $\mathfrak{S} = IV(H)$ correspond through the $*$ -isomorphism Φ_1^{-1} to the elements e_j in $L_1(E, \omega^M)$, and it is clear that $L_1(E, \omega) \cong L_1(E, \omega^M)/K$, where K is the $*$ -ideal of $L_1(E, \omega^M)$ generated by e_j , with $\omega(e) + 1 \leq j \leq \omega(s(e))$. We thus obtain a $*$ -isomorphism, also denoted by Φ_1 , from $L_1(E, \omega)$ onto $LV(E(\omega)_1, C(\omega)^1)$, as desired. \square

4. The algebra $L^{\text{ab}}(E, \omega)$

In this section, we introduce the $*$ -algebra $L^{\text{ab}}(E, \omega)$ for a locally finite weighted graph (E, ω) , and we show it can be written as a full corner of a direct limit of a sequence of Leavitt path algebras of separated graphs.

We begin with some preliminary definitions, see [10, Definition 12.9].

DEFINITION 4.1. A set F of partial isometries of a $*$ -algebra A is said to be *tame* if for any two elements $u, u' \in U$, where U is the multiplicative $*$ -subsemigroup of A generated by F , we have that $e(u)$ and $e(u')$ are commuting elements of A , where $e(v) = vv^*$ for $v \in U$. Note that if F is a tame set of partial isometries, then all elements of U are indeed partial isometries, and therefore the elements $e(u)$, with $u \in U$, are mutually commuting projections in A .

A $*$ -algebra A is said to be *tame* if it is generated as $*$ -algebra by a tame set of partial isometries. If A is a $*$ -algebra generated by a subset F of partial isometries, there is a universal tame $*$ -algebra A^{ab} associated to F . By definition, there is a surjective $*$ -homomorphism $\pi : A \rightarrow A^{\text{ab}}$ so that $\pi(F)$ is a tame set of partial isometries generating A^{ab} , and such that for any $*$ -homomorphism $\psi : A \rightarrow B$ of A to a $*$ -algebra B such that $\psi(F)$ is a tame set of partial isometries in B , there is a unique $*$ -homomorphism $\bar{\psi} : A^{\text{ab}} \rightarrow B$ such that $\psi = \bar{\psi} \circ \pi$. Indeed, we have $A^{\text{ab}} = A/J$, where J is the ideal of A generated by all the commutators $[e(u), e(u')]$, where $u, u' \in U$.

If (E, C) is a separated graph, then the universal tame $*$ -algebra associated to the generating set $E^0 \sqcup E^1$ of partial isometries of $L(E, C)$ is denoted by $L^{\text{ab}}(E, C)$.

Similarly, if (E, C) is a row-finite bipartite separated graph, the universal tame $*$ -algebra associated to the generating set $E^{0,0} \sqcup \{\tau(x, y) : x, y \in E^1, r(x) = r(y)\}$ of partial isometries of $LV(E, C)$ is denoted by $LV^{\text{ab}}(E, C)$.

We can now define the abelianized Leavitt path algebra of a weighted graph. Recall from Remark 3.2(2) that the elements e_i , for $e \in E^1$ and $1 \leq i \leq \omega(e)$ are partial isometries in $L_1(E, \omega)$.

DEFINITION 4.2. Let (E, ω) be a weighted graph and K a field. The *abelianized Leavitt path algebra* of (E, ω) , denoted by $L^{\text{ab}}(E, \omega)$ is the universal tame $*$ -algebra associated to the generating set $E^0 \cup \{e_i : e \in E^1, 1 \leq i \leq \omega(e)\}$ of partial isometries of $L_1(E, \omega)$. Let U be the $*$ -subsemigroup of $L_1(E, \omega)$ generated by $\{e_i : e \in E^1, 1 \leq i \leq \omega(e)\}$, and let J be the ideal of $L_1(E, \omega)$ generated by all the commutators $[e(u), e(u')]$, where $u, u' \in U$. Then $L^{\text{ab}}(E, \omega) = L_1(E, C)/J$. (Note that vertices commute with all elements $e(u)$.)

Recall from Definition 3.3 the canonical sequence $\{(E_n, C^n)\}_{n \geq 0}$ of locally finite bipartite separated graphs associated to a locally finite bipartite separated graph (E, C) . We denote by $\pi_n : L(E, C) \rightarrow L(E_n, C^n)$ and $\pi_\infty : L(E, C) \rightarrow L^{\text{ab}}(E, C)$ the canonical surjective $*$ -homomorphisms. Note that $\pi_{2n}(V) = V_{2n}$, where $V_{2n} = \sum_{v \in E_{2n}^{0,0}} v \in M(L(E_{2n}, C^{2n}))$.

For a locally finite weighted graph (E, ω) , we want to relate $L^{\text{ab}}(E, \omega)$ with the algebra $L^{\text{ab}}(E(\omega)_1, C(\omega)^1)$, where $(E(\omega)_1, C(\omega)^1)$ is the bipartite separated graph introduced in Section 3. For this, we need the following general lemma.

LEMMA 4.3. *Let (E, C) be a locally finite bipartite separated graph and set $V = \sum_{v \in E^{0,0}} v \in M(L(E, C))$. Let $\{(E_n, C^n)\}_{n \geq 0}$ be the canonical sequence of bipartite separated graphs associated to (E, C) . Then we have*

$$LV^{\text{ab}}(E, C) \cong \pi_\infty(V)L^{\text{ab}}(E, C)\pi_\infty(V) \cong \varinjlim \pi_{2n}(V)L(E_{2n}, C^{2n})\pi_{2n}(V).$$

PROOF. By [4, Theorem 5.7], we have

$$L^{\text{ab}}(E, C) \cong \varinjlim L(E_{2n}, C^{2n})$$

naturally, and hence

$$\pi_\infty(V)L^{\text{ab}}(E, C)\pi_\infty(V) \cong \varinjlim \pi_{2n}(V)L(E_{2n}, C^{2n})\pi_{2n}(V).$$

By definition, $LV^{\text{ab}}(E, C)$ is the universal tame $*$ -algebra with respect to the set

$$E^{0,0} \sqcup \{\tau(x, y) : x, y \in E^1, r(x) = r(y)\}$$

of partial isometries of $LV(E, C) = VL(E, C)V$ (see Proposition 2.8). Let U be the $*$ -subsemigroup of $LV(E, C)$ generated by $F := \{\tau(x, y) : x, y \in E^1, r(x) = r(y)\}$, and let J be the ideal of $LV(E, C)$ generated by all the commutators $[e(u), e(u')]$, with $u, u' \in U$. Let $\pi : LV(E, C) \rightarrow LV^{\text{ab}}(E, C) = LV(E, C)/J$ be the projection map.

Since $\tau(x, y) = xy^*$ in $L(E, C)$, it is clear that $\pi_\infty(F)$ becomes a tame set of partial isometries in $\pi_\infty(V)L^{\text{ab}}(E, C)\pi_\infty(V)$. Hence, there is a unique surjective $*$ -homomorphism

$$\mu : LV^{\text{ab}}(E, C) \longrightarrow \pi_{\infty}(V)L^{\text{ab}}(E, C)\pi_{\infty}(V)$$

such that $\pi_{\infty}| = \mu \circ \pi$. It remains to show that μ is injective.

Let U' be the $*$ -subsemigroup of $L(E, C)$ generated by E^1 and let J' be the ideal generated by all the commutators $[e(u), e(u')]$ for $u, u' \in U'$. Then the kernel of π_{∞} is the ideal J' and our task is to show that $J' \cap VL(E, C)V = J$. As we observe above, $J \subseteq J' \cap VL(E, C)V = VJ'V$. For the reverse inclusion, note that $VJ'V$ is generated by the following types of elements.

- (a) Elements of the form $[e(u), e(u')]$, where

$$u = x_1x_2^*x_3x_4^* \cdots x_{2n-1}, \quad u' = y_1y_2^*y_3y_4^* \cdots y_{2m-1}$$

with $x_i, y_j \in E^1$ and $s(x_1) = s(y_1)$.

- (b) Elements of the form $z[e(u), e(u')]t^*$, where

$$u = x_1^*x_2x_3^*x_4 \cdots x_{2n-1}^*x_{2n}, \quad u' = y_1^*y_2y_3^*y_4 \cdots y_{2m-1}^*y_{2m}$$

with $z, t, x_i, y_j \in E^1$ and $r(x_1) = r(y_1) = r(z) = r(t)$.

For elements u, u' as in (a), observe that $e(u) = e(ux_{2n-1}^*)$ and $e(u') = e(u'y_{2m-1}^*)$, so that the corresponding element $[e(u), e(u')]$ belongs to J . For elements u, u' as in (b), note that $x_1[e(u), e(u')]x_1^* = [e(u_2), e(u_3)]$, where

$$u_2 = x_1u = (x_1x_1^*)(x_2x_3^*) \cdots (x_{2n-2}x_{2n-1}^*)x_{2n},$$

$$u_3 = x_1u' = (x_1y_1^*)(y_2y_3^*) \cdots (y_{2m-2}y_{2m-1}^*)y_{2m},$$

so that u_2, u_3 are as in (a), so we get that $x_1[e(u), e(u')]x_1^* \in J$. However, now

$$z[e(u), e(u')]t^* = (zx_1^*)(x_1[e(u_2), e(u_3)]x_1^*)(x_1t^*) \in J.$$

We conclude that $VJ'V \subseteq J$, as desired. This completes the proof. □

THEOREM 4.4. *Let (E, ω) be a locally finite weighted graph and let $(E(\omega)_1, C(\omega)^1)$ be the corresponding bipartite separated graph (Definition 3.7). Then there are natural $*$ -isomorphisms*

$$L^{\text{ab}}(E, \omega) \cong LV^{\text{ab}}(E(\omega)_1, C(\omega)^1) \cong \overline{VL^{\text{ab}}(E(\omega)_1, C(\omega)^1)\overline{V}}, \tag{4-1}$$

where $\overline{V} = \pi_{\infty}(V) \in M(L^{\text{ab}}(E(\omega)_1, C(\omega)^1))$.

PROOF. Let $\Phi_1 : L_1(E, \omega) \rightarrow LV(E(\omega)_1, C(\omega)^1)$ be the $*$ -isomorphism from Theorem 3.9. It is clear that Φ_1 sends the canonical set of generators of $L_1(E, \omega)$ to the canonical set of generators of $LV(E(\omega)_1, C(\omega)^1)$. Hence, we get that Φ_1 induces a $*$ -isomorphism $\Phi^{\text{ab}} : L^{\text{ab}}(E, \omega) \rightarrow LV^{\text{ab}}(E(\omega)_1, C(\omega)^1)$. The second isomorphism in (4-1) follows from Lemma 4.3. □

We finally point out that the representation of $L^{\text{ab}}(E, C)$ as a partial crossed product

$$L^{\text{ab}}(E, C) \cong C_K(\Omega(E, C)) \rtimes \mathbb{F}$$

for a finite bipartite separated graph [4, Corollary 6.12(1)] can be easily adapted to obtain a corresponding representation for the abelianized Leavitt path algebra $L^{\text{ab}}(E, \omega)$ of any *finite* weighted graph. Although we think that a suitable version holds for any locally finite weighted graph, we restrict here to the finite case, because only finite bipartite separated graphs are considered in [4].

We refer the reader to [4, 10] for the background definitions on crossed products of partial actions of groups.

We first state the universal property of the dynamical system associated to a weighted graph. This is basically an internal version of the corresponding property for the associated bipartite separated graph $(E(\omega)_1, C(\omega)^1)$.

DEFINITION 4.5. Let (E, ω) be a finite weighted graph. An (E, ω) -dynamical system consists of a compact Hausdorff space Ω , with a family of clopen sets $\{\Omega_v\}_{v \in E^0}$ and, for each $v \in E^0$, a family of clopen subsets $H_{\alpha^i(e)}$ for $e \in s^{-1}(v)$ and $1 \leq i \leq \omega(e)$, and $H_{\alpha^e(i)}$ for each $e \in r^{-1}(v)$ and $1 \leq i \leq \omega(e)$, such that

$$\Omega_v = \bigsqcup_{e \in s^{-1}(v): \omega(e) \geq i} H_{\alpha^i(e)}, \quad (v \in E^0, 1 \leq i \leq \omega(v)),$$

$$\Omega_v = \bigsqcup_{i=1}^{\omega(e)} H_{\alpha^e(i)}, \quad (v \in E^0, e \in r^{-1}(v)),$$

together with a family of homeomorphisms $\theta_{e_i} : H_{\alpha^e(i)} \rightarrow H_{\alpha^i(e)}$ for each $e \in E^1$ and $1 \leq i \leq \omega(e)$.

Given two (E, ω) -dynamical systems (Ω, θ) and (Ω', θ') , an *equivariant map* is a map $f : \Omega \rightarrow \Omega'$ such that $f(\Omega_v) \subseteq \Omega'_v$ for each $v \in E^0$ and $f(H_{\alpha^i(e)}) \subseteq H'_{\alpha^i(e)}$, $f(H_{\alpha^e(i)}) \subseteq H'_{\alpha^e(i)}$ for each $e \in E^1$ and $1 \leq i \leq \omega(e)$, and $\theta'_{e_i}(f(x)) = f(\theta_{e_i}(x))$ for all $x \in H_{\alpha^e(i)}$.

An (E, ω) -dynamical system (Ω, θ) is said to be *universal* if for each other (E, ω) -dynamical system (Ω', θ') , there exists a unique equivariant continuous map $f : \Omega' \rightarrow \Omega$. Of course, if such a universal space exists, it is unique up to a unique equivariant homeomorphism.

THEOREM 4.6. *Let (E, ω) be a finite weighted graph. Then there exists a universal (E, ω) -dynamical system $(\Omega(E, \omega), \theta)$. Moreover, θ can be extended to a partial action of the free group \mathbb{F} on $\{e_i : e \in E^1, 1 \leq i \leq \omega(e)\}$, and we have*

$$L^{\text{ab}}(E, \omega) \cong C_K(\Omega(E, \omega)) \rtimes_{\theta} \mathbb{F}.$$

PROOF. Let $(\Omega(E(\omega)_1, C(\omega)^1), \theta)$ be the universal $(E(\omega)_1, C(\omega)^1)$ -dynamical system from [4, Corollary 6.11]. (See [4, Definition 6.10] for the definition of an (E, C) -universal dynamical system.)

Writing $\Omega := \Omega(E(\omega)_1, C(\omega)^1)$, we have homeomorphisms

$$\theta_{\alpha^i(e)} : \Omega_{v(e,i)} \rightarrow H_{\alpha^i(e)}, \quad \theta_{\alpha^e(i)} : \Omega_{v(e,i)} \rightarrow H_{\alpha^e(i)}$$

for each $e \in E^1$ and $1 \leq i \leq \omega(e)$. It is easy to check that $\theta_{e_i} := \theta_{\alpha^i(e)} \circ \theta_{\alpha^e(i)}^{-1} : H_{\alpha^e(i)} \rightarrow H_{\alpha^i(e)}$ provide the homeomorphisms making $\Omega(E, \omega) : \bigsqcup_{v \in E^0} \Omega_v$ the universal (E, ω) -dynamical system.

The $*$ -isomorphism $L^{\text{ab}}(E, \omega) \cong C_K(\Omega(E, \omega)) \rtimes_{\theta} \mathbb{F}$ follows from the above observation, Theorem 4.4 and [4, Theorem 6.12(1)]. □

5. The \mathcal{V} -monoid and structure of ideals

In this section, we study the \mathcal{V} -monoid and the structure of ideals of the algebras $L_1(E, \omega)$ and $L^{\text{ab}}(E, \omega)$ introduced above. The results follow immediately from known results in [4, 6, 7] and our previous work in the paper. We point out that the \mathcal{V} -monoid of $L(E, \omega)$, where (E, ω) is an arbitrary weighted graph, has been determined by Preusser in [15].

We refer the reader to [1, Definition 3.2.1] for the definition of the \mathcal{V} -monoid $\mathcal{V}(R)$ of a ring R . We use the idempotent picture of $\mathcal{V}(R)$, in which an element of $\mathcal{V}(R)$ is given by the Murray–von Neumann equivalence class $[e]$ of an idempotent matrix e over R .

For a commutative monoid M , we denote by $\mathcal{L}(M)$ its lattice of order-ideals, and for a (nonnecessarily unital) ring R , we denote by $\mathcal{L}(R)$ its lattice of (two-sided) ideals, and by $\text{Tr}(R)$ its lattice of trace ideals. See [6, Section 10] for these notions. It is shown in [6, Theorem 10.10] that for any ring R , there is a lattice isomorphism $\mathcal{L}(\mathcal{V}(R)) \cong \text{Tr}(R)$. (The unital case of this result is due to Facchini and Halter-Koch [11, Theorem 2.1(c)].) We denote by $\text{Idem}(R)$ the lattice of idempotent-generated ideals of a ring R .

Let R be a ring with local units [1, Definition 1.2.10], and let e be an idempotent in the multiplier ring $M(R)$ of R . We say that eRe is a (generalized) *corner ring* of R . The ring eRe is a *full corner* of R if $ReR = R$, that is, for each element $x \in R$, there are elements $r_i, s_i \in R, i = 1, \dots, n$ such that $x = \sum_{i=1}^n r_i e s_i$.

PROPOSITION 5.1. *Let (E, C) be a row-finite bipartite separated graph such that $s(E^1) = E^{0,0}$ and $r(E^1) = E^{0,1}$. Then both $LV(E, C)$ and $LW(E, C)$ are full corners of $L(E, C)$. In particular, the inclusion $LV(E, C) \subseteq L(E, C)$ induces a monoid isomorphism*

$$\mathcal{V}(LV(E, C)) \cong \mathcal{V}(L(E, C)),$$

and the usual restriction/extension process gives lattice isomorphisms

$$\mathcal{L}(LV(E, C)) \cong \mathcal{L}(L(E, C)), \quad \text{Tr}(LV(E, C)) \cong \text{Tr}(L(E, C)).$$

Similar statements hold for $LW(E, C)$ and $L(E, C)$, and also for the corresponding abelianized algebras $LV^{\text{ab}}(E, C)$ and $L^{\text{ab}}(E, C)$.

PROOF. Recall from Proposition 2.8 that $LV(E, C) = VL(E, C)V$ and $LW(E, C) = WL(E, C)W$, where $V = \sum_{v \in E^{0,0}} v \in M(L(E, C))$ and $W = \sum_{w \in E^{0,1}} w \in M(L(E, C))$. The fact that these are full corners follows immediately from the defining relations of $L(E, C)$, since $E^{0,0} = s(E^1)$ and $E^{0,1} = r(E^1)$.

Since all the involved algebras have local units, the result for the lattices of ideals follows from [12, Proposition 3.5]. The proof for the \mathcal{V} -monoids follows from the first paragraph of the proof of [5, Lemma 7.3]. \square

We first define abstractly the monoid $M_1(E, \omega)$ and then we prove below that $M_1(E, \omega)$ is isomorphic to $\mathcal{V}(L_1(E, \omega))$.

DEFINITION 5.2. Let (E, ω) be a locally finite weighted graph. Then $M_1(E, \omega)$ is the commutative monoid with generators $\{a_v : v \in E^0\} \cup \{a_{v(e,i)} : e \in E^1, 1 \leq i \leq \omega(e)\}$ with the defining relations

$$a_v = \sum_{e \in s^{-1}(v), \omega(e) \geq i} a_{v(e,i)} \quad (v \in E_{\text{reg}}^0, 1 \leq i \leq \omega(v)) \tag{5-1}$$

$$a_{r(e)} = \sum_{i=1}^{\omega(e)} a_{v(e,i)} \quad (e \in E^1). \tag{5-2}$$

Observe that relations (5-1) and (5-2) give a refinement of the relation $\omega(v)a_v = \sum_{e \in s^{-1}(v)} a_{r(e)}$ for each $v \in E_{\text{reg}}^0$. Observe also that there is a well-defined monoid homomorphism

$$\gamma_{(E,\omega)} : M_1(E, \omega) \longrightarrow \mathcal{V}(L_1(E, \omega))$$

given by $\gamma(a_v) = [v]$ and $\gamma(a_{v(e,i)}) = [e_i e_i^*] = [e_i^* e_i]$.

THEOREM 5.3. Let (E, ω) be a locally finite weighted graph. Then the natural homomorphism $\gamma_{(E,\omega)} : M_1(E, \omega) \rightarrow \mathcal{V}(L_1(E, \omega))$ is an isomorphism.

PROOF. Note that since $LV(E(\omega)_1, C(\omega)^1)$ is a full corner of $L(E(\omega)_1, C(\omega)^1)$, we have that the inclusion $\iota : LV(E(\omega)_1, C(\omega)^1) \subseteq L(E(\omega)_1, C(\omega)^1)$ induces an isomorphism of the corresponding \mathcal{V} -monoids. It is clear that $M_1(E, \omega) = M(E(\omega)_1, C(\omega)^1)$ (see [6, Definition 4.1] for the definition of the graph monoid $M(E, C)$ of a separated graph).

The composition of the maps

$$M_1(E, \omega) \xrightarrow{\gamma} \mathcal{V}(L(E, \omega)) \xrightarrow{\mathcal{V}(\Phi_1)} \mathcal{V}(LV(E(\omega)_1, C(\omega)^1)) \xrightarrow{\mathcal{V}(\iota)} \mathcal{V}(L(E(\omega)_1, C(\omega)^1))$$

agrees with the canonical map $M(E(w)_1, C(w)^1) \rightarrow \mathcal{V}(L(E(w)_1, C(w)^1))$, which is an isomorphism by [6, Theorem 4.3]. Since both $\mathcal{V}(\Phi_1)$ and $\mathcal{V}(\iota)$ are isomorphisms, we obtain that γ is also an isomorphism. \square

THEOREM 5.4. Let (E, ω) be a locally finite weighted graph. Then $\text{Idem}(L_1(E, \omega)) = \text{Tr}(L_1(E, \omega))$ and we have lattice isomorphisms

$$\text{Idem}(L_1(E, \omega)) \cong \mathcal{L}(M_1(E, \omega)) \cong \mathcal{H}(E(w)_1, C(w)^1).$$

PROOF. Since $\gamma : M_1(E, C) \rightarrow \mathcal{V}(L_1(E, \omega))$ is an isomorphism, the monoid $\mathcal{V}(L_1(E, \omega))$ is generated by equivalence classes of idempotents in $L_1(E, \omega)$ and hence

$\text{Tr}(L_1(E, \omega)) = \text{Idem}(L_1(E, \omega))$ (see the proof of [6, Proposition 6.2]). By Theorem 5.3 and [6, Theorem 10.10],

$$\text{Idem}(L_1(E, \omega)) = \text{Tr}(L_1(E, \omega)) \cong \mathcal{L}(\mathcal{V}(L_1(E, \omega))) \cong \mathcal{L}(M_1(E, \omega)).$$

Since $M_1(E, \omega) = M(E(\omega)_1, C(\omega)^1)$, it follows from [6, Corollary 6.10] that $\mathcal{L}(M_1(E, \omega)) \cong \mathcal{H}(E(\omega)_1, C(\omega)^1)$, completing the proof. \square

A detailed study of the structure of ideals of the algebras $L^{\text{ab}}(E, C)$, for a finite bipartite separated graph (E, C) , has been performed in [7]. The main point is that the lattice of *trace ideals* of $L^{\text{ab}}(E, C)$ is isomorphic to the lattice of hereditary D_∞ -saturated subsets of F_∞^0 , where (F_∞, D^∞) is the separated Bratteli diagram of (E, C) (Definition 3.3). Using these results and the fact that the lattices of ideals are preserved under Morita equivalence of rings with local units, one can translate all these results to the algebras $L^{\text{ab}}(E, \omega)$ for any finite weighted graph (E, ω) . Observe that the ideals of $L^{\text{ab}}(E, \omega)$ can be pulled back to the original algebra $L(E, \omega)$, because $L^{\text{ab}}(E, \omega)$ is a quotient algebra of $L(E, \omega)$.

THEOREM 5.5. *Let (E, ω) be a finite weighted graph. Let $(E(\omega)_1, C(\omega)^1)$ be the bipartite separated graph from Definition 3.7 and let $(E(\omega)_\infty, C(\omega)^\infty)$ be the corresponding separated Bratteli diagram. Then there are lattice isomorphisms*

$$\text{Idem}(L^{\text{ab}}(E, \omega)) \cong \mathcal{L}(\mathcal{V}(L^{\text{ab}}(E, \omega))) \cong \mathcal{H}(E(\omega)_\infty, C(\omega)^\infty).$$

PROOF. Apply Theorem 4.4 and [7, Theorem 4.5]. \square

REMARK 5.6

- (a) Observe that all ideals of $L(E, \omega)$ obtained by pulling back the ideals described in Theorem 5.5 are graded ideals. Hence, we obtain a large family of graded ideals of $L(E, \omega)$, shedding some light on the second Open Problem in [16, Section 12].
- (b) Nongraded ideals of $L^{\text{ab}}(E, C)$ are studied in [7, Section 7]. This gives information on nongraded ideals of $L(E, \omega)$.

5.1. Remarks on the ideal structure of $L(m, n)$. We close the paper with some remarks on the ideal structure of the Leavitt algebra $L(m, n)$. Since the algebra $L(1, n)$ is simple for all $n \geq 2$, we concentrate here on the remaining cases, so we assume throughout this subsection that $1 < m \leq n$. Recall that $L(m, n)$ is the $*$ -algebra with generators x_{ij} , $1 \leq i \leq m$, $1 \leq j \leq n$, subject to the relations given by the matricial equations $XX^* = I_m$, $X^*X = I_n$, where $X = (x_{ij})$ and X^* is the $*$ -transpose of X .

It is an interesting and challenging problem to construct maximal ideals of $L(m, n)$ and study the corresponding simple factor rings. In particular, we do not know any maximal ideal of $L(m, n)$ such that the corresponding simple factor algebra retains the Leavitt type $(m, n - m)$ of $L(m, n)$, although we suspect such maximal ideals exist.

We construct here two maximal ideals of $L(m, n)$, and we relate one of them to the Leavitt path algebra of the minimal weighted graph of shape (m, n) , defined below.

Recall that a *partition* of a positive integer l is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ such that $\lambda_i \geq \lambda_{i+1}$ for $i = 1, \dots, r - 1$ and $l = \lambda_1 + \dots + \lambda_r$. We use the concept of the *shape* of a partition [17, Definition 2.1.1].

DEFINITION 5.7. Let $1 < m < n$ be integers. We say that (E, ω) is an (m, n) -*weighted graph* if it is a weighted graph with one vertex v , n edges and $\omega(v) = m$.

An (m, n) -weighted graph (E, ω) is completely determined, up to permutation of the edges, by its *shape*, which is constructed as follows. Choose an enumeration of the edges $e^{(1)}, \dots, e^{(n)}$ of E such that $m = \omega(e^{(1)}) \geq \omega(e^{(2)}) \geq \dots \geq \omega(e^{(n)}) \geq 1$. Then the *shape* of (E, ω) is the shape of the partition $(\lambda_1, \lambda_2, \dots, \lambda_m)$ of $\lambda_1 + \lambda_2 + \dots + \lambda_m$, where $\lambda_i := |\{e \in E^1 : \omega(e) \geq i\}|$ for $i = 1, \dots, m$. Observe that (up to permutation of edges) any partition $(\lambda_1, \lambda_2, \dots, \lambda_m)$ such that $\lambda_1 = n$ determines a unique (m, n) -weighted graph, setting $\omega(e^{(i)})$ equal to the length of the i th column of the shape of λ . We call such a partition an (m, n) -*partition*.

Say that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \leq \mu = (\mu_1, \mu_2, \dots, \mu_m)$ if $\lambda_i \leq \mu_i$ for all $i = 1, 2, \dots, m$. (Observe that this is *not* the dominance ordering introduced in [17, Definition 2.2.2].) With this order, the set of (m, n) -partitions is a lattice, with maximum element (n, n, \dots, n) (m times) and minimum element $(n, 1, 1, \dots, 1)$ (with $m - 1$ one's).

We may think of the shape of an (m, n) -partition λ as a $\{0, 1\}$ $m \times n$ -matrix. For instance, the shape of the $(3, 4)$ -partition $(4, 2, 2)$ is the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

and we have $\omega(e^{(1)}) = \omega(e^{(2)}) = 3$ and $\omega(e^{(3)}) = \omega(e^{(4)}) = 1$. We can also associate to λ the corresponding matrix of the generators (x_{ij}) of $L(E, \omega)$, where $x_{ij} = e_i^{(j)}$, which is the matrix obtained from the full matrix

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$$

by substituting by 0 the variables that are not in the positions allowed by the shape of the partition. Looking at the algebra $L_1(E, \omega)$ of the weighted graph associated to the (m, n) -partition λ , we can interpret the shape of λ as the refinement matrix R that defines the \mathcal{V} -monoid $M_1(E, \omega)$ of $L_1(E, \omega)$ (see Theorem 5.3). For instance, for the above partition $\lambda = (4, 2, 2)$, we have the refinement matrix

$$R = \begin{pmatrix} v(e^{(1)}, 1) & v(e^{(2)}, 1) & v(e^{(3)}, 1) & v(e^{(4)}, 1) \\ v(e^{(1)}, 2) & v(e^{(2)}, 2) & 0 & 0 \\ v(e^{(1)}, 3) & v(e^{(2)}, 3) & 0 & 0 \end{pmatrix}.$$

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & 0 & \cdots & 0 \\ x_{31} & 0 & \cdots & 0 \\ & \cdots & \cdots & \\ x_{m1} & 0 & \cdots & 0 \end{pmatrix}$$

with relations $XX^* = I_m$ and $X^*X = I_n$. Observe that in this case, we have $L(E, \omega_0) = L_1(E, \omega_0)$, so that we exactly recover the weighted Leavitt path algebra with our construction. The monoid $M_1(E, \omega_0) \cong \mathcal{V}(L(E, \omega_0))$ is given by

$$M_1(E, \omega_0) = \langle a, x \mid a = x + (m - 1)a = x + (n - 1)a \rangle.$$

(Here, a corresponds to a_v and x to $a_{v(e^{(1)}, 1)}$.) It is easy to see that the Leavitt type of $L(E, \omega_0)$ is $(1, n - m)$. Indeed, observe that

$$(n - m + 1)a = a + (n - m)a = x + (m - 1)a + (n - m)a = x + (n - 1)a = a,$$

and the Grothendieck group of $M_1(E, \omega_0)$ is $\mathbb{Z}/(n - m)$ (with generator a), so that the Leavitt type is $(1, n - m)$ as claimed.

The algebra $L(E, \omega_0)$ has a unique nontrivial trace ideal M , corresponding to setting $x_{11} = 0$. We have $L(E, \omega_0)/M \cong L(1, m - 1) * L(1, n - 1)$, the coproduct of the two simple Leavitt algebras $L(1, m - 1)$ and $L(1, n - 1)$. This is indeed the Leavitt path algebra of the separated graph (E', C') with a unique vertex v and with $C' = \{X, Y\}$, $X = \{x_1, \dots, x_{n-1}\}$ and $Y = \{y_1, \dots, y_{m-1}\}$. The $*$ -isomorphism is given by the assignment

$$\begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ x_{21} & 0 & \cdots & 0 \\ x_{31} & 0 & \cdots & 0 \\ & \cdots & \cdots & \\ x_{m1} & 0 & \cdots & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & x_1 & \cdots & x_{n-1} \\ y_1^* & 0 & \cdots & 0 \\ y_2^* & 0 & \cdots & 0 \\ & \cdots & \cdots & \\ y_{m-1}^* & 0 & \cdots & 0 \end{pmatrix}.$$

We then have

$$\mathcal{V}(L(E, \omega_0)/M) \cong \langle a \mid a = (n - 1)a = (m - 1)a \rangle$$

and we get another drop in the Leavitt type, because this algebra has Leavitt type $(1, d)$ where $d := \gcd(m - 2, n - 2)$.

We obtain a maximal ideal $\mathfrak{m}' \supset M$ such that $L(E, \omega_0)/\mathfrak{m}' \cong L(1, m - 1) \otimes L(1, n - 1)$, and pulling back this ideal to $L(m, n)$, we obtain the desired ideal \mathfrak{m} . The last statement follows from [3, Theorem 5.1]. □

References

- [1] G. Abrams, P. Ara and M. Siles Molina, *Leavitt Path Algebras*, Lecture Notes in Mathematics, 2191 (Springer, London, 2017).
- [2] G. Abrams and R. Hazrat, ‘Connections between abelian sandpile models and the K -theory of weighted Leavitt path algebras’, Preprint, 2022, [arXiv:2112.09218v1](https://arxiv.org/abs/2112.09218v1) [math.RA].

- [3] P. Ara and G. Cortiñas, 'Tensor products of Leavitt path algebras', *Proc. Amer. Math. Soc.* **141** (2013), 2629–2639.
- [4] P. Ara and R. Exel, 'Dynamical systems associated to separated graphs, graph algebras, and paradoxical decompositions', *Adv. Math.* **252** (2014), 748–804.
- [5] P. Ara and A. Facchini, 'Direct sum decompositions of modules, almost trace ideals, and pullbacks of monoids', *Forum Math.* **18** (2006), 365–389.
- [6] P. Ara and K. R. Goodearl, 'Leavitt path algebras of separated graphs', *J. reine angew. Math.* **669** (2012), 165–224.
- [7] P. Ara and M. Lolk, 'On the ideal structure of tame separated graph C^* -algebras', *Adv. Math.* **328** (2018), 367–435.
- [8] P. Ara and F. Perera, 'Multipliers of von Neumann regular rings', *Comm. Algebra* **28** (2000), 3359–3385.
- [9] G. M. Bergman, 'Coproducts and some universal ring constructions', *Trans. Amer. Math. Soc.* **200** (1974), 33–88.
- [10] R. Exel, *Partial Dynamical Systems, Fell Bundles, and Applications*, Mathematical Surveys and Monographs, 224 (American Mathematical Society, Providence, RI, 2017).
- [11] A. Facchini and F. Halter-Koch, 'Projective modules and divisor homomorphisms', *J. Algebra Appl.* **2** (2003), 435–449.
- [12] J. L. García and J. J. Simón, 'Morita equivalence for idempotent rings', *J. Pure Appl. Algebra* **76** (1991), 39–56.
- [13] R. Hazrat, 'The graded structure of Leavitt path algebras', *Israel J. Math.* **195** (2013), 833–895.
- [14] W. G. Leavitt, 'The module type of a ring', *Trans. Amer. Math. Soc.* **103** (1962), 113–130.
- [15] R. Preusser, 'The V-monoid of a weighted Leavitt path algebra', *Israel J. Math.* **234** (2019), 125–147.
- [16] R. Preusser, 'Weighted Leavitt path algebras—An overview', Preprint, 2021, [arXiv:2109.00434v1](https://arxiv.org/abs/2109.00434v1) [math.RA].
- [17] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd edn, Graduate Texts in Mathematics, 203 (Springer-Verlag, New York, 2001).

PERE ARA, Departament de Matemàtiques, Edifici Cc,
Universitat Autònoma de Barcelona, 08193 Cerdanyola del Vallès (Barcelona), Spain
and
Centre de Recerca Matemàtica, Edifici Cc, Campus de Bellaterra, 08193
Cerdanyola del Vallès (Barcelona), Spain
e-mail: para@mat.uab.cat