

AN EXPONENTIAL CONTINUOUS-TIME GARCH PROCESS

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Abstract

In this paper we introduce an exponential continuous-time GARCH(p, q) process. It is defined in such a way that it is a continuous-time extension of the discrete-time EGARCH(p, q) process. We investigate stationarity, mixing, and moment properties of the new model. An instantaneous leverage effect can be shown for the exponential continuous-time GARCH(p, p) model.

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1. Introduction

GARCH-type processes have become very popular in financial econometrics to model returns of stocks, exchange rates, and other series observed at equidistant time points. They have been designed (see [9] and [3]) to capture so-called *stylised facts* of such data, which are, for example, volatility clustering, dependence without correlation, and tail heaviness. Another characteristic is that stock returns seem to be negatively correlated with changes in the volatility, i.e. that volatility tends to increase after negative shocks and to decrease after positive shocks. This effect is called the *leverage effect* and cannot be modelled by a GARCH-type process without further extensions. This finding led Nelson [19] to introduce the exponential GARCH process, which is able to model this asymmetry in stock returns. The log-volatility of the EGARCH(p, q) process was modelled as an ARMA($q, p - 1$) process. Another popular model in this context is the linear ARCH process suggested by Robinson [21]. Here the volatility process is modelled as the square of an MA(∞) process driven by past observations. In [10] it was shown that the model contains, in addition to a long memory property, a leverage effect.

The availability of high-frequency data, which has increased enormously in the last few years, is one reason for considering continuous-time models with similar behaviour to discrete-time GARCH models. The reason for this is that at the highest available frequency, the observations of the price process occur at irregularly spaced time points and, therefore, it is natural to assume an underlying continuous-time model. Different approaches have been taken to set up a continuous-time model, which has the same features as discrete-time GARCH processes. Recently Klüppelberg *et al.* [13] developed a continuous-time GARCH(1, 1) model, abbreviated to a COGARCH(1, 1) model in the sequel. Their approach fundamentally differed from previous attempts, which could be summarised as diffusion approximations (see, e.g. [18]),

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by the fact that their model was driven by only one source of randomness (as in discrete-time GARCH models) instead of two (as in the diffusion approximations). They replaced the noise process of discrete-time GARCH models by the jumps of a Lévy process. The COGARCH(1, 1) model was then extended by Brockwell *et al.* [6] to a continuous-time GARCH(p, q) process for general orders $p, q \in \mathbb{N}, q \geq p$; henceforth called a COGARCH(p, q) process.

In this paper a continuous-time analogue of the EGARCH(p, q) model is introduced. The noise processes will also be modelled by the increments of a Lévy process. As in the discrete-time case, we describe the log-volatility process as a linear process, more precisely a continuous-time ARMA($q, p - 1$) process.

The paper is organised as follows. In Section 2 we review the definition of the discrete-time EGARCH process. After a short review of the elementary properties of Lévy processes, we define the exponential continuous-time GARCH(p, q) process at the beginning of Section 3. In addition, we state stationarity conditions for the log-volatility and volatility processes of our model. Afterwards, the leverage effect in our model is considered. We close Section 3 with an investigation of the mixing properties of the (log-)volatility and return processes. In Section 4 we derive second-order properties of the volatility process. Section 5 is devoted to the analysis of the second-order behaviour of the return process. We derive expressions for the first and second moment of the return process. The stylised fact of zero correlation in the return process, but correlation of the squared returns, is also shown.

2. The discrete-time EGARCH process

Motivated by empirical evidence that stock returns are negatively correlated with changes in returns volatility, Nelson [19] defined the exponential GARCH process (EGARCH) to model this effect, which is called the leverage effect (see also Section 3.1).

Definition 2.1. The process $(X_n)_{n \in \mathbb{Z}}$ of the form $X_n = \sigma_n \varepsilon_n, n \in \mathbb{Z}$, where $(\varepsilon_n)_{n \in \mathbb{Z}}$ is an independent and identically distributed (i.i.d.) sequence with $E(\varepsilon_1) = 0$ and $\text{var}(\varepsilon_1) = 1$, is called an *EGARCH process*, if the volatility process $(\sigma_n^2)_{n \in \mathbb{Z}}$ satisfies

$$\log(\sigma_n^2) = \mu + \sum_{k=1}^{\infty} \beta_k f(\varepsilon_{n-k}),$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ is some measurable real-valued deterministic function, $\mu \in \mathbb{R}$, and $(\beta_k)_{k \in \mathbb{N}}$ are real coefficients such that $E(|f(\varepsilon_n)|) < \infty, \text{var}(f(\varepsilon_n)) < \infty$, and $\sum_{k=1}^{\infty} |\beta_k| < \infty$.

Nelson [19] also suggested a finite-parameter model by modelling the log-volatility as an ARMA($q, p - 1$) process instead of an infinite-moving average process. This leads to the EGARCH(p, q) model, which is defined in the following way.

Definition 2.2. Let $p, q \in \mathbb{N}$ and $\mu, \alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p \in \mathbb{R}$. Suppose that $\alpha_q \neq 0, \beta_p \neq 0$, that the *autoregressive polynomial* $\phi(z) := 1 - \alpha_1 z - \dots - \alpha_q z^q$ and the *moving average polynomial* $\psi(z) := \beta_1 + \beta_2 z + \dots + \beta_p z^{p-1}$ have no common zeros, and that $\phi(z) \neq 0$ on $\{z \in \mathbb{C} \mid |z| \leq 1\}$. Let $(\varepsilon_n)_{n \in \mathbb{Z}}$ be an i.i.d. sequence with $E(\varepsilon_1) = 0$ and $\text{var}(\varepsilon_1) = 1$, and let $f(\cdot)$ be such that $E(|f(\varepsilon_n)|) < \infty$ and $\text{var}(f(\varepsilon_n)) < \infty$. Then $(X_n)_{n \in \mathbb{Z}}$, where $X_n = \sigma_n \varepsilon_n$ and

$$\log(\sigma_n^2) = \mu + \sum_{k=1}^p \beta_k f(\varepsilon_{n-k}) + \sum_{k=1}^q \alpha_k \log(\sigma_{n-k}^2),$$

is called an EGARCH(p, q) process.

To achieve the asymmetric relation between the stock returns and the volatility, $f(\varepsilon_n)$ must be a function of the magnitude and the sign of ε_n , as noted by Nelson [19]. Therefore, he proposed the following function:

$$f(\varepsilon_n) := \theta\varepsilon_n + \gamma(|\varepsilon_n| - E(|\varepsilon_n|)),$$

with real coefficients θ and γ . We see that $f(\varepsilon_n)$ is piecewise linear in ε_n and has slope $\theta + \gamma$ for positive shocks ε_n and slope $\theta - \gamma$ for negative shocks. Therefore, $f(\varepsilon_n)$ allows the volatility process $(\sigma_n^2)_{n \in \mathbb{Z}}$ to respond asymmetrically to positive and negative jumps in the stock price.

3. The exponential COGARCH process

The goal of this section is to construct a continuous-time analogue of the discrete-time EGARCH(p, q) process. Therefore, we will use the idea of Klüppelberg *et al.* [13] to replace the noise variables ε_n by the increments of a Lévy process $L = (L_t)_{t \geq 0}$. Any Lévy process L on \mathbb{R} has a characteristic function of the form $E(\exp(iuL_t)) = \exp\{t\psi_L(u)\}$, $t \geq 0$, with

$$\psi_L(u) := i\gamma_L u - \frac{\tau_L^2}{2} u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\chi_{(-1,1)}(x))\nu_L(dx), \quad u \in \mathbb{R},$$

where $\tau_L^2 \geq 0$, $\gamma_L \in \mathbb{R}$, the measure ν_L satisfies

$$\nu_L(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min(x^2, 1)\nu_L(dx) < \infty,$$

and $\chi_A(\cdot)$ denotes the indicator function of the set $A \subset \mathbb{R}$. The measure ν_L is called the *Lévy measure* of L , and the triplet $(\gamma_L, \tau_L^2, \nu_L)$ is called the *characteristic triplet* of L . The map ψ_L is called the *Lévy symbol*. For more details on Lévy processes, we refer the reader to [1] or [22].

We consider Lévy processes L defined on a probability space (Ω, \mathcal{F}, P) with jumps $\Delta L_t := L_t - L_{t-}$, zero mean, and finite variance. In this case the Lévy–Itô decomposition (see, e.g. [1, Theorem 2.4.16]) of L is

$$L_t = B_t + \int_{\mathbb{R} \setminus \{0\}} x \tilde{N}_L(t, dx), \quad t \geq 0,$$

where B is a Brownian motion with variance τ_L^2 , and $\tilde{N}_L(t, dx) = N_L(t, dx) - t\nu_L(dx)$, $t \geq 0$, is the compensated random measure associated to the Poisson random measure

$$N_L(t, A) = \#\{0 \leq s < t; \Delta L_s \in A\} = \sum_{0 < s \leq t} \chi_A(\Delta L_s), \quad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}),$$

on $\mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$, which is independent of B .

The driving-noise process of the log-volatility in this continuous-time model will be constructed similarly to the discrete-time case. In particular, for a zero-mean Lévy process L , with $E(L_1^2) < \infty$ and parameters $(\theta, \gamma)^\top \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, we define the driving process M of the log-volatility process by

$$M_t := \int_{\mathbb{R} \setminus \{0\}} h(x)\tilde{N}_L(t, dx), \quad t \geq 0, \tag{3.1}$$

with $h(x) := \theta x + \gamma|x|$.

Remark 3.1. (i) The process M defined by (3.1) is by construction a process with independent and stationary increments and by Theorem 4.3.4 of [1] well defined if

$$\int_{\mathbb{R}} |h(x)|^2 \nu_L(dx) < \infty. \tag{3.2}$$

Condition (3.2) is satisfied since ν_L is a Lévy measure and L has finite variance. By Equation (2.9) of [1], the characteristic triplet of M is $(\gamma_M, 0, \nu_M)$, where $\nu_M := \nu_L \circ h^{-1}$ is the Lévy measure of M and $\gamma_M := -\int_{|x|>1} x \nu_M(dx)$. The precise form of ν_M depends on the sign and size of θ and γ , and is given in the following formulae:

$$\nu_M((-\infty, -x]) = \begin{cases} \nu_L\left(\left[-\frac{x}{\theta + \gamma}, \infty\right)\right) + \nu_L\left(\left(-\infty, -\frac{x}{\theta - \gamma}\right]\right), & -\gamma > \theta > \gamma, \\ \nu_L\left(\left(-\infty, -\frac{x}{\theta - \gamma}\right]\right), & -\theta < \gamma < \theta, \\ \nu_L\left(\left[-\frac{x}{\theta + \gamma}, \infty\right)\right), & -\theta > \gamma > \theta, \\ 0, & -\gamma < \theta < \gamma, \end{cases}$$

and

$$\nu_M([x, \infty)) = \begin{cases} \nu_L\left(\left[\frac{x}{\theta + \gamma}, \infty\right)\right) + \nu_L\left(\left(-\infty, \frac{x}{\theta - \gamma}\right]\right), & -\gamma < \theta < \gamma, \\ \nu_L\left(\left(-\infty, \frac{x}{\theta - \gamma}\right]\right), & -\theta > \gamma > \theta, \\ \nu_L\left(\left[\frac{x}{\theta + \gamma}, \infty\right)\right), & -\theta < \gamma < \theta, \\ 0, & -\gamma > \theta > \gamma, \end{cases}$$

for $x > 0$. We recognise that M is a spectrally negative Lévy process for $\gamma < \theta < -\gamma$, i.e. M has only negative jumps, and a spectrally positive Lévy process for $-\gamma < \theta < \gamma$.

(ii) In case the jump part of L is of finite variation, M is a Lévy process of finite variation with Lévy–Itô decomposition

$$M_t := \sum_{0 < s \leq t} [\theta \Delta L_s + \gamma |\Delta L_s|] - Ct, \quad t > 0,$$

where $C := \gamma \int_{\mathbb{R}} |x| \nu_L(dx)$.

Now we define the exponential continuous-time GARCH(p, q) process by specifying the log-volatility process as a continuous-time ARMA($q, p - 1$) process; henceforth called a CARMA($q, p - 1$) process (see, e.g. [5] for details on CARMA processes), which is the continuous-time analogue of an ARMA($q, p - 1$) process.

Definition 3.1. Let $L = (L_t)_{t \geq 0}$ be a zero-mean Lévy process with Lévy measure ν_L such that $\int_{|x| \geq 1} x^2 \nu_L(dx) < \infty$. Then we define the exponential COGARCH(p, q) process G , abbreviated to ECOGARCH(p, q), as the stochastic process satisfying

$$dG_t := \sigma_{t-} dL_t, \quad t > 0, G_0 = 0,$$

where the log-volatility process $\log(\sigma^2) = (\log(\sigma_t^2))_{t \geq 0}$ is a CARMA($q, p - 1$) process, $1 \leq p \leq q$, with mean $\mu \in \mathbb{R}$ and state space representation

$$\log(\sigma_t^2) := \mu + \mathbf{b}^\top \mathbf{X}_t, \quad t \geq 0, \tag{3.3}$$

$$d\mathbf{X}_t = \mathbf{A}\mathbf{X}_t dt + \mathbf{1}_q dM_t, \quad t > 0, \tag{3.4}$$

where $\mathbf{X}_0 \in \mathbb{R}^q$ is independent of the driving Lévy process M . The $q \times q$ matrix \mathbf{A} and the vectors $\mathbf{b} \in \mathbb{R}^q$ and $\mathbf{1}_q \in \mathbb{R}^q$ are defined by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_q & -a_{q-1} & -a_{q-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{q-1} \\ b_q \end{bmatrix}, \quad \mathbf{1}_q = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

with coefficients $a_1, \dots, a_q, b_1, \dots, b_p \in \mathbb{R}$, where $a_q \neq 0, b_p \neq 0$, and $b_{p+1} = \dots = b_q = 0$.

Returns over a time interval of length $r > 0$ are described by the increments of G :

$$G_t^{(r)} := G_t - G_{t-r} = \int_{(t-r, t]} \sigma_{s-} dL_s, \quad t \geq r > 0. \tag{3.5}$$

Thus, this gives us the possibility to model ultra-high-frequency data, which consists of returns over varying time intervals. Conversely, an equidistant sequence of such nonoverlapping returns of length r is given by $(G_{nr}^{(r)})_{n \in \mathbb{N}}$.

In the sequel we refer to G and $G^{(r)}$ as the *(log-)price process* and the *(log-)return process*, respectively. Also, we call σ^2 and $\log(\sigma^2)$ the *volatility process* and the *log-volatility process*, respectively.

Remark 3.2. ([5, Proposition 2].) (i) As the log-volatility process is a CARMA($q, p - 1$) process, it is strictly stationary if the eigenvalues of \mathbf{A} all have negative real parts and \mathbf{X}_0 has the same distribution as $\int_0^\infty e^{Au} \mathbf{1}_q dM_u$. If σ^2 is as defined in Definition 3.1 then σ^2 is also strictly stationary, as this property is preserved under measurable mappings.

(ii) The solution of the continuous-time state-space model, (3.3) and (3.4), has the representation

$$\log(\sigma_t^2) = \mu + \mathbf{b}^\top e^{At} \mathbf{X}_0 + \int_0^t \mathbf{b}^\top e^{A(t-u)} \mathbf{1}_q dM_u, \quad t > 0.$$

If we choose a second Lévy process, $(\tilde{L}_t)_{t \geq 0}$, independent of L and with the same distribution as L , then we can define an extension $(L_t^*)_{t \in \mathbb{R}}$ of L to the real line by

$$L_t^* := L_t \chi_{[0, \infty)}(t) - \tilde{L}_{-t} \chi_{(-\infty, 0)}(t), \quad t \in \mathbb{R}.$$

Using L^* instead of L in (3.1), we obtain an extension M^* of M . For simplicity, in the following we will write L and M instead of L^* and M^* . In the strictly stationary case the log-volatility process can be defined on the whole real line by

$$\log(\sigma_t^2) = \mu + \int_{-\infty}^t g(t-u) dM_u, \quad t \in \mathbb{R}, \tag{3.6}$$

with kernel function

$$g(t) = \mathbf{b}^\top e^{At} \mathbf{1}_q \chi_{(0, \infty)}(t); \tag{3.7}$$

see [5, Section 2] for further details.

(iii) From (3.5) it directly follows that the increments $G^{(r)} = \int_{(t-r, t]} \sigma_{s-} dL_s$ of G are stationary if the volatility σ^2 is stationary, as the increments of L are stationary and independent by definition.

(iv) If $q \geq p + 1$, the log-volatility process is $(q - p - 1)$ times differentiable, which follows from the state-space representation of $\log(\sigma^2)$ and, hence, the volatility process has continuous sample paths. In particular, the volatility will contain jumps only for $p = q$.

3.1. The leverage effect

For empirical return data, researchers have found evidence that (see, e.g. [19, Section 1]) current returns are negatively correlated with future volatility. We distinguish two cases. Either a negative jump in the log-price process G increases the future volatility more than a positive jump of the same size (compare to (i), below, with $-\gamma < \theta < 0$), or a negative jump in G increases the future volatility while a positive jump even decreases it (compare to (iv), below). This phenomenon is called the leverage effect in the literature.

If we take a look at the jumps of the state process \mathbf{X} in the ECOGARCH(p, q) model,

$$\Delta M_t = \begin{cases} (\theta + \gamma)\Delta L_t, & \Delta L_t \geq 0, \\ (\theta - \gamma)\Delta L_t, & \Delta L_t < 0, \end{cases}$$

we see that

- (i) for $-\gamma < \theta < 0$ ($0 < \theta < \gamma$), a positive jump ΔL_t leads to a smaller (greater) positive jump ΔM_t compared to a negative jump ΔL_t of the same size;
- (ii) for $\theta > |\gamma|$, a positive jump ΔL_t leads to a positive jump ΔM_t , while a negative jump of the same size results in a negative jump ΔM_t ;
- (iii) for $0 < \theta < -\gamma$ ($\gamma < \theta < 0$), a positive jump ΔL_t leads to a smaller (greater) negative jump ΔM_t compared to a negative jump ΔL_t of the same size;
- (iv) for $\theta < -|\gamma|$, a positive jump ΔL_t leads to a negative jump ΔM_t , while a negative jump of the same size results in a positive jump ΔM_t .

If we compare this to the COGARCH(p, q) process, we see that in the COGARCH model the innovations of the volatility process at time t are given by the squared innovations of the log-price process (see [6, Section 2]). Hence, the volatility process of the COGARCH model reacts in the same way to positive and negative jumps. Now we will consider the *instantaneous leverage effect*, which is defined as

$$\text{cov}(\Delta G_t, \sigma_t^2 \mid |\Delta L_t| > \varepsilon)$$

being negative. Intuitively it is clear that this correlation can only be different from 0 if the sample paths of σ^2 exhibit jumps. But, from Remark 3.2(iv) we know that this is the case only for $p = q$. The reason is that, for $p < q$, the parameter b_q will be 0 and, therefore, the jump ΔL_t at time t contributes only to the $(q - 1)$ th derivative of the state process \mathbf{X} , but is not taken into account for the log-volatility at that time point. Thus, we will expect an instantaneous

leverage effect only for the ECOGARCH(p, p) models. This will be shown in Proposition 3.1, below; in particular, we will show that the sign of the correlation is equal to the sign of θb_q . This result is similar to the discrete-time case; see [23, Proposition 2.9].

Proposition 3.1. *Assume that the distribution of the jumps of L is symmetric, i.e. for all $\varepsilon > 0$,*

$$P(\Delta L_t \in dx \mid |\Delta L_t| > \varepsilon) = P(\Delta L_t \in -dx \mid |\Delta L_t| > \varepsilon), \quad t \geq 0.$$

Conditionally on the event that $|\Delta L_t| > \varepsilon$, the sign of $\text{cov}(\Delta G_t, \sigma_t^2)$ is equal to the sign of θb_q .

Proof. Since the distribution of the jumps of L is symmetric, we obtain

$$E(\Delta G_t \mid |\Delta L_t| > \varepsilon) = E(\sigma_{t-}) E(\Delta L_t \mid |\Delta L_t| > \varepsilon) = 0.$$

This then implies that

$$\begin{aligned} \text{cov}(\Delta G_t, \sigma_t^2 \mid |\Delta L_t| > \varepsilon) &= E(\Delta G_t \sigma_t^2 \mid |\Delta L_t| > \varepsilon) \\ &= E(\Delta G_t \exp(\log(\sigma_{t-}^2) + b_q \Delta M_t) \mid |\Delta L_t| > \varepsilon) \\ &= E(\sigma_{t-}^3 \Delta L_t \exp(b_q(\theta \Delta L_t + \gamma |\Delta L_t|)) \mid |\Delta L_t| > \varepsilon). \end{aligned}$$

Since ΔL_t is independent of σ_{t-}^3 , we obtain

$$\begin{aligned} \text{cov}(\Delta G_t, \sigma_t^2 \mid |\Delta L_t| > \varepsilon) &= E(\sigma_{t-}^3) E(\Delta L_t \exp(b_q(\theta \Delta L_t + \gamma |\Delta L_t|)) \mid |\Delta L_t| > \varepsilon) \\ &= E(\sigma_{t-}^3) \int_{x>\varepsilon} x \exp(b_q \gamma x) (\exp(\theta b_q x) - \exp(-\theta b_q x)) P(\Delta L_t \in dx \mid |\Delta L_t| > \varepsilon). \end{aligned}$$

From the fact that $\text{sgn}(\exp(\theta b_q x) - \exp(-\theta b_q x)) = \text{sgn}(\theta b_q)$ for all $x > \varepsilon$, the desired result follows.

From Proposition 3.1 we see that the existence of an instantaneous leverage effect depends only on the parameters θ and b_q . Hence, we will have a leverage effect for all orders $p \geq 1$ as long as the condition on θ and b_q is satisfied. In simulations of the ECOGARCH(p, p) process we see similar results concerning the leverage effect for different orders $p \geq 1$. As an example, we will now consider an ECOGARCH(1, 1) process.

Example 3.1. In this example we consider an ECOGARCH(1, 1) process driven by a Lévy process L with Lévy symbol

$$\psi_L(u) = -\frac{u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1) \lambda \Phi_{0,1/\lambda}(dx),$$

where $\Phi_{0,1/\lambda}(\cdot)$ is the distribution function of a normal distribution with zero mean and variance $1/\lambda$. This means that L is the sum of a standard Brownian motion W and the compound Poisson process $J_t = \sum_{k=1}^{N_t} Z_k$, $t \geq 0$, where $(N_t)_{t \in \mathbb{R}}$ is an independent Poisson process with intensity $\lambda > 0$ and jump times $(T_k)_{k \in \mathbb{Z}}$. The Poisson process N is also independent from the i.i.d. sequence of jump sizes $(Z_k)_{k \in \mathbb{Z}}$, with $Z_1 \sim N(0, 1/\lambda)$. The Lévy process M is, in this case,

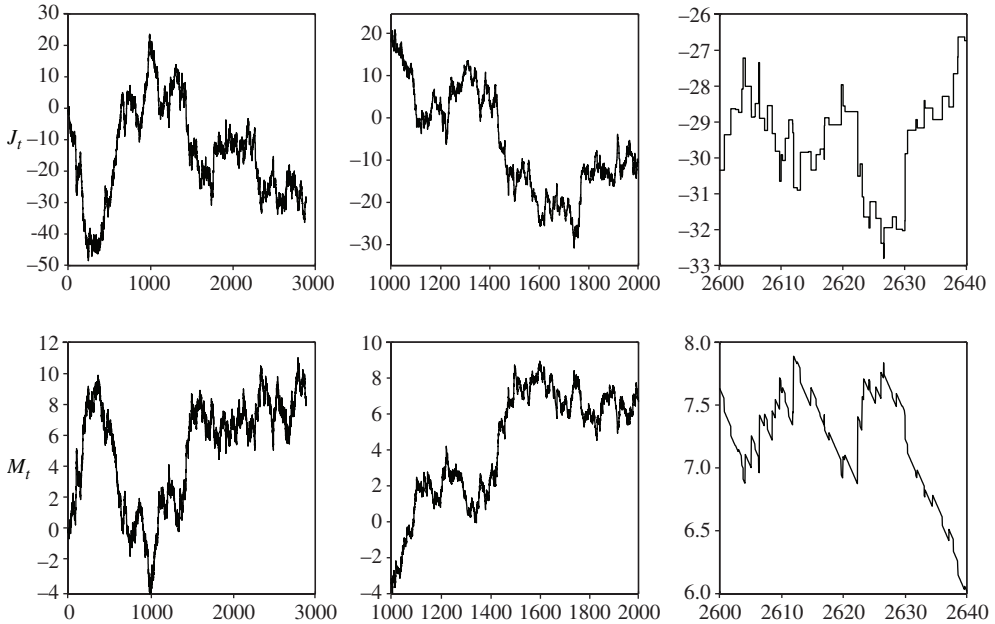


FIGURE 1: Simulated sample paths of J (top row) and M (bottom row), with parameters $\theta = -0.2$ and $\gamma = 0.1$, over three different time scales.

given by the following expression:

$$M_t = \sum_{k=1}^{N_t} [\theta Z_k + \gamma |Z_k|] - Ct, \quad t > 0,$$

with

$$C = \gamma \int_{\mathbb{R}} |x| \lambda \Phi_{0,1/\lambda}(dx) = \gamma \sqrt{\frac{2\lambda}{\pi}}.$$

If we just consider the case in which $\theta < -\gamma < 0$ then the Lévy measure ν_M of M is defined by

$$\nu_M((-\infty, -x]) = \lambda \Phi_{0,1/\lambda} \left(\left[-\frac{x}{\theta + \gamma}, \infty \right) \right), \quad x > 0,$$

on the negative half real line and by

$$\nu_M([x, \infty)) = \lambda \Phi_{0,1/\lambda} \left(\left(-\infty, \frac{x}{\theta - \gamma} \right] \right), \quad x > 0,$$

on the positive half real line. In the top row of Figure 1 a simulated sample path of the compound Poisson process J , with $N(0, \frac{1}{2})$ distributed jumps, can be seen over three time scales. The corresponding Lévy process M , with parameters $\theta = -0.2$ and $\gamma = 0.1$, can be seen in the bottom row. Over all three time intervals we can recognise the desired asymmetry for this set of parameters. If J jumps up then M jumps down and vice versa. If J does not move then we observe the downwards drift of M , which can be seen on the right-hand side of Figure 1.

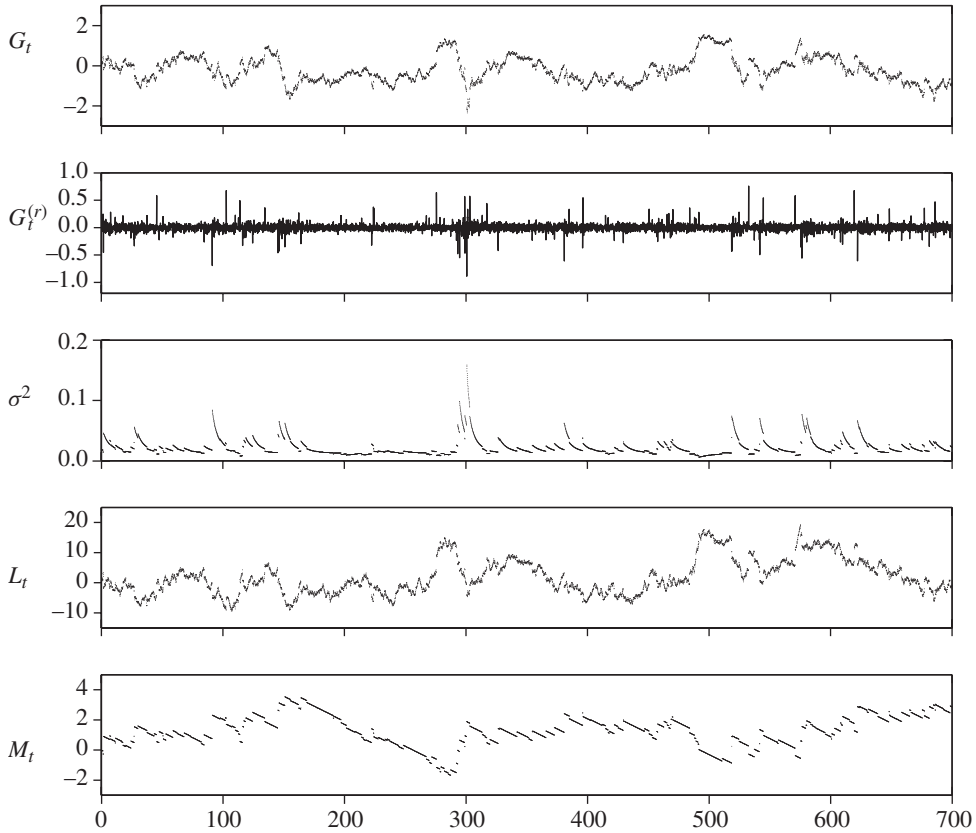


FIGURE 2: Observations of the log-price process G_t (top row), the return process $G_t^{(r)}$ (second row), the volatility process σ_t^2 (third row), with parameters $b_1 = 1, a_1 = 0.1, \mu = -4, \theta = -0.2$, and $\gamma = 0.1$, the driving Lévy process L_t (fourth row), and the Lévy process M_t (bottom row) in the time interval $(0, 700]$.

The log-volatility process is then of the form

$$\begin{aligned} \log(\sigma_t^2) &= \mu + b_1 \exp(-a_1 t) X_0 + \int_0^t b_1 \exp(-a_1(t-s)) dM_s \\ &= \mu + b_1 \exp(-a_1 t) X_0 + \sum_{k=1}^{N_t} b_1 \exp(-a_1(t-T_k)) [\theta Z_k + \gamma |Z_k|] \\ &\quad - C \frac{b_1}{a_1} (1 - \exp(-a_1 t)) \quad \text{for } t > 0, \end{aligned}$$

and the log-price process is given by

$$G_t = \int_0^t \sigma_{s-} dW_s + \sum_{k=1}^{N_t} \sigma_{T_k-} Z_k, \quad t > 0, \quad G_0 = 0,$$

with jump times $T_k, k \in \mathbb{N}$.

Generally the simulation of a sample path of the log-price process, G , and the log-volatility process, $\log(\sigma^2)$, over a time interval $[0, T]$ is carried out in the following steps.

1. Choose observation times $0 = t_0 < t_1 < \dots < t_n \leq T$, possibly random.
2. Simulate the jump times (T_k) , $k = 1, \dots, n_T$, with $n_T := \max\{k \in \mathbb{N} : T_k \leq T\}$, of the compound Poisson process J .
3. Approximate the state process (3.4) of the log-volatility by a stochastic Euler scheme.
4. Compute an approximation \widehat{G} via the recursion

$$\widehat{G}_{t_i} = \widehat{G}_{t_{i-1}} + \sqrt{\exp(\mu + \mathbf{b}^\top \widehat{\mathbf{X}}_{t_{i-1}})} \widetilde{W}_i + \sum_{k=N_{t_{i-1}}+1}^{N_{t_i}} \sqrt{\exp(\mu + \mathbf{b}^\top \widehat{\mathbf{X}}_{T_{k-}})} Z_k,$$

where $\widetilde{W}_i \sim N(0, t_i - t_{i-1})$, and $\widehat{\mathbf{X}}_{T_{k-}}$ is the Euler approximation without the jump ΔM_{T_k} .

In Figure 2 the results of the above simulation procedure are shown. The jump rate λ is now chosen to be $\frac{1}{4}$, which implies a variance of the jump sizes Z_i of 4. For i.i.d. exponentially distributed interarrival times $\Delta t_i := t_i - t_{i-1}$, with $E(\Delta t_i) = 1$, the sample path of the log-price G , the return process $G^{(\Delta t)}$, and the volatility process σ^2 are displayed in the first three rows of Figure 2. The sample paths of the Lévy processes L and M are shown in the bottom two rows. From the plots of the return and volatility processes, we see the negative correlation between the two processes. On the one hand, we recognise increases in the volatility after large negative returns, and on the other hand, we recognise a decrease in the volatility after a larger positive return. This displays the leverage effect explained in Subsection 3.1.

3.2. Mixing

Mixing properties are useful for a number of applications including asymptotic statistics, since a central limit theorem holds for mixing processes (see [8] for a comprehensive treatment of mixing properties). For an application in this continuous-time GARCH setting, see, e.g. Theorem 3 of Haug *et al.* [12]. Thus, we will derive mixing properties of the strictly stationary volatility process and the return process in the ECOGARCH(p, q) model.

First we recall the definition of strong mixing, which is also called α -mixing for a process with a continuous-time parameter.

Definition 3.2. ([7].) For a process $Y = (Y_s)_{s \geq 0}$, define the σ -algebras $\mathcal{F}_{[0,u]}^Y := \sigma((Y_s)_{s \in [0,u]})$ and $\mathcal{F}_{[u+t,\infty)}^Y := \sigma((Y_s)_{s \geq u+t})$ for all $u \geq 0$. Then Y is called strongly or α -mixing if

$$\alpha_Y(t) := \sup_{u \geq 0} \alpha(\mathcal{F}_{[0,u]}^Y, \mathcal{F}_{[u+t,\infty)}^Y) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where $\alpha(\mathcal{F}_{[0,u]}^Y, \mathcal{F}_{[u+t,\infty)}^Y) := \sup\{ |P(A \cap B) - P(A)P(B)| : A \in \mathcal{F}_{[0,u]}^Y, B \in \mathcal{F}_{[u+t,\infty)}^Y \}$.

Above we denote by $\sigma(\cdot)$ the generated completed σ -algebra. The strong mixing property with exponential rate of the log-volatility, volatility, and return processes is the subject of Theorem 3.1, below. Here strong mixing with exponential rate (exponentially α -mixing) means that the α -mixing coefficient $\alpha_Y(t)$ decays to 0 exponentially fast as t tends to ∞ .

Theorem 3.1. *Let $\log(\sigma^2)$ be as defined by (3.3) and (3.4) with both θ and γ not equal to 0. Assume that $E(L_1^2) < \infty$, that the eigenvalues of A all have negative real parts, and that X_0 has the same distribution as $\int_0^\infty e^{Au} \mathbf{1}_q dM_u$; hence, $\log(\sigma^2)$ and σ^2 are strictly stationary.*

(i) *Then there exist constants $K > 0$ and $a > 0$ such that*

$$\alpha_{\log(\sigma^2)}(t) \leq K e^{-at} \quad \text{and} \quad \alpha_{\sigma^2}(t) \leq K e^{-at} \quad \text{as } t \rightarrow \infty,$$

where $\alpha_{\log(\sigma^2)}(t)$ and $\alpha_{\sigma^2}(t)$ are the α -mixing coefficients of the log-volatility and volatility processes, respectively.

(ii) *Then the discrete-time process $(G_{nr}^{(r)})_{n \in \mathbb{N}}$, where $G_{nr}^{(r)}$ is defined in (3.5), is strongly mixing with exponential rate and ergodic.*

Proof. (i) The log-volatility process is a CARMA($q, p - 1$) process, which is equal to the first component of the q -dimensional Ornstein–Uhlenbeck process $V := (V^1, \dots, V^q)^\top \in \mathbb{R}^q$ (see, e.g. [4, Section 4]), where, for fixed t ,

$$V_t = \exp(\mathbf{B} \mathbf{A} \mathbf{B}^{-1}(t - s)) V_s + \int_s^t \exp(\mathbf{A}(t - u)) \mathbf{B} \mathbf{1}_q dM_u \quad \text{a.s.,}$$

with

$$\mathbf{B} = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_q \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Since L , hence M , has finite second moment, V also has finite second moment. Therefore, condition (4.5) of [15] is satisfied. By Theorem 4.3 of [15], V is then exponentially α -mixing. Since every component of a multidimensional exponentially strong mixing process is exponentially strong mixing, the log-volatility process is also exponentially α -mixing. The property of α -mixing is invariant under continuous transformations, which implies that σ^2 also has this property.

(ii) Define the σ -algebras

$$\mathcal{F}_I^{\sigma^2, dL} := \sigma(\sigma_t^2, L_t - L_s : s, t \in I) \quad \text{for } I \subset \mathbb{R}$$

and

$$\mathcal{F}_J^{G^{(r)}} := \sigma(G_{kr}^{(r)} : k \in J) \quad \text{for } J \subset \mathbb{N}.$$

From (3.5) it follows that

$$\mathcal{F}_{\{1, 2, \dots, l\}}^{G^{(r)}} \subset \mathcal{F}_{[0, lr]}^{\sigma^2, dL} \quad \text{and} \quad \mathcal{F}_{\{k+l, k+l+1, \dots\}}^{G^{(r)}} \subset \mathcal{F}_{[(k+l-1)r, \infty)}^{\sigma^2, dL}. \tag{3.8}$$

To show the strong mixing property of the return process we will use the following relation:

$$\alpha(\mathcal{F}_1, \mathcal{F}_2) \leq \tilde{\alpha}(\mathcal{F}_1, \mathcal{F}_2) \leq 6\alpha(\mathcal{F}_1, \mathcal{F}_2), \tag{3.9}$$

where \mathcal{F}_1 and \mathcal{F}_2 are σ -algebras,

$$\tilde{\alpha}(\mathcal{F}_1, \mathcal{F}_2) := \sup\{\|E(f \mid \mathcal{F}_1) - E(f)\|_{L^1(P)} : f \in b\mathcal{F}_2, \|f\|_\infty \leq 1\},$$

$\|\cdot\|_{L^1(P)}$ and $\|\cdot\|_\infty$ denote the L^1 -norm under P and the sup-norm, respectively, and $b\mathcal{F}$ denotes the set of all bounded \mathcal{F} -measurable random variables. The left-hand inequality of (3.9) is easy to see (cf. [12, Lemma B.2]) and the right-hand inequality follows from Lemma 3.5 of [17]. For a stochastic process Y , the corresponding $\tilde{\alpha}$ -mixing coefficient is defined as

$$\tilde{\alpha}_Y(t) := \sup_{s \in \mathbb{R}_+} \tilde{\alpha}(\mathcal{F}_{[0,s]}^Y, \mathcal{F}_{[s+t,\infty)}^Y), \quad t \in \mathbb{R}_+;$$

see, e.g. [16, Section 2.1]. Now since $(G_{nr}^{(r)})_{n \in \mathbb{N}}$ is strictly stationary, we have the following:

$$\begin{aligned} \tilde{\alpha}_{G^{(r)}}(k-l) &= \sup\{\|E(f \mid \mathcal{F}_{\{1,2,\dots,l\}}^{G^{(r)}}) - E(f)\|_{L^1(P)} : f \in b\mathcal{F}_{\{k,k+1,\dots\}}^{G^{(r)}}, \|f\|_\infty \leq 1\} \\ &\leq \sup\{\|E(f \mid \mathcal{F}_{[0,lr]}^{\sigma^2, dL}) - E(f)\|_{L^1(P)} : f \in b\mathcal{F}_{[(k-1)r,\infty)}^{\sigma^2, dL}, \|f\|_\infty \leq 1\}, \end{aligned}$$

where the inequality follows from (3.8) and an application of Jensen’s inequality (see also [16, Remark 1]). From the exponentially α -mixing property of σ^2 and (3.9), we find that there exists a constant $K_{\sigma^2} > 0$ such that

$$\tilde{\alpha}_{\sigma^2}(t-s) = \sup\{\|E(f \mid \mathcal{F}_{[0,s]}^{\sigma^2}) - E(f)\|_{L^1(P)} : f \in b\mathcal{F}_{[t,\infty)}^{\sigma^2, dL}, \|f\|_\infty \leq 1\} \leq K_{\sigma^2} e^{-a(t-s)}$$

for all $0 \leq s \leq t < \infty$ and $\|f\|_\infty \leq 1$. Now analogously to the proof of Lemma 1 of [14], it follows that

$$\|E(f \mid \mathcal{F}_{[0,lr]}^{\sigma^2, dL}) - E(f)\|_{L^1(P)} \leq K_{\sigma^2} e^{-a((k-1-l)r)} \|f\|_\infty$$

for all $f \in b\mathcal{F}_{[(k-1)r,\infty)}^{\sigma^2, dL}$. The only difference is that we do not have a Markov process; hence, we have to condition on the information over the whole time interval $[0, lr]$ and not just on the information at the time point lr . This implies that we have

$$\tilde{\alpha}_{G^{(r)}}(k-l) \leq K_{\sigma^2} e^{-a((k-1-l)r)},$$

which means that $(G_{nr}^{(r)})_{n \in \mathbb{N}}$ is exponentially α -mixing by (3.9). Since strict stationarity and strong mixing imply ergodicity the result follows.

4. Second-order properties of the volatility process

In this section we derive moments and the autocovariance function of the volatility process σ^2 . Since it is a nonlinear transformation of a CARMA($q, p-1$) process, we will first recall the moment structure and conditions for weak stationarity of a CARMA($q, p-1$) process.

Proposition 4.1. *If X_0 has the same mean vector and covariance matrix as $\int_0^\infty e^{Au} \mathbf{1}_q dM_u$, then $\log(\sigma^2)$ is weakly stationary. In the weakly stationary case the mean and autocovariance function of $\log(\sigma^2)$ are given by*

$$E(\log(\sigma_t^2)) = \mu \quad \text{and} \quad \text{cov}(\log(\sigma_{t+h}^2), \log(\sigma_t^2)) = E(M_1^2) \mathbf{b}^\top e^{Ah} \Sigma \mathbf{b}, \quad t, h \geq 0,$$

where

$$\Sigma := \int_0^\infty e^{As} \mathbf{1}_q \mathbf{1}_q^\top \exp(\mathbf{A}^\top s) ds.$$

The results follow from Proposition 1 and Remark 4 of [5] and the fact that $\int_{\mathbb{R}} g(u - h) \times g(u) du = \mathbf{b}^\top e^{A h} \Sigma \mathbf{b}$, with g as defined in (3.7).

The moments of the strictly stationary volatility process are exponential moments of the stationary distribution of the log-volatility process. In Remark 3.2(i) we gave conditions for the existence of a stationary distribution F of the log-volatility process. In Proposition 4.2, below, we want to further characterise this distribution.

Proposition 4.2. *Let $(\gamma_M, 0, \nu_M)$ be the characteristic triplet of the Lévy process M , where M is as defined in (3.1), and F is the stationary distribution of the log-volatility process. Then F is infinitely divisible with characteristic triplet $(\gamma_\infty, 0, \nu_\infty)$, where*

$$\begin{aligned} \gamma_\infty &= \mu + \int_0^\infty g(s) \gamma_M ds + \int_0^\infty \int_{\mathbb{R}} g(s)x [\chi_{(-1,1)}(g(s)x) - \chi_{(-1,1)}(x)] \nu_M(dx) ds, \\ \nu_\infty(B) &= \int_0^\infty \int_{\mathbb{R}} \chi_B(g(s)x) \nu_M(dx) ds, \quad B \in \mathcal{B}(\mathbb{R}), \end{aligned}$$

with $g(s) = \mathbf{b}^\top e^{As} \mathbf{1}_q \chi_{(0,\infty)}(s)$.

Proof. In the strictly stationary case the log-volatility process is the continuous-time moving average process (3.6). Since M has finite variance, the kernel g and the driving Lévy process M satisfy the conditions in Theorem 2.7 of [20] which are:

- $\int_{\mathbb{R}} |\gamma_M g(s) + \int_{\mathbb{R}} x g(s) [\chi_{(-1,1)}(x g(s)) - \chi_{(-1,1)}(x)] \nu_M(dx)| ds < \infty$,
- $\int_{\mathbb{R}} \int_{\mathbb{R}} \min(|g(s)x|^2, 1) \nu_M(dx) ds < \infty$.

Therefore, the stationary distribution F of the log-volatility process is infinitely divisible with characteristic triplet $(\gamma_\infty, 0, \nu_\infty)$.

Let $\log(\sigma_\infty^2)$ be a random variable with distribution F . Since F is infinitely divisible, we can now apply Theorem 25.17 of [22] to calculate the exponential moments of $\log(\sigma_\infty^2)$, i.e. the moments of σ_∞^2 , in Proposition 4.3, below.

Proposition 4.3. *Let F be the stationary distribution of $\log(\sigma^2)$ with characteristic triplet $(\gamma_\infty, 0, \nu_\infty)$. Then the k th moment of σ_t^2 is finite if*

$$\begin{aligned} k \in K_\infty &= \left\{ s \in \mathbb{R} : \int_{|x|>1} e^{sx} \nu_\infty(dx) < \infty \right\} \\ &= \left\{ s \in \mathbb{R} : \int_0^\infty \int_{x \in \mathbb{R}, |h(x)|>1} e^{s g(u)x} \nu_L(dx) du < \infty \right\}. \end{aligned}$$

In this case

$$\Psi_\infty(k) := \gamma_\infty k + \int_{\mathbb{R}} (e^{kx} - 1 - kx \chi_{(-1,1)}(x)) \nu_\infty(dx) \tag{4.1}$$

is well defined and

$$E(\sigma_t^{2k}) = \exp(\Psi_\infty(k)) \quad \text{for all } t \geq 0. \tag{4.2}$$

Proposition 4.4. *Let $\log(\sigma_t^2)$ be the strictly stationary solution of (3.3) and (3.4). Assume that $E(\sigma_t^4) < \infty$ for all $t \geq 0$. Let $\Psi_\infty^h(\cdot)$ and $\Psi^h(\cdot)$ be as defined by (4.1) with kernel function g replaced by $g_\infty^h(s) = \mathbf{b}^\top (I_q + e^{Ah}) e^{As} \mathbf{1}_q$ and $g^h(s) = \mathbf{b}^\top e^{As} \mathbf{1}_q \chi_{(0,h)}(s)$, respectively. Then the autocovariance function of σ^2 is given by the following expression:*

$$\text{cov}(\sigma_{t+h}^2, \sigma_t^2) = \exp(\Psi_\infty^h(1)) \exp(\Psi^h(1)) - \exp(2\Psi_\infty(1)), \quad h > 0, t \geq 0. \tag{4.3}$$

Proof. Let $\mathcal{F}_t^M = \sigma(M_s, -\infty < s \leq t)$ be the σ -algebra generated by the Lévy process M up to time t , then

$$\begin{aligned} E(\sigma_{t+h}^2 \mid \mathcal{F}_t^M) &= E\left(\exp\left(\mu + \int_{-\infty}^{t+h} g(t+h-s) dM_s\right) \mid \mathcal{F}_t^M\right) \\ &= \exp\left(\mu + \int_{-\infty}^t \mathbf{b}^\top e^{Ah} e^{A(t-s)} \mathbf{1}_q dM_s\right) \\ &\quad \times E\left(\exp\left(\int_t^{t+h} g(t+h-s) dM_s\right)\right). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} E(\sigma_{t+h}^2 \sigma_t^2) &= E(E(\sigma_{t+h}^2 \sigma_t^2 \mid \mathcal{F}_t^M)) \\ &= E(\sigma_t^2 E(\sigma_{t+h}^2 \mid \mathcal{F}_t^M)) \\ &= E\left(\sigma_t^2 \exp\left(\mu + \int_{-\infty}^t \mathbf{b}^\top e^{Ah} e^{A(t-s)} \mathbf{1}_q dM_s\right) \right. \\ &\quad \left. \times E\left(\exp\left(\int_t^{t+h} g(t+h-s) dM_s\right)\right)\right) \\ &= E\left(\exp\left(2\mu + \int_{-\infty}^t \mathbf{b}^\top (I_q + e^{Ah}) e^{A(t-s)} \mathbf{1}_q dM_s\right)\right) E\left(\exp\left(\int_0^h g(s) dM_s\right)\right) \\ &= E\left(\exp\left(\mu + \int_0^\infty \mathbf{b}^\top (I_q + e^{Ah}) e^{As} \mathbf{1}_q dM_s\right)\right) \\ &\quad \times E\left(\exp\left(\mu + \int_0^\infty \mathbf{b}^\top e^{As} \mathbf{1}_q \chi_{(0,h)}(s) dM_s\right)\right) \\ &= \exp(\Psi_\infty^h(1)) \exp(\Psi^h(1)), \end{aligned}$$

where the last equality follows from (4.2) when we substitute the kernel g in (3.6) by $g_\infty^h(s)$ and g^h , respectively. This together with (4.2) yields (4.3).

5. Second-order properties of the return process

In this section we derive the moment structure of the *return process*

$$G_t^{(r)} = G_t - G_{t-r} = \int_{(t-r,t]} \sigma_{s-} dL_s, \quad t \geq r > 0.$$

We will consider only the case of a strictly stationary volatility process.

5.1. Moments and autocovariance function of the return process

Proposition 5.1. *Let L be a Lévy process with $E(L_1) = 0$ and $E(L_1^2) < \infty$. Assume that the volatility process σ^2 is strictly stationary with finite mean. Then $E(G_t^{(r)}) < \infty$ for all $t \geq 0$ and, for every $t, h \geq r > 0$, it holds that*

$$E G_t^{(r)} = 0, \tag{5.1}$$

$$E(G_t^{(r)})^2 = \exp(\Psi_\infty(1))r E(L_1^2), \tag{5.2}$$

$$\text{cov}(G_t^{(r)}, G_{t+h}^{(r)}) = 0. \tag{5.3}$$

Furthermore, if $E(L_1^4) < \infty$ and the volatility process has finite second moment then $E(G_t^4) < \infty$ for all $t \geq 0$ and, for every $t, h \geq r > 0$, we have

$$\text{cov}((G_t^{(r)})^2, (G_{t+h}^{(r)})^2) = E(L_1^2) \int_h^{h+r} \text{cov}(G_r^2, \sigma_s^2) ds.$$

Proof. If L has no Brownian component, the proof of (5.1)–(5.3) is analogous to the proof of Proposition 5.1 of [13] and can be extended in the same way as in the proof of Proposition 2.1 of [12] for the case in which L has a Brownian component. Since G is a square integrable martingale, we obtain

$$E((G_r^{(r)})^2(G_{h+r}^{(r)})^2) = E(G_r^2(G_{h+r} - G_h)^2) = E(G_r^2(G_{h+r}^2 - G_h^2)).$$

Using this result,

$$G_t^2 = 2 \int_0^t G_{s-} \sigma_{s-} dL_s + \int_0^t \sigma_{s-}^2 d[L, L]_s, \quad t \geq 0,$$

and the compensation formula (see, e.g. [2, Section 0.5]), we obtain

$$\begin{aligned} E((G_r^{(r)})^2(G_{h+r}^{(r)})^2) &= E\left(2 \int_h^{h+r} G_r^2 G_{s-} \sigma_{s-} dL_s + \int_h^{h+r} G_r^2 \sigma_{s-}^2 d[L, L]_s\right) \\ &= E\left(\int_h^{h+r} G_r^2 \sigma_{s-}^2 d[L, L]_s\right) \\ &= \int_h^{h+r} E(G_r^2 \sigma_s^2) \tau_L^2 ds + \int_h^{h+r} E(G_r^2 \sigma_s^2) ds \int_{\mathbb{R}} x^2 \nu_L(dx) \\ &= E(L_1^2) \int_h^{h+r} E(G_r^2 \sigma_s^2) ds. \end{aligned}$$

Hence, the covariance is equal to

$$\begin{aligned} \text{cov}((G_r^{(r)})^2, (G_{h+r}^{(r)})^2) &= E((G_r^{(r)})^2(G_{h+r}^{(r)})^2) - (E(G_t^{(r)})^2)^2 \\ &= E(L_1^2) \int_h^{h+r} (\text{cov}(G_r^2, \sigma_s^2) + E(G_r^2) E(\sigma_s^2)) ds - (E(G_t^{(r)})^2)^2 \\ &= E(L_1^2) \int_h^{h+r} \text{cov}(G_r^2, \sigma_s^2) ds. \end{aligned}$$

The covariance is finite if $E(G_t^4) < \infty$ for all $t \geq 0$, and this follows with $E(L_1^4) < \infty$ and $2 \in K_\infty$ analogously to the proof of Proposition 1.1 of [12].

Example 5.1. Let us consider again Example 3.1. From 50 000 equidistant observations of the simulated log-price process we computed the empirical autocorrelation function of the returns and squared returns. In Figure 3 the first 40 lags of both empirical autocorrelation functions are shown. We recognise the GARCH-like behaviour of zero correlation of the returns and significant correlation of the squared returns.

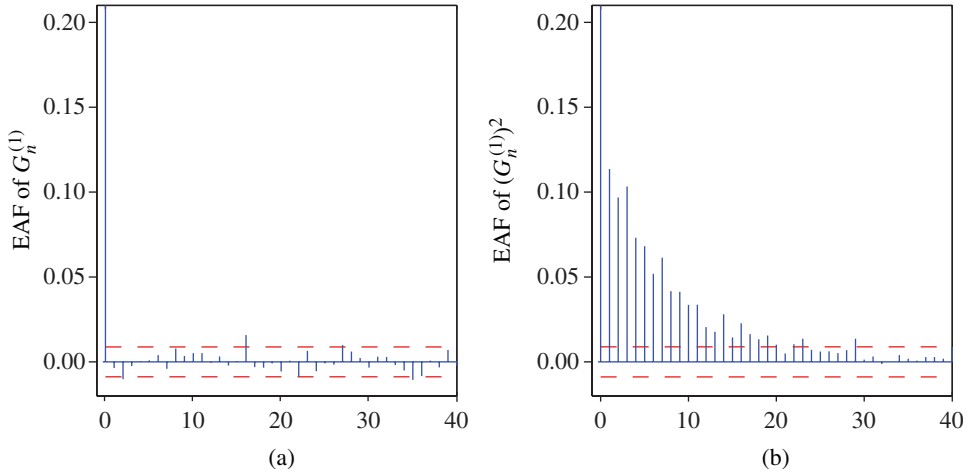


FIGURE 3: The first 40 lags of the empirical autocorrelation function (EAF) of (a) the return process and (b) the squared return process.

Remark 5.1. In Theorem 3.1 we have seen that the volatility and return processes are strongly mixing with exponential rate. A consequence of this property (see, e.g. [8, Section 1.2.2]) is that there exist constants $K_1, K_2 > 0$ such that

$$|\text{cov}(\sigma_{t+h}^2, \sigma_t^2)| \leq K_1 e^{-ah} \quad \text{and} \quad |\text{cov}((G_{(n+h)r}^{(r)})^2, (G_{nr}^{(r)})^2)| \leq K_2 e^{-ah}$$

for all $h > 0$, with $a > 0$ as given in Theorem 3.1. In particular, this means that the autocovariance function of the volatility and squared returns will decay to 0 at an exponential rate. Therefore, we will speak of a short memory process in both cases. The model can be extended to incorporate long memory effects, by specifying the log-volatility process by a fractionally integrated CARMA($q, p - 1$) process. For further details we refer the reader to [11].

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References

- [1] APPLEBAUM, D. (2004). *Lévy Processes and Stochastic Calculus*. Cambridge University Press.
- [2] BERTOIN, J. (1996). *Lévy Processes*. Cambridge University Press.
- [3] BOLLERSLEV, T. (1986). Generalized autoregressive conditional heteroskedasticity. *J. Econometrics* **31**, 307–327.
- [4] BROCKWELL, P. J. (2001). Lévy driven CARMA processes. *Ann. Inst. Statist. Math.* **53**, 113–124.
- [5] BROCKWELL, P. J. AND MARQUARDT, T. (2005). Lévy-driven and fractionally integrated ARMA processes with continuous time parameter. *Statist. Sinica* **15**, 477–494.
- [6] BROCKWELL, P. J., CHADRAA, E. AND LINDNER, A. M. (2006). Continuous time GARCH processes. *Ann. Appl. Prob.* **16**, 790–826.
- [7] DAVYDOV, Y. A. (1973). Mixing conditions for Markov chains. *Theory Prob. Appl.* **18**, 312–328.

- [8] DOUKHAN, P. (1994). *Mixing: Properties and Examples* (Lecture Notes Statist. **85**). Springer, New York.
- [9] ENGLE, R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of UK inflation. *Econometrica* **50**, 987–1008.
- [10] GIRAITIS, L., LEIPUS, R., ROBINSON, P. M. AND SURGAILIS, D. (2004). LARCH, leverage, and long memory. *J. Financial Econometrics* **2**, 177–210.
- [11] HAUG, S. AND CZADO, C. (2006). A fractionally integrated ECOGARCH process. *Discussion paper 484, SFB386*.
- [12] HAUG, S., KLÜPPELBERG, C., LINDNER, A. M. AND ZAPP, M. (2007). Method of moment estimation in the COGARCH(1, 1) model. *Econometrics J.* **10**, 320–341.
- [13] KLÜPPELBERG, C., LINDNER, A. AND MALLER, R. (2004). A continuous time GARCH process driven by a Lévy process: stationarity and second-order behaviour. *J. Appl. Prob.* **41**, 601–622.
- [14] KUSUOKA, S. AND YOSHIDA, N. (2000). Malliavin calculus, geometric mixing, and expansion of diffusion functionals. *Prob. Theory Relat. Fields* **116**, 457–484.
- [15] MASUDA, H. (2004). On multidimensional Ornstein–Uhlenbeck processes driven by a general Lévy process. *Bernoulli* **10**, 97–120.
- [16] MASUDA, H. (2005). Classical method of moments for partially and discretely observed ergodic models. *Statist. Infer. Stoch. Process.* **8**, 25–50.
- [17] MCLEISH, D. L. (1975). Invariance principles for dependent variables. *Z. Wahrscheinlichkeitsthe.* **32**, 165–175.
- [18] NELSON, D. B. (1990). ARCH models as diffusion approximations. *J. Econometrics* **45**, 7–38.
- [19] NELSON, D. B. (1991). Conditional heteroskedasticity in asset returns: a new approach. *Econometrica* **59**, 347–370.
- [20] RAJPUT, B. S. AND ROSIŃSKI, J. (1989). Spectral representations of infinitely divisible processes. *Prob. Theory Relat. Fields* **82**, 451–487.
- [21] ROBINSON, P. M. (1991). Testing for strong serial correlation and dynamic conditional heteroskedasticity in multiple regression. *J. Econometrics* **47**, 67–84.
- [22] SATO, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.
- [23] SURGAILIS, D. AND VIANO, M. C. (2002). Long memory properties and covariance structure of the EGARCH model. *ESAIM Prob. Statist.* **6**, 311–329.