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Existence of Leray's Self-Similar Solutions of the Navier-Stokes Equations In $\mathcal{D} \subset \mathbb{R}^3$

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Abstract. Leray's self-similar solution of the Navier-Stokes equations is defined by

$$u(x,t) = U(y)/\sqrt{2\sigma(t^* - t)},$$

where $y = x/\sqrt{2\sigma(t^* - t)}$, $\sigma > 0$. Consider the equation for $U(y)$ in a smooth bounded domain $\mathcal{D}$ of $\mathbb{R}^3$ with non-zero boundary condition:

$$\begin{aligned} -\nu \bigtriangleup U + \sigma U + \sigma y \cdot \nabla U + U \cdot \nabla U + \nabla P = 0, &\quad y \in \mathcal{D}, \\ \nabla \cdot U = 0, &\quad y \in \mathcal{D}, \\ U = \mathcal{G}(y), &\quad y \in \partial\mathcal{D}. \end{aligned}$$

We prove an existence theorem for the Dirichlet problem in Sobolev space $W^{1,2}(\mathcal{D})$. This implies the local existence of a self-similar solution of the Navier-Stokes equations which blows up at $t = t^*$ with $t^* < +\infty$, provided the function $\mathcal{G}(y)$ is permissible.

1 Introduction

Consider the incompressible Navier-Stokes equations in $\mathbb{R}^3 \times [0, \infty)$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \nu \bigtriangleup u, \qquad \nabla \cdot u = 0, \tag{1.1}$$

where $u(x,t) = (u_1, u_2, u_3)$ denotes the velocity field, p the pressure scalar, and the constant viscosity ν is positive. In 1934 Leray [L] suggested (backward) self-similar solutions of (1.1) of the form:

$$u(x,t) = \frac{1}{\sqrt{2\sigma(t^* - t)}} U(y), \quad y = \frac{x}{\sqrt{2\sigma(t^* - t)}} \in \mathbb{R}^3, \tag{1.2}$$

where $0 < \sigma \leq 1$ is a constant, and $0 < t < t^* < \infty$. If a solution $U \not\equiv 0$ is found, then the system (1.1) develops a finite-time singularity at $(0, t^*)$.

For the Leray system in an unbounded domain, it was proved in [NRŠ] and [T] that if a weak solution $U(y)$ is in $L_p(\mathbb{R}^3)$ for $3 \leq p \leq \infty$, then U must be trivial.

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Note that in their proofs, it was implicitly assumed that the self-similar solution satisfies (1.1) for all $x \in \mathbb{R}^3$. There is also a recent work [NOZ] on the similarity solution in a domain bounded in one direction: the blowup problem is connected to existence of the non-trivial steady state which has the unstable manifold.

On the other hand, some works (for instance [BP], [M], and [P]) have raised a subtle question: could the Leray's solutions exist only locally? That is, could a solution of (1.1) be self-similar only in a bounded domain Ω and be regular in Ω^c, where $\Omega = \{(x,t) : |x| \leq y_0\sqrt{2\sigma(t^* - t)}\}$ for some $y_0 > 0$? Interestingly, such a local self-similar blowup is found for the nonlinear Schrödinger equation by rigorous and numerical analysis (see [SS] for a review). For example in the supercritical dimension, it is found that the domain of the similarity solution is bounded above in the x-coordinate by some $x_0 < K < \infty$, with three regions: (i) in the inner, the exact self-similarity remains in a domain bounded in the y-coordinate as defined in (1.2); (ii) in the intermediate, the domain of non-stationary self-similar solution tends to infinity in the y-coordinate at the singular time; (iii) in the outer, there is a solution of the linear Schrödinger equation, decaying rapidly as $|x| \to \infty$ to ensure the finite wave energy. On each of the boundaries the solutions are continued smoothly, and depending on initial conditions, $x_0(t)$ can be a constant.

Pondering these works and the analogy of the Navier-Stokes equation to the nonlinear Schrödinger equation, it seems natural to explore the possibilities of local Leray solution (1.2). As a small step, we shall study the following problem for U in a smooth bounded domain $\mathcal{D}$:

$$-\nu \bigtriangleup U + \sigma U + \sigma y \cdot \nabla U + U \cdot \nabla U + \nabla P = 0, \quad y \in \mathcal{D}, \tag{1.3}$$

$$\nabla \cdot U = 0, \quad y \in \mathcal{D},$$

$$U = \mathcal{G}(y), \quad y \in \partial\mathcal{D}, \tag{1.4}$$

where $\mathcal{G}$ is a prescribed function. A physical motivation of the present study is that if this solution exists, it may serve as a central part of the blowup solution, and a continuation to the other regions. The boundedness of $\mathcal{D}$ is assumed on the base of numerical evidence [K], [OG], which indicates multiple length scales in the singularity formation.

In this paper, we show that Leray's solutions do exist for the above Dirichlet problem. To prove the theorem, an inequality has to hold: $4\sigma d^2 < \nu$, where d = diameter of $\mathcal{D}$. For a "turbulent" Navier-Stokes flow of small ν, the inequality requires that either σ or d be small. In the construction, the boundary condition $\mathcal{G}$ appears to be a forcing term, which is also critical for the proof. For general works related to the interest here, see for example, [CFM], [CKN], and [G]. There are also results on Leray's forward solutions [GM], [CP], which are regular on $\mathbb{R}^3 \times (0, \infty)$ and singular at $(0, 0)$.

Remark Suppose the self-similar solution is local, so it is defined in a bounded domain $\Omega = \{(x,t) : |x| \leq y_0\sqrt{2\sigma(t^* - t)}\}$. Let $x_0 = y_0\sqrt{2\sigma(t^* - t)}$ denote its extension. It is clear from the transformation (1.2) that for any x arbitrarily close to x_0, we have $y \to \infty$ as $t \to t^*$. This shows that although the self-similarity could be local in the x-coordinate, it cannot be local in the y-coordinate in any case. Hence to have a complete solution to the system (1.1), a solution of the problem (1.3) and (1.4) needs to be continuously connected to other regions. For a solution at the final self-similar region, it must match smoothly at the point $y = \infty$ to an outer solution u_{out}, where u_{out} is a regular solution to (1.1). Note that u_{out} should take a finite but non-zero value on the boundary, as there are incoming and outgoing motions in the vortex interaction zone (see Figure 4 of [P], and Section 7 of [M] for a discussion of the matching condition). We thus observe that this local self-similar problem is different from the one treated in [NRŠ] and [T] due to the above subtleties, though, at this stage we cannot state whether the complete solution exists or it is non-trivial.

2 Preliminaries

Let the domain $\mathcal{D} \subset \mathbb{R}^3$ be an open set with smooth boundary. Let $C_0^k(\mathcal{D})$ be the class of C^k real functions f with compact support on $\mathcal{D}$. Unless otherwise stated, a vector function f is divergence-free. Let ∇f denote $\partial_i f$ or $\partial_i f_j$, and write

$$\langle f, g\rangle = \int_{\mathcal{D}} f \cdot g \, dy, \quad \langle f, g, h\rangle = \int_{\mathcal{D}} f_i g_j \partial_j h_i \, dy.$$

Denote a Hilbert space $H^m(\mathcal{D}) = W^{m,2}(\mathcal{D})$, where $W^{m,p}$ are the Sobolev spaces. Let $H_0^1(\mathcal{D})$ be the closure of $C_0^\infty(\mathcal{D})$ in $H^1(\mathcal{D})$, with the norm on $H_0^1(\mathcal{D})$ by

$$\|f\|_{H_0^1} = (f, f) = \left\{ \int_{\mathcal{D}} |\nabla f|^2 \, dy \right\}^{1/2}.$$

Let $d =$ diameter of the domain $\mathcal{D}$.

For a weak solution $U \in H^1(\mathcal{D})$, multiplying (1.3) by a divergence-free vector function $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C_0^\infty(\mathcal{D})$, integrating by parts we have

$$\nu\langle \nabla\varphi, \nabla U\rangle + \sigma\langle \varphi, U\rangle + \sigma\langle \varphi, y, U\rangle + \langle \varphi, U, U\rangle = 0.$$

To treat (1.4), let G be a divergence-free extension to $\mathcal{D}$ of the given function $\mathcal{G}$ at the boundary, satisfying

$$G(y) \in C^2(\overline{\mathcal{D}}), \quad \nabla \cdot G = 0, \ |G(y)| \leq \kappa, \quad G = \mathcal{G}(y) \text{ on } \partial\mathcal{D}, \tag{2.1}$$

where κ is a positive constant. We further assume κ to be small, in the sense $\kappa \leq d$, d being the diameter of $\mathcal{D}$. This implies the smallness of the boundary data. Note that in (2.1), $\mathcal{G}(y)$ must satisfy the compatibility condition:

$$\int_{\partial\mathcal{D}} n \cdot \mathcal{G} \, dS = \int_{\mathcal{D}} \nabla \cdot G \, dy = 0.$$

Then we set

$$U = v(y) + G(y), \quad \nabla \cdot v = 0, \ v = 0 \text{ on } \partial\mathcal{D}, \ v \in H_0^1(\mathcal{D}).$$

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So solving (1.3) and (1.4) weakly is equivalent to finding a v such that

$$\nu\langle \nabla v, \nabla \varphi\rangle - 2\sigma\langle v, \varphi\rangle - \sigma\langle v, y, \varphi\rangle - \langle v, v+G, \varphi\rangle - \langle G, v, \varphi\rangle = \langle f_G, \varphi\rangle, \tag{2.2}$$

where $f_G(y) = \nu \bigtriangleup G - \sigma G - \sigma y \cdot \nabla G - G \cdot \nabla G$, $f_G \not\equiv 0$.

We now present a few results to be used in the sequel.

Lemma 2.1 *Let $\mathcal{D}$ be a bounded set in $\mathbb{R}^3$. Then $\forall f \in H_0^1(\mathcal{D})$,*

$$\|f\|_{L_2(\mathcal{D})} \le c_1\|\nabla f\|_{L_2(\mathcal{D})}, \quad \|f\|_{L_4(\mathcal{D})} \le c_2\|\nabla f\|_{L_2(\mathcal{D})},$$

where c_1, c_2 are constants depending on $\mathcal{D}$ and are bounded above by d.

Proof The result is the well-known Poincaré Lemma. ∎

Lemma 2.2 *Let $\mathcal{D}$ be bounded in $\mathbb{R}^3$. Let G be as in (2.1), where $\kappa \le d$. Then there exists some constant $c_3 \le \kappa d$ such that*

$$|\langle G, f, f\rangle| \le c_3\|\nabla f\|^2_{L_2(\mathcal{D})} \quad \forall f \in H_0^1(\mathcal{D}).$$

Proof By the assumption and Cauchy-Schwarz's inequality,

$$|\langle G, f, f\rangle| \le \kappa\langle\, |f|, |\nabla f|\,\rangle \le \kappa\|f\|_{L_2}\|\nabla f\|_{L_2}.$$

Using Lemma 2.1, we obtain

$$|\langle G, f, f\rangle| \le c_3\|\nabla f\|^2_{L_2(\mathcal{D})}, \quad c_3 = \kappa c_1.$$

The assertion is proved. ∎

Concerning (2.2), we define operators for fixed v on the Hilbert space $H_0^1(\mathcal{D})$:

$$T_1(\varphi) = 2\sigma\langle v, \varphi\rangle, \quad T_2(\varphi) = \sigma\langle v, y, \varphi\rangle,$$
$$T_3(\varphi) = \langle v, v+G, \varphi\rangle, \quad T_4(\varphi) = \langle G, v, \varphi\rangle.$$

Define

$$T = T_1 + T_2 + T_3 + T_4. \tag{2.3}$$

Lemma 2.3 *Let $\mathcal{D} \subset \mathbb{R}^3$. For each fixed v, T in (2.3) is a bounded linear functional on $H_0^1(\mathcal{D})$.*

Proof Let $v \in H_0^1(\mathcal{D})$ be fixed. Note that $|y| \le d$, and $G \in L_2(\mathcal{D})$ by (2.1). Applying Lemma 2.1, we observe that there exists finite $N > 0$ such that:

$$|T_1(\varphi)| \le C_1 \|\nabla v\|_{L_2(\mathcal{D})} \|\varphi\|_{H_0^1} \le N \|\varphi\|_{H_0^1},$$
$$|T_2(\varphi)| \le C_2 \|\nabla v\|_{L_2(\mathcal{D})} \|\varphi\|_{H_0^1} \le N \|\varphi\|_{H_0^1},$$
$$|T_3(\varphi)| \le C_3 \|v\|_{L_4(\mathcal{D})} \{\|v\|_{L_4(\mathcal{D})} + \|G\|_{L_4(\mathcal{D})}\} \|\varphi\|_{H_0^1} \le N \|\varphi\|_{H_0^1},$$
$$|T_4(\varphi)| \le C_4 \|G\|_{L_2(\mathcal{D})} \|\nabla v\|_{L_2(\mathcal{D})} \|\varphi\|_{H_0^1} \le N \|\varphi\|_{H_0^1}.$$

Hence the functional T is bounded on the Hilbert space $H_0^1(\mathcal{D})$. ■

Corollary 2.4 *Equation (2.2) can be reduced to a mapping equation:*

$$v - \lambda(Tv + F) = 0, \quad \lambda = 1/\nu,\ Tv,\ F \in H_0^1(\mathcal{D}). \tag{2.4}$$

Proof Since for fixed v, T is a bounded linear functional, Riesz's representation theorem guarantees there exists an element $Tv \in H_0^1$ such that $T(\varphi) = (Tv, \varphi)$, where $(\cdot, \cdot)$ is the inner product. The $\langle f_G, \varphi \rangle$ also define a linear functional, and its boundedness follows from (2.1). Hence there exists an element $F \in H_0^1$ such that $T_{f_G}(\varphi) = (F, \varphi)$. This gives $(\nu v - Tv - F, \varphi) = 0\ \forall \varphi \in H_0^1$. ■

3 Existence Theorem

Theorem 3.1 *Let $\mathcal{D} \subset \mathbb{R}^3$ be open with $\partial\mathcal{D} \in C^2$. Let $\kappa = \sigma d$ be the constant in Lemma 2.2, where $\sigma \in (0, 1]$ as in (1.2) and d = diameter of $\mathcal{D}$. Assume ν, σ and d satisfy the condition $4\sigma d^2 < \nu$. Then there exists a weak solution $U \in H^1(\mathcal{D})$ for the Dirichlet problem (1.3) and (1.4).*

Proof The weak form of (1.3) and (1.4) is equivalent to (2.2). According to Corollary 2.4, solvability of (2.2) is reduced to that of (2.4) in the Hilbert space H_0^1. We shall apply Leray-Schauder's fixed point theorem [LS] to prove the solvability. The proof proceeds in 2 steps.

Step 1. Compactness of the Operator T: Recall an operator A is compact on the Hilbert space if for every bounded sequence $\{v_n\} \subset H_0^1$, the sequence $\{Av_n\}$ has a convergent subsequence. It suffices to show the operators T_i, $i = 1, \dots, 4$ are compact, as T in (2.3) is a linear combination of them. We start with

$$|(T_1 v_m - T_1 v_n, \varphi)| = \left| 2\sigma \int_{\mathcal{D}} (v_m - v_n) \cdot \varphi \, dy \right| \le C \|v_m - v_n\|_{L_2(\mathcal{D})} \|\varphi\|_{H_0^1}.$$

By Rellich's selection theorem, the v_m converges strongly in $L_2(\mathcal{D})$. Setting $\varphi = T_1 v_m - T_1 v_n$, we thus obtain

$$\|T_1 v_m - T_1 v_n\|_{H_0^1} \le C' \|v_m - v_n\|_{L_2(\mathcal{D})} \to 0 \quad \text{as } m, n \to \infty.$$

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Similarly,

$$|(T_2 v_m - T_2 v_n, \varphi)| = \left| \sigma \int_{\mathcal{D}} (v_m - v_n) \cdot y \cdot \nabla \varphi \, dy \right| \le C \|v_m - v_n\|_{L_2(\mathcal{D})} \|\varphi\|_{H_0^1},$$

i.e., $\|T_2 v_m - T_2 v_n\|_{H_0^1} \le C' \|v_m - v_n\|_{L_2(\mathcal{D})} \to 0$ as $m, n \to \infty$.

$$|(T_3 v_m - T_3 v_n, \varphi)| = \left| \int_{\mathcal{D}} [v_m(v_m + G) - v_n(v_n + G)] \cdot \nabla \varphi \, dy \right|$$
$$\le \left\{ \int_{\mathcal{D}} |v_m(v_m + G) - v_n(v_n + G)|^2 \, dy \right\}^{1/2} \|\varphi\|_{H_0^1}.$$

We make use of the identity: $|v_m(v_m + G) - v_n(v_n + G)| = |(v_m - v_n)(v_m + G) + (v_m - v_n)v_n|$. By Lemma 2.1, the v_m also converges strongly in the $L_4(\mathcal{D})$ norm. Set $\varphi = T_3 v_m - T_3 v_n$. Then

$$\|T_3 v_m - T_3 v_n\|_{H_0^1} \le C \|v_m - v_n\|_{L_4(\mathcal{D})} \to 0 \quad \text{as } m, n \to \infty.$$

Finally,

$$|(T_4 v_m - T_4 v_n, \varphi)| = \left| \int_{\mathcal{D}} G \cdot (v_m - v_n) \cdot \nabla \varphi \, dy \right| \le C \|v_m - v_n\|_{L_2(\mathcal{D})} \|\varphi\|_{H_0^1},$$

i.e., $\|T_4 v_m - T_4 v_n\|_{H_0^1} \le C' \|v_m - v_n\|_{L_2(\mathcal{D})} \to 0$ as $m, n \to \infty$.

It has been shown that T is compact.

Step 2. Bound On $\|v\|_{H_0^1}$ ***:*** To show all possible solutions of (2.4) are uniformly bounded in $H_0^1(\mathcal{D})$, we return to (2.2) by putting $\varphi = v$. Noticing $\langle v, v + G, v \rangle = 0$, we get

$$\nu \|v\|_{H_0^1}^2 - 2\sigma \|v\|_{L_2}^2 - \sigma \langle v, y, v \rangle - \langle G, v, v \rangle = \langle f_G, v \rangle.$$

Here f_G is introduced in (2.2). Using Lemma 2.1, we estimate

$$\|v\|_{L_2}^2 \le d^2 \|v\|_{H_0^1}^2, \quad |\langle v, y, v \rangle| \le d^2 \|v\|_{H_0^1}^2, \quad |\langle f_G, v \rangle| \le d |f_G| \, \|v\|_{H_0^1}.$$

Taking $\kappa = \sigma d$ in Lemma 2.2 leads to

$$|\langle G, v, v \rangle| \le \sigma d^2 \|v\|_{H_0^1}^2.$$

Collecting these together, we have

$$(\nu - 4\sigma d^2) \|v\|_{H_0^1}^2 \le C \|v\|_{H_0^1}, \quad C = d |f_G|.$$

The assumption on smallness of either σ or d then yields the bound:

$$\|v\|_{H_0^1(\mathcal{D})} \le M, \quad M = C/(\nu - 4\sigma d^2).$$

We now appeal to the Leray-Schauder theorem. Since T is compact, then $S = T+F$ is also compact, where F in (2.4). Write (2.4) as $v = \lambda S v$, $\lambda \in [0, 1/\nu]$. It is true that $\|v\|_{H_0^1(\mathcal{D})} \le M \; \forall v \in H_0^1(\mathcal{D})$ satisfying the equation, so S has a fixed point. ∎

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4 Remarks

(a) By the regularity theory for the steady Navier-Stokes equations, it can be shown that the weak solution U is smooth (*cf.* [G]). The pressure P can be obtained by solving the divergence equation in the sense of distributions. The solution is unique if $(\nu - 4\sigma d^2)^2 > d^3|f_G|$; clearly from the definition of f_G in (2.2), the uniqueness requires the smallness of boundary data $\mathcal{G}(y)$. We do not know whether solutions proved in Theorem 3.1 are stable.

(b) For a meaningful singular solution to (1.1), its L_2-norm should be bounded. Suppose we are looking for a complete local self-similar solution, which has its spatial extension $x_0 = y_0\sqrt{2\sigma(t^* - t)}$. Using (1.2), we can formally write down the kinetic energy,

$$
\begin{aligned}
E &:= \frac{1}{2}\|u\|_{L_2(\mathbb{R}^3)} = \frac{1}{2}\int_{|x|\le x_0} |u|^2\,dx + \frac{1}{2}\int_{|x|>x_0} |u|^2\,dx \\
&= \sqrt{\frac{1}{2}\sigma(t^* - t)}\int_{|y|\le x_0/\sqrt{2\sigma(t^*-t)}} |U|^2\,dy + \frac{1}{2}\int_{|x|>x_0} |u|^2\,dx.
\end{aligned}
\tag{4.1}
$$

The first term on the r.h.s of (4.1) is the self-similar energy:

$$
\begin{aligned}
E_{\text{self-similar}} &:= \sqrt{\frac{1}{2}\sigma(t^* - t)}\int_{|y|\le x_0/\sqrt{2\sigma(t^*-t)}} |U|^2\,dy \\
&= \sqrt{\frac{1}{2}\sigma(t^* - t)}\Big(\int_{|y|\le d} |U|^2\,dy + \int_{d<|y|\le x_0/\sqrt{2\sigma(t^*-t)}} |U|^2\,dy\Big),
\end{aligned}
\tag{4.2}
$$

where d = diameter of the bounded domain $\mathcal{D}$. Let U be a solution as in Theorem 3.1, then we have in (4.2) $\int_{|y|\le d} |U|^2\,dy < K < +\infty$. So the self-similar energy in this central region tends to zero as $t \to t^*$ because of the factor $\sqrt{\frac{1}{2}\sigma(t^* - t)}$. This would imply that some energy is going back to the second integral in (4.2), and further to the second integral in (4.1). These two integrals have to be bounded at $t = t^*$ to guarantee the finiteness of the energy.

(c) For the existence, the inequality $4\sigma d^2 < \nu$ must hold for a fixed ν. Since σ in (1.2) is a free parameter, it takes arbitrary values in $(0, 1]$. Let $\sigma = \nu^2/4$. Now the inequality becomes $\nu d^2 < 1$. In this case for a turbulent flow with small ν, a finite d would satisfy the condition. One could think of the boundedness of d in the y-coordinate as a ball uniformly shrinking in the x-coordinate: the singular solutions appear to be very localised in the x-variable if they do occur.

(d) If $\mathcal{G}(y) = 0$ on $\partial\mathcal{D}$, then $f_G \equiv 0$. We would have a trivial solution in this case. This shows the prescribed data is necessary for proving the existence result, which supplies the necessary energy and vorticity into the central self-similar region. But our work does not show how to construct the imposed function $\mathcal{G}$, nor how it may be connected with an outer solution (suppose the outer flow is not self-similar). These interesting questions are left for future studies.

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