THE HOOK GRAPHS OF THE SYMMETRIC GROUP

J. S. FRAME, G. DE B. ROBINSON, AND R. M. THRALL

1. The hook graph. Each irreducible representation $[\lambda]$ of the symmetric group S_n may be identified by a partition $[\lambda]$ of n into non-negative integral parts $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \lambda_n \geqslant 0$, of which the first λ'_j parts are $\geqslant j$, or by a right (Young) diagram also called $[\lambda]$, that contains λ_i nodes in its ith row and λ'_j nodes in its jth column. An interchange of rows and columns in the diagram $[\lambda]$ converts it to the associated diagram $[\lambda']$ belonging to the associated representation $[\lambda']$ of the same degree f_λ.

The node in the ith row and jth column of $[\lambda]$ is called its ij-node. It is called the corner of the ij right hook (7) that consists of this node and all nodes to the right of it or below it. The $\lambda_i - j$ nodes on the right (in the ith row) are called the arm of the ij right hook, and the right end node is called the head. The $\lambda'_j - i$ nodes below the ij node (in the jth column) are called the leg of the right hook, and the bottom node is called the foot. The total hook length h_{ij} is

$$h_{ij} = 1 + (\lambda_i - j) + (\lambda'_j - i). \tag{1.1}$$

The hook graph $H[\lambda]$ belonging to $[\lambda]$ is an array of positive integers obtained by placing each of the n hook lengths h_{ij} at the corresponding ij-node of the diagram $[\lambda]$. The hook product H_λ is the product of the n integers h_{ij} in the hook graph.

The ij-node is called a q-node if and only if h_{ij} is divisible by q. It is called a q-node of residue r, or simply a (q, r)-node, if the integers λ_i and λ'_j satisfy the congruences

$$\lambda_i - i + 1 \equiv j - \lambda'_j \equiv r \pmod{q}, \qquad 1 \leqslant r \leqslant q. \tag{1.2}$$

Clearly a (q, r)-node is also a q-node.

In this paper we present several properties of the representation $[\lambda]$ that can be stated and proved more simply than heretofore by using the hook graph and related concepts.

Following a preliminary lemma about the hook numbers associated with the nodes of any right hook, we prove in §2 that the degree f_λ of the representation $[\lambda]$ is equal to the group order $n!$ divided by the hook product H_λ if the diagram $[\lambda]$ is either a right diagram or a direct sum of disjoint right diagrams. We also characterize irreducible representations $[\lambda]$ of defect 0 (mod p) by the absence of multiples of p in the hook graph.

In §3 we show that the simply constructed q-quotient diagram $[\lambda]_q$ obtained by deleting all nodes of $[\lambda]$ except q-nodes is the same except for rearrangement of disjoint constituents as the star diagram of Robinson, Staal and others

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(1; 8; 9; 10; 11; 12), and we give simple proofs of Staal's Theorem B concerning the removal of kq-hooks from $[\lambda]$. The relationship of Littlewood's p-quotient **(5)** to Robinson's star diagram has been discussed by Farahat **(2)**.

For each integer r from 1 to q the (q, r)-nodes of $[\lambda]$ (if such exist) are shown in §4 to form one of the disjoint constituents of the (rearranged) star diagram. From this follows a short proof of Staal's Theorem C.

In §5 we give a short hook graph proof of Staal's Theorem A concerning the exponent of p in the degree of $[\lambda]$ and in §6 we describe a constructive method for determining the q-core from the hook graph without actually removing hooks. Finally, in §7 we show how the leg length of a removable hook is determined from the hook-graph.

2. The degree of the representation $[\lambda]$. The st-node is called a rim node of $[\lambda]$ if there is no node of $[\lambda]$ in the $s+1$, $t+1$ position. Counting along the rim from the head of the right hook with corner at the ij-node, (located at the right of the ith row) to its foot at the bottom of the jth column, there are h_{ij} rim nodes forming what we call the ij skew or rim hook. Consider the two pieces of the ij rim hook obtained by cutting it between the mth and $(m+1)$th node, counting from the head. If these nodes are in the same row, and a vertical cut is made between the tth and $(t-1)$th columns, the upper right part ends in a foot and is a rim hook of length h_{it}, but the lower left part does not start with a head node and is not a rim hook. If, however, the two nodes are in the same column, and a horizontal cut is made between the $(s-1)$th and sth rows, the lower left part starts with a head node and forms a rim hook of length h_{sj}, whereas the upper right part with $h_{ij} - h_{sj}$ nodes does not end in a foot and does not form a rim hook. As m varies from 1 to h_{ij}, these lengths m of the upper right parts assume as values either h_{it} $(t \geqslant j)$, or $h_{ij} - h_{sj}$ $(s > i)$ but not both. Thus we establish the lemma

LEMMA 1. *If* h_{it} $(j \leqslant t \leqslant \lambda_i)$ *and* h_{sj} $(i < s \leqslant \lambda'_j)$ *are the* h_{ij} *integers in the ij-right hook of the hook graph* $H[\lambda]$, *then the integers* h_{it} *and* $h_{ij} - h_{sj}$ *are distinct and form a permutation of the integers* $1, 2, \ldots h_{ij}$.

It is clear from Lemma 1 that the product of all hook lengths in the ith row of $H[\lambda]$ is given by the formula

2.1
$$P_i = (h_{i1})! \Big/ \prod_{s>i} (h_{i1} - h_{s1}).$$

Now the first column hook lengths $h_{i1} = \lambda_i - i + \lambda'_1$ are precisely the numbers l_i that appear in the Frobenius formula **(3; 4; 6)** for the degree f_λ of $[\lambda]$, namely:

2.2
$$f_\lambda = n! \prod_i \frac{1}{(l_i)!} \prod_{s>i} (l_i - l_s).$$

This formula was discovered independently by A. Young **(14)**.

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Setting $l_i = h_{i1}$ and substituting from 2.1 in 2.2 we obtain

2.3 $$f_\lambda = n!\prod_i (1/P_i).$$

The product of all the P_i is the complete hook product H_λ. Thus we have proved our first main theorem.

THEOREM 1. *Let H_λ be the product of the hook numbers h_{ij} in the hook graph of an irreducible representation* $[\lambda]$ *of the symmetric group S_n. Then the degree f_λ of* $[\lambda]$ *is given by the formula*

2.4 $$f_\lambda = n!/H_\lambda.$$

Example 1. The irreducible representation [6,4,2] of S_{12} has the following right diagram $[\lambda]$ and hook graph $H[\lambda]$.

2.5 $$[\lambda]:\ \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & & & & \end{matrix} \qquad H[\lambda]:\ \begin{matrix} 8 & 7 & 5 & 4 & 2 & 1 \\ 5 & 4 & 2 & 1 & & \\ 2 & 1 & & & & \end{matrix}$$

Its degree is computed as follows:

2.6 $$f_{[6,4,2]} = \frac{(12)!}{H_{[6,4,2]}} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 6\cdot 3}{5\cdot 4\cdot 2\cdot 1\cdot 2\cdot 1} = 11\cdot 3^5 = 2673.$$

COROLLARY 1. *Let H_λ be the product of the hook numbers h_{ij} in the hook graph of the reducible representation* $[\lambda]$ *of S_n that corresponds to a diagram consisting of a number of disjoint right diagrams having b_r nodes in the rth constituent. Then the degree of* $[\lambda]$ *is given by the same formula* 2.4.

Proof. The degree f_λ is the number of standard orderings of the n nodes of the diagram such that the numbers increase from left to right within any row, and from top to bottom within any column **(9)**. There are $n!/\Pi(b_r!)$ ways in which the numbers 1 to n can be assigned to the various constituents, and by Theorem 1 there are $b_r!/H_{\lambda,r}$ ways of ordering them within the rth constituent, if $H_{\lambda,r}$ is the hook product for the rth constituent. Hence

2.7 $$f_\lambda = \frac{n!}{\prod_r (b_r)!}\cdot \prod_r \frac{b_r!}{H_{\lambda,r}} = \frac{n.}{\prod_r H_{\lambda,r}} = \frac{n!}{H_\lambda}$$

where the hook product H_λ for $[\lambda]$ is the product of the hook products of its constituents.

Another simple consequence of Theorem 1 is the following known result about p-hooks for any prime p. An ordinary irreducible representation of a group G is said to be of defect 0 (mod p) when its degree is divisible by the highest power of p that divides the group order **(1)**. Such a representation is an indecomposable and irreducible modular component of the regular representation and its character vanishes for p-singular classes. For the symmetric group

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S_n such representations of defect 0 are found by inspection of the hook graph as follows.

COROLLARY 2. *If p is any prime, an ordinary irreducible representation $[\lambda]$ is of defect* 0 (*mod p*) *if and only if its hook graph contains no multiple of p.*

In Example 1 above we see that [6,4,2] is of defect 0 (mod 3) and (mod 11). However it is of maximum defect for $p = 2$, 5, or 7.

3. The hook graph of the q-quotient (**or star**) **diagram.** Given a diagram $[\lambda]$ whose nodes include b q-nodes, we define the q-quotient diagram $[\lambda]_q$ to be the diagram of b nodes obtained by deleting all the nodes of $[\lambda]$ except the q-nodes.

LEMMA 2. *If the hook number of the ij-node in $[\lambda]$ is $h_{ij} = kq$, the hook number of the corresponding node in $[\lambda]_q$ is $k = h_{ij}/q$.*

Proof. In Lemma 1 it was proved that each number from 1 to h_{ij} occurs exactly once among the integers $h_{it}(t \geqslant j)$ and $h_{ij} - h_{sj}$, $(s > i)$. Thus if $h_{ij} = kq$ it follows that exactly k multiples of q appear among the numbers h_{it} $(t \geqslant j)$ and $kq - h_{sj}$ $(s > i)$. Hence exactly k of the hook numbers h_{it} $(t \geqslant j)$ and h_{sj} $(s > i)$ are divisible by q, and there are k different q-nodes in the ij right hook of $[\lambda]$. Only these k nodes are retained in the corresponding right hook of $[\lambda]_q$, so the corresponding hook number is k.

Thus we obtain the hook graph $H[\lambda]_q$ of the q-quotient if we divide each hook number in $H[\lambda]$ by q and retain only the integers. Lemma 2 shows that our easily constructed q-quotient diagram is equivalent to the star diagram of Robinson and Staal whose existence is proved in Staal's Theorem B, which he stated as follows:

STAAL'S THEOREM B. *Given the right diagram λ, and a positive integer q, there exists a diagram λ^** (*called the "star diagram" of λ*) *such that there is a one-to-one correspondence between the kq-hooks of λ and the k-hooks of λ^*.*

Staal's diagram λ^* and our diagram $[\lambda]_q$ differ at most in the rearrangement of disjoint constituents, but the order of rows and columns within each constituent is the same for λ^* and $[\lambda]_q$.

We next give a simpler proof of Staal's Theorem B, applied to $[\lambda]_q$ and rephrased in our notation.

STAAL'S THEOREM B′. *If a k-hook is removed from $[\lambda]_q$, leaving $[\mu]$, and if the corresponding kq-hook is removed from $[\lambda]$ leaving $[\bar{\lambda}]$, then $[\bar{\lambda}]_q = [\mu]$.*

New Proof. We shall study the effect of the respective hook removals on the hook graphs of $[\lambda]$ and of $[\lambda]_q$. Let $[\bar{\lambda}]$ be obtained from $[\lambda]$ by removing either the right kq-hook with corner at the ij-node or the corresponding ij rim hook. We may obtain $H[\bar{\lambda}]$ from $H[\lambda]$ in three steps.

1. We delete from $H[\lambda]$ the kq hook numbers in the ij right hook.

2. We diminish by kq each of the integers h_{sj} $(s < i)$ standing above h_{ij} and move this reduced jth column of $H[\lambda]$ past the $\lambda_i - j$ columns of the hook arm to form the λ_ith column of $H[\bar{\lambda}]$.

3. We diminish by kq each of the integers h_{it} $(t < j)$ standing to the left of h_{ij} and move this reduced ith row down past the $\lambda'_j - i$ rows of the hook leg to form the λ'_jth row of $H[\bar{\lambda}]$.

The effect of these three operations on the q-nodes of $[\lambda]$, which are the nodes of $[\lambda]_q$, is to remove the k nodes belonging to the corresponding right k-hook of $[\lambda]_q$, to reduce by k the hook quotient numbers h_{it}/q or h_{sj}/q of q-nodes above or to the left of h_{ij}/q, and to move them past the arm or leg of the k-hook. Hence $[\bar{\lambda}]_q = [\mu]$.

Example 2. To illustrate the effect of hook removal on the hook graph, we remove from $H[7,6,5,3]$ a right 6-hook with corner at the 23 node. The six rim nodes are shown by dots at the right. Then we form the 2-quotient hook graph $H[\lambda]_2$ and remove the corresponding 3 hook.

$$
\begin{array}{ccccccc}
\multicolumn{7}{c}{H[\lambda]} \\
10 & 9 & 8 & 6 & 5 & 3 & 1 \\
8 & 7 & 6 & 4 & 3 & 1 & \\
6 & 5 & 4 & 2 & 1 & & \\
3 & 2 & 1 & & & &
\end{array}
\qquad
\begin{array}{cc|ccccc}
10 & 9 & 2 & 6 & 5 & 3 & 1 \\
\cline{3-5}
2 & 1 & & & & & \\
6 & 5 & & 2 & 1 & & \\
3 & 2 & & & & &
\end{array}
\qquad
\begin{array}{ccccccc}
\multicolumn{7}{c}{H[\bar{\lambda}]} \\
10 & 9 & 6 & 5 & 3 & 2 & 1 \\
6 & 5 & 2 & 1 & . & . & \\
3 & 2 & . & . & . & & \\
2 & 1 & . & & & &
\end{array}
$$

$$
\begin{array}{cccc}
\multicolumn{4}{c}{H[\lambda]_2} \\
5 & & 4 & 3 \\
4 & & 3 & 2 \\
3 & & 2 & 1 \\
 & 1 & &
\end{array}
\qquad
\begin{array}{cc|cc}
5 & & 1 & 3 \\
\cline{3-4}
1 & & & \\
3 & & & 1 \\
 & 1 & &
\end{array}
\qquad
\begin{array}{cccccc}
\multicolumn{6}{c}{H[\bar{\lambda}]_2} \\
5 & & 3 & & & 1 \\
3 & & 1 & & & . \\
 & 1 & & & & \\
1 & & . & & & .
\end{array}
$$

4. The disjoint constituents of the q-quotient (or star) diagram. It is clear from 1.2 that two q-nodes in the same row or the same column of $[\lambda]$ have the same residue r (mod q), where $1 \leqslant r \leqslant q$. We shall call the ith row a (q, r)-row and the jth column a (q, r)-column if and only if i and j satisfy 1.2. The (q, r)-nodes of a right diagram of $[\lambda]$ (if any exist) are those in $[\lambda]$ which lie at intersections of (q, r)-rows and (q, r)-columns of $[\lambda]$, and they form a right diagram $[\lambda]_{q,r}$ which is a disjoint constituent of $[\lambda]_q$. As r varies from 1 to q we obtain at most q such constituents, but some may be vacuous. Thus we obtain

THEOREM 3. *The q-quotient diagram $[\lambda]_q$ derived from a right diagram $[\lambda]$ is composed of at most q disjoint right diagrams, of which the rth is composed of the (q, r)-nodes of $[\lambda]$ if any exist.*

It is easily seen that one or more rows (or columns) of any of the disjoint constituents of the star diagram may be moved past any or all of the rows (or columns) of a different constituent without affecting the hook numbers of this diagram or most of its other essential properties. However, among equivalent star diagrams of $[\lambda]$ the easiest from which to see the one-to-one correspondence of its k-hooks with kq-hooks of $[\lambda]$ is the q-quotient diagram $[\lambda]_q$ defined above in §3.

Staal's Theorem C follows immediately from our Theorem 3. His δ's are our first row hook numbers h_{1j}, his λ^* our $[\lambda]_q$.

STAAL'S THEOREM C. *Gather the h_{1j}'s of $[\lambda]$ into classes which are congruent (mod q). For each such class of congruent h_{1j}'s form the diagram having these as first row hook numbers. The diagrams thus formed will be the constituents of the star diagram $[\lambda]_q$.*

New Proof. We see from 1.1 and 1.2 that

4.1 $$h_{1j} \equiv \lambda_1 - r \pmod q \text{ if and only if } j - \lambda'_j \equiv r \pmod q.$$

Hence the q-nodes in columns headed by hook numbers congruent to $\lambda_1 - r$ are (q, r)-nodes, and they form the rth constituent of $[\lambda]_q$. If we form a new hook graph $H[\mu]$ by retaining only those top hook numbers in $H[\lambda]$ that are congruent to $\lambda_1 - r \pmod q$, then by Lemma 1 the column of $H[\mu]$ headed by h_{1j} will have in addition to the hook numbers of the jth column of $H[\lambda]$ those numbers $h_{1j} - h_{1t}$ such that $h_{1j} > h_{1t}$ but

$$h_{1j} \not\equiv h_{1t} \pmod q.$$

No new q-nodes are present, so the q-quotient $[\mu]_q$ is equivalent to $[\lambda]_{q,r}$.

THEOREM 4. *If p and q are any integers and $[\lambda]$ any diagram, the pq-quotient of $[\lambda]$ is the q-quotient of $[\lambda]_p$.*

Proof. This analogue of Robinson's theorem **(10)** for the star diagram is a trivial consequence of the fact stated in Lemma 2 that pq-nodes of $[\lambda]$ correspond to q-nodes of $[\lambda]_p$.

Example 3. We illustrate Theorems 3 and 4 by showing the hook graphs $H[\lambda]_q$ for $[\lambda] = [8,7,5,3,2]$ and $q = 1, 2, 3, 4, 6$. Residues (mod 12) given at the left should be reduced (mod q) to identify the various disjoint constituents. Dots are for spacing only.

r	$q = 1$	$q = 2$	$q = 3$	$q = 4$	$q = 6$
8	12 11 9 7 6 4 3 1	6 . . . 3 2	4 . 3 . 2 . 1	3 1	2 . . . 1
6	10 9 7 5 4 2 1	5 . . . 2 1	. 3	. . . .	. .
3	7 6 4 2 1	. 3 2 1	. 2	. . 1	. 1
12	4 3 1	2	. 1	1	
10	2 1	1			

5. The p-exponent in the degree. For any irreducible representation $[\lambda]$ and for any prime p, it is easy to determine from the hook graph the exponent

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$e(f_\lambda)$ of the highest power of p dividing the degree f_λ. In fact, equation 2.4 shows immediately that

5.1 $$e(f_\lambda) = e(n!) - e(H_\lambda).$$

A similar formula holds moreover, by Corollary 1, for the degree $f_\lambda{}^*$ of the reducible representation of S_b associated with the p-quotient diagram $[\lambda]_p$, namely

5.2 $$e(f_\lambda{}^*) = e(b!) - e(H_{\lambda,p}).$$

These facts make possible a simple proof of Robinson's version **(1)** of Nakayama's formula **(7)**, given as Theorem A in Staal's paper **(12)**. Other proofs of this result have recently been given by Nakayama and Osima **(8)** and Farahat **(2)**.

THEOREM A. *If a denotes the number of nodes in the p-core of $[\lambda]$ then*

5.3 $$e(f_\lambda) = e(n!) - e(n-a)! + e(f_\lambda{}^*).$$

Proof. Since $n - a = bp$, equations 5.1 and 5.2 enable us to rewrite 5.3 in the form

5.4 $$e(H_\lambda) - e(H_{\lambda,p}) = e((bp)!) - e(b!).$$

Each side of 5.4 reduces to b, since there are exactly b explicit factors in the indicated products H_λ and $(bp)!$ that are divisible by p, and the quotients of these factors by p are the factors of $H_{\lambda,p}$ and $b!$ respectively.

6. Construction of the q-core. The q-core of $[\lambda]$ is the diagram $[\alpha]$ that remains after all q-hooks have been removed. Its partition numbers α_k can be constructed from the hook numbers h_{i1} as follows:

THEOREM 6. *Let β_i be the number of q-nodes in the ith row of $[\lambda]$, and let α_i be the number of nodes in the ith row of the q-core of $[\lambda]$. Then the numbers $h_{i1} - q\beta_i$ are distinct non-negative integers that form a permutation of the integers $\alpha_i - i + \lambda'_1$, if we set $\alpha_i = 0$ for $i > \alpha'_1$. The sign customarily attached to the q-core is the sign of this permutation.*

Proof. The effect on the hook graph of the removal of a kq-hook was described in the proof of Theorem B′ (§3). The effect on the first column hook numbers h_{i1} of removing from $[\lambda]$ a q-hook whose corner is not in the first column, is simply to diminish one of the h_{i1} by q and rearrange the order of rows. Let all such be removed from $[\lambda]$, beginning with the bottom rows and working up, so the only remaining q-nodes are a set of k contained in the first column. Their hook numbers are $q, 2q, \ldots kq$, counting from the bottom. Reducing each of these by q is equivalent to deleting the number kq. The number of rows lost in removing the final kq-hook is the smaller of the two numbers kq and $h_{11} - h_{12}$ and this is equal to $\lambda'_1 - \alpha'_1$. If kq is the smaller, the first column hook numbers which are greater than kq are each reduced by $\lambda'_1 - \alpha'_1 = kq$

and become the hook numbers $\alpha_i - i + \alpha'_1$ of the q-core. If $h_{11} - h_{12} = d < kq$, the second column hook numbers $h_{i1} - d$ (for $h_{i1} \neq kq$ and $h_{i1} > d$) become the first column hook numbers $\alpha_i - i + \alpha'_1$ of the q-core $[\alpha]$. In each case the numbers $h_{i1} - q\beta_i$ are distinct and form a permutation of the integers $\alpha_i - i + \lambda'_1$, and the row numbers α_i can be calculated. The sign attached to the combined permutation of rows is the factor by which the Young Symmetrizers N_i are altered in the reduction process, and this sign is customarily attached to the q-core.

Example 4. Find the 3-core of $[\lambda] = [9,7,4,3,2]$.

$H[\lambda]$	$h_{i1}-q\beta_i$	$\alpha_i-i+\lambda'_1$	α_i	$\alpha_i-i+\alpha'_1$	$H[\alpha]$
13 12 10 8 6 5 4 2 1	7	7	$7-4=3$	4	4 2 1
10 9 7 5 3 2 1	4	4	$4-3=1$	1	1
6 5 3 1	0	2	$2-2=0$		
4 3 1	1	1	$1-1=0$		
2 1	2	0	$0-0=0$		

7. Leg lengths. In the computation of characters **(9)** it is the leg length rather than the class of a hook which is important. So far we have attached significance only to (q, r)-nodes and their *alignment* in the rows and columns of $H[\lambda]$. We state the following

THEOREM 7. *The leg length of the ij-hook in* $[\lambda]$ *is the number of missing integers less than* h_{ij} *and to the right of it in* $H[\lambda]$.

Proof. Since each such missing integer indicates one step down in the rim of $[\lambda]$, the total number of such steps down is the leg length in question.

Nakayama studied the effect of interchanging the order of removing two successive hooks in some detail **(7**, I, §§3, 4**)**. It will be sufficient if we consider only the "interlocking" case of an ij-hook and an st-hook, such that $\lambda_i \geqslant t$, $i > s$, $j < t$. Applying the three steps of §3 it is evident that removing the ij-hook first *shortens* the leg length of the st-hook by 1, while removing the st-hook first *lengthens* the leg length of the ij-hook by 1. This makes explicit the consideration of this same problem in **(9**, p. 289**)**. If one hook is completely contained within the other the problem does not arise, and if the two hooks do not interlock then the leg lengths are unaffected.

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Michigan State College
University of Toronto
University of Michigan