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A THREE-FOLD NON-LATTICE COVERING

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Let $\bar{D}_k$ be the density of thinnest k-fold covering of the plane by equal circles (of radius 1, say). Let D_k be the corresponding density when the centres of the circles are at the points of a lattice Λ. It is clear that $\bar{D}_k \le D_k$.

The problem of finding $\bar{D}_k$ for any given k is much more difficult than that of finding D_k. For single covering Kerschner [3] has proved that $\bar{D}_1 = D_1$ $(= D$, say$) = 2\pi/3\sqrt{3}$. As for lattice coverings, Blundon [1] has proved $D_2/2D = 1$ and that

$$D_3/3D = \frac{1}{288}(15+\sqrt{97})\sqrt{(22+10\sqrt{97})} = 0.94708\ldots,$$

and also that $D_4/4D = 25\sqrt{3}/48 = 0.90210\ldots.$
As for non-lattice coverings, the only known significant result is that of Danzer [2], namely, $\bar{D}_2/2D \le 0.97087\ldots$

The purpose of the present paper is to exhibit a rather simple configuration which gives an estimate for $\bar{D}_3$ much stronger than the corresponding lattice result. We prove the following theorem.

THEOREM. $\bar{D}_3/3D \le 9(5-2\sqrt{6}) = 0.90918\ldots$, *and the corresponding configuration is the union of three translates of an equilateral lattice of side* $1+\frac{1}{3}\sqrt{6} = 1.81649\ldots.$

Proof. Let Λ be an equilateral lattice of side $1+\frac{1}{3}\sqrt{6}$ with fundamental triangle OAB where O is the origin. Let Λ_1, Λ_2, Λ_3 denote the translates of Λ in the directions AB, BO, OA respectively by a distance 1/3 in each case. Then $O_iA_iB_i$ is a fundamental triangle of the lattice Λ_i $(i = 1, 2, 3)$.

Let circles of radius 1 be centered at the points of the lattices Λ_1, Λ_2, Λ_3. Since the configuration of the fundamental parallelograms of Λ, Λ_1, Λ_2, Λ_3 as a whole is symmetrical about the line AB, it is sufficient to ensure that every point of the triangle OAB is covered by at least three of the circles centred at the vertices of the triangles $O_1A_1B_1$, $O_2A_2B_2$, $O_3A_3B_3$.

Construct an equilateral triangle $P_1Q_1R_1$ with points P_1, Q_1 on the line O_3B_3 such that R_1, A_1 are on opposite sides of the line O_3B_3 and such that $O_3Q_1 = B_3P_1 = 1$. This equilateral triangle has side $1-\frac{1}{3}\sqrt{6}$. Construct corresponding triangles $P_2Q_2R_2$, $P_3Q_3R_3$ as in the diagram.

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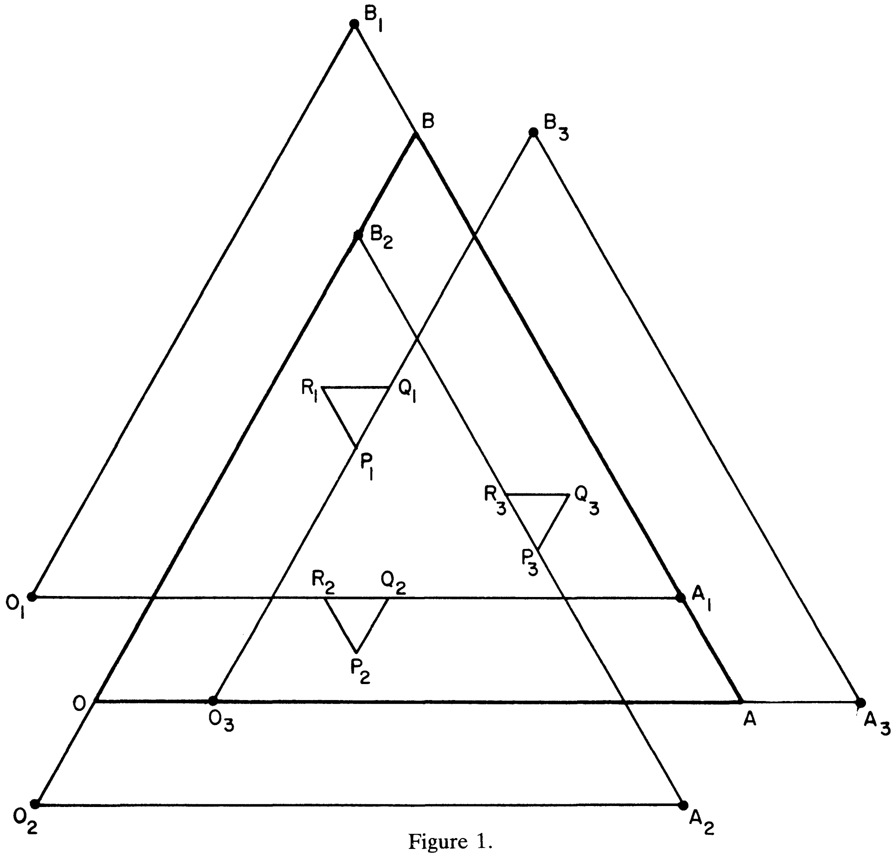

Figure 1.

Consider the covering of a triangle OAB by circles centred at the vertices of the triangle $O_1A_1B_1$. It is easily verified that each of the segments O_1P_1, O_1R_1, B_1R_1, B_1Q_1, A_1Q_1, A_1P_1 has length 1. Hence all points of the triangle $O_1A_1B_1$ (with the exception of points in the triangle $P_1Q_1R_1$) are covered at least once by the circles at O_1, A_1, B_1. Also, since the distance from each of O_1, A_1 to the foot of the perpendicular from P_2 to OA is less than 1, it follows that each of the circle at O_1 and the circle at A_1 covers all those points of the triangle which are outside the triangle $O_1A_1B_1$. Further, each of these two circles covers the triangle $P_2Q_2R_2$, since $O_1Q_2 = A_1R_2 = 1$.

To sum up, the circles of Λ_1 do not cover at all the points of the triangle $P_1Q_1R_1$ but they do cover every other point of the triangle OAB including a double covering of the triangle $P_2Q_2R_2$ and a single covering of the triangle $P_3Q_3R_3$. Repeat the same argument for the circles of Λ_2 and of Λ_3.

Thus each of the triangles $P_1Q_1R_1$, $P_2Q_2R_2$, $P_3Q_3R_3$ is covered not at all by one of the lattices, once by a second lattice, and twice by the third lattice. Every other point of the triangle OAB is covered at least once by each of the three lattices.

Therefore the given configuration does indeed provide a three-fold covering of the plane. Let Δ be the determinant of the lattice Λ. Then $\Delta=(\frac{1}{2}\sqrt{3})(1+\frac{1}{3}\sqrt{6})^2$ and $\bar{D}_3\le 3\pi/\Delta$. Since $D=2\pi/3\sqrt{3}$, it follows that $D_3/3D\le 9(5-2\sqrt{6})=0.90918\ldots$. This completes the proof of the theorem.

References

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