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RESEARCH ARTICLE

A note on quantum K-theory of root constructions

Hsian-Hua Tseng

Department of Mathematics, Ohio State University, 100 Math Tower, 231 West 18th Ave., Columbus, OH, 43210, USA
Email: hhtseng@math.ohio-state.edu

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Abstract

We consider K-theoretic Gromov-Witten theory of root constructions. We calculate some genus 0 K-theoretic Gromov-Witten invariants of a root gerbe. We also obtain a K-theoretic relative/orbifold correspondence in genus 0.

1. Introduction

1.1. Étale gerbes

Let X be a smooth projective variety over the complex numbers. An étale gerbe $\mathcal{G}$ over X may be thought of as a fiber bundle over X whose fibers are the classifying stack BG of a certain finite group G. Geometric properties of $\mathcal{G}$ are of purely stack-theoretic nature.

In ref. [17], physical theories on an étale gerbe $\mathcal{G}$ are considered, leading to the formulation of *decomposition conjecture* (also known as *gerbe duality*). Interpreted in mathematics, the decomposition conjecture for $\mathcal{G}$ asserts that the geometry of $\mathcal{G}$ is equivalent to the geometry of a disconnected space $\widehat{\mathcal{G}}$ equipped with a $\mathbb{C}^*$-gerbe. The decomposition conjecture has been proven in several mathematical aspects in ref. [22].

1.2. Gromov-Witten theory

Gromov-Witten theory of a target Z is defined using moduli stacks $\mathcal{K}_{g,n}(Z, d)$ of stable maps to Z. Gromov-Witten invariants of Z are integrals of natural cohomology classes on $\mathcal{K}_{g,n}(Z, d)$ against the virtual fundamental class of $\mathcal{K}_{g,n}(Z, d)$.

The Gromov-Witten theory of an étale gerbe $\mathcal{G}$ has been studied, with a point of view toward the decomposition conjecture, in various generalities in refs. [5], [6], [19], [23], [25].

1.3. Quantum K-theory

Quantum K-theory, introduced in refs. [13], [20], is the K-theoretic counterpart of Gromov-Witten theory. K-theoretic Gromov-Witten invariants of a target Z are Euler characteristics of natural K-theory classes on $\mathcal{K}_{g,n}(Z, d)$ tensored with the *virtual structure sheaf* $\mathcal{O}^{vir}_{\mathcal{K}_{g,n}(Z,d)}$. An extension of quantum K-theory to target Deligne-Mumford stacks is given in ref. [24].

Quantum Hirzebruch-Riemann-Roch theorems [16], [24], [14], [15] imply that quantum K-theory can be determined by (cohomological) Gromov-Witten theory. Since (cohomological) Gromov-Witten theory of étale gerbes has been shown to satisfy the decomposition conjecture in many cases, it is natural to ask if quantum K-theory of an étale gerbe $\mathcal{G}$ can be studied with a viewpoint toward the decomposition conjecture. This note contains an attempt to address this for *root gerbes* over X in genus 0.

1.4. Root gerbes

Given a line bundle $L \to X$ and an integer $r > 0$, one can associate the stack $\sqrt[r]{L/X}$ of r-th roots of L, which is a smooth Deligne-Mumford stack whose points over an X-scheme $f : S \to X$ are

$$\sqrt[r]{L/X}(S) = \{(M, \phi) \mid M \to S \text{ line bundle}, \phi : M^{\otimes r} \xrightarrow{\simeq} f^*L\}.$$

The coarse moduli space of $\sqrt[r]{L/X}$ is X. Furthermore, the natural map $\rho : \sqrt[r]{L/X} \to X$ has the structure of a μ_r-gerbe.

The strategy employed to study quantum K-theory of $\sqrt[r]{L/X}$ in this note is the same as that of [5]. Namely, we examine the structure of moduli stacks of genus 0 stable maps to $\sqrt[r]{L/X}$ and apply pushforward results for virtual structure sheaves. The main result of this note is Proposition 2.2.

1.5. Root stacks

Given a smooth irreducible divisor $D \subset X$ and an integer $r > 0$, one can associate the stack $X_{D,r}$ of r-th roots of X along D. In [1], genus 0 *relative* Gromov-Witten invariants of (X, D) and Gromov-Witten invariants of $X_{D,r}$ are shown to be the same when r is sufficiently large. Their proof uses pushforwards of virtual fundamental classes and an intermediate moduli space. In Section 2.6, we explain how to adapt their argument to obtain a similar result for genus 0 K-theoretic Gromov-Witten invariants, see (2.13).

1.6. Outline

The rest of this note is organized as follows. Section 2.1 recalls notations used the definition of K-theoretic Gromov-Witten invariants of Deligne-Mumford stacks. In Section 2.2, we discuss properties of the structure morphism for moduli stacks of genus 0 stable maps to a root gerbe. In Section 2.3, we discuss pushforwards of virtual structure sheaves. Section 2.4 contains the proof of our main result and Section 2.5 discusses an extension of the main result to a more general class of gerbes. In Section 2.6, we discuss a K-theoretic version of relative/orbifold correspondence. In Section 3, we discuss some related questions.

1.7. Acknowledgment

This note is inspired by the results on virtual pushforwards in K-theory in refs. [10] and [11]. It is a pleasure to thank the authors Y.-C. Chou, L. Herr, and Y.-P. Lee. The author also thanks E. Sharpe for discussions. The author is supported in part by Simons Foundation Collaboration Grant.

2. Results

2.1. Quantum K-theory of target stacks

We begin with recalling the definition of K-theoretic Gromov-Witten invariants of Deligne-Mumford stacks, as given in ref. [24]. Let $\mathcal{Z}$ be a smooth proper Deligne-Mumford stack with projective coarse moduli space Z. The moduli stack of n-pointed genus g degree d stable maps to $\mathcal{Z}$ is denoted by $\mathcal{K}_{g,n}(\mathcal{Z}, d)$. The detailed definition can be found in [4]. It is known that $\mathcal{K}_{g,n}(\mathcal{Z}, d)$ is a proper Deligne-Mumford stack equipped with a perfect obstruction theory, see [4], [3]. Applying the recipe of [20] to this perfect obstruction theory yields a virtual structure sheaf $\mathcal{O}^{vir}_{\mathcal{K}_{g,n}(\mathcal{Z},d)}$. There are evaluation maps $ev_i : \mathcal{K}_{g,n}(\mathcal{Z}, d) \to \bar{I}\mathcal{Z}$, where $\bar{I}\mathcal{Z}$ is the *rigidified inertia stack* of $\mathcal{Z}$. See [3] for more details on the construction of evaluation maps.

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K-theoretic Gromov-Witten invariants of $\mathcal{Z}$ are Euler characteristics of the following form:

$$\chi\left(\mathcal{K}_{g,n}(\mathcal{Z},d),\mathcal{O}^{vir}_{\mathcal{K}_{g,n}(\mathcal{Z},d)}\otimes\bigotimes_{i=1}^{n}ev_i^*\alpha_i\right)\in\mathbb{Z},\quad \alpha_1,\dots,\alpha_n\in K^*(\bar{I}\mathcal{Z}). \tag{2.1}$$

2.2. Structure morphism

Sending a stable map $f:\left(\mathcal{C},\{\Sigma_i\}_{i=1}^n\right)/S\to\mathcal{Z}$ to the induced map $\bar{f}:\left(C,\{\bar{\Sigma}_i\}_{i=1}^n\right)/S\to Z$ between coarse moduli spaces yields a morphism

$$\mathcal{K}_{g,n}(\mathcal{Z},d)\to\mathcal{K}_{g,n}(Z,d). \tag{2.2}$$

We examine (2.2) in the special case $\mathcal{Z}=\sqrt[r]{L/X}$ and $g=0$.

As explained in [5, Section 3.1], the rigidified inertia stack of $\sqrt[r]{L/X}$ is a disjoint union of components $\bar{I}\left(\sqrt[r]{L/X}\right)_g$ indexed by $g\in\mu_r$. As in [5, Definition 3.3], for $g_1,\dots,g_n\in\mu_r$, put

$$\mathcal{K}_{0,n}\left(\sqrt[r]{L/X},d\right)^{\vec{g}}=\bigcap_{i=1}^{n}ev_i^{-1}\left(\bar{I}(\sqrt[r]{L/X})_{g_i}\right).$$

In order for $\mathcal{K}_{0,n}\left(\sqrt[r]{L/X},d\right)^{\vec{g}}$ to be non-empty, the elements $g_1,\dots,g_n$ are required to satisfy certain condition, see [5, Section 3.1].

We consider the restriction of (2.2) to $\mathcal{K}_{0,n}\left(\sqrt[r]{L/X},d\right)^{\vec{g}}$:

$$p:\mathcal{K}_{0,n}\left(\sqrt[r]{L/X},d\right)^{\vec{g}}\to\mathcal{K}_{0,n}(X,d). \tag{2.3}$$

The structure of the map p has been analyzed in ref. [5]. We reproduce [5, Diagram (26)] as follows:

$$\begin{tikzcd}
\mathcal{K}_{0,n}(\sqrt[r]{L/X},d)^{\vec{g}} \arrow[r, "t"'] \arrow[dr, "s"'] \arrow[rrr, bend left, "p"] & P_n^{\vec{g}} \arrow[r, "r'"'] \arrow[d] \arrow[dr, phantom, "\square"] & P \arrow[r, "r"'] \arrow[d, "q'"] \arrow[dr, phantom, "\square"] & \mathcal{K}_{0,n}(X,d) \arrow[d, "q"] \\
& \mathfrak{Y}^{\vec{g}}_{0,n,d} \arrow[r] & \mathfrak{M}^{tw}_{0,n,d} \arrow[r] \arrow[d, "s'"] \arrow[dr, phantom, "\square"] & \mathfrak{M}_{0,n,d} \arrow[d, "s''"] \\
& & \mathfrak{M}^{tw}_{0,n} \arrow[r] & \mathfrak{M}_{0,n}.
\end{tikzcd} \tag{2.4}$$

Here, $\mathfrak{M}_{0,n}$ is the stack of n-pointed genus 0 prestable curves (see e.g. [7] for a discussion), and $\mathfrak{M}^{tw}_{0,n}$ is the stack of n-pointed genus 0 prestable twisted curves (see [21]). $\mathfrak{M}_{0,n,d}$ and $\mathfrak{M}^{tw}_{0,n,d}$ are variants of $\mathfrak{M}_{0,n}$ and $\mathfrak{M}^{tw}_{0,n}$ parametrizing prestable (twisted) curves weighted by $d\in H^2(X,\mathbb{Z})$, see [5, Section 3.2] for an introduction and [7] and [26] for further details.

In (2.4), the stack $\mathfrak{Y}^{\vec{g}}_{0,n,d}$ is constructed in [5, Definition 3.12] by applying the root construction to a certain divisor of $\mathfrak{M}_{0,n,d}$. It follows that the composition $\mathfrak{Y}^{\vec{g}}_{0,n,d}\to\mathfrak{M}^{tw}_{0,n,d}\to\mathfrak{M}_{0,n,d}$ is proper and birational. The stacks P and $P_n^{\vec{g}}$ are defined by cartesian squares. The map s is defined by [5, Lemma 3.18].

Example 2.1. *When X is a point, the line bundle L is necessarily trivial. In this case, $\sqrt[r]{L/X}=B\mu_r$. The moduli stacks $\mathcal{K}_{0,n}(B\mu_r)^{\vec{g}}$ and $\mathcal{K}_{0,n}(\mathrm{pt})=\overline{\mathcal{M}}_{0,n}$ are smooth of expected dimensions. The morphism (2.3) in this case has been studied in [8]. It is shown in [8] that there is a factorization $\mathcal{K}_{0,n}(B\mu_r)^{\vec{g}}\to\mathcal{N}\to\overline{\mathcal{M}}_{0,n}$, where $\mathcal{K}_{0,n}(B\mu_r)^{\vec{g}}\to\mathcal{N}$ is the stack of r-th roots of certain line bundle, and $\mathcal{N}\to\overline{\mathcal{M}}_{0,n}$ is a root construction.*

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2.3. Pushforward

We now examine obstruction theories. Since the map s'' is étale, the standard obstruction theory on $\mathcal{K}_{0,n}(X,d)$ relative to $\mathfrak{M}_{0,n}$ can be viewed as a obstruction theory $E^\bullet_{\mathcal{K}_{0,n}(X,d)} \to L^\bullet_q$ on $\mathcal{K}_{0,n}(X,d)$ relative to the morphism q. The stack P can be equipped with an obstruction theory relative to the morphism q' by pulling back $E^\bullet_{\mathcal{K}_{0,n}(X,d)}$. The stack $P_n^{\vec{g}}$ can be equipped with an obstruction theory relative to $\mathfrak{Y}^{\vec{g}}_{0,n,d}$ by pulling back the obstruction theory on P.

Since both maps s' and $\mathfrak{Y}^{\vec{g}}_{0,n,d} \to \mathfrak{M}^{tw}_{0,n,d}$ are étale [5, Lemma 3.15], the standard obstruction theory on $\mathcal{K}_{0,n}\big(\sqrt[r]{L/X}, d\big)^{\vec{g}}$ relative to $\mathfrak{M}^{tw}_{0,n}$ can be viewed as an obstruction theory $E^\bullet_{\mathcal{K}_{0,n}\left(\sqrt[r]{L/X},d\right)^{\vec{g}}} \to L^\bullet_s$ on $\mathcal{K}_{0,n}\big(\sqrt[r]{L/X}, d\big)^{\vec{g}}$ relative to the morphism s.

By [5, Lemma 4.1], $E^\bullet_{\mathcal{K}_{0,n}(X,d)}$ pulls back to $E^\bullet_{\mathcal{K}_{0,n}\left(\sqrt[r]{L/X},d\right)^{\vec{g}}}$. We then have the following results on virtual structure sheaves.

1. Since $\mathfrak{Y}^{\vec{g}}_{0,n,d} \to \mathfrak{M}_{0,n,d}$ is proper and birational, by [11, Theorem 1.12], we have
$$(r \circ r')_*\left[\mathcal{O}^{vir}_{P_n^{\vec{g}}}\right] = \left[\mathcal{O}^{vir}_{\mathcal{K}_{0,n}(X,d)}\right]. \tag{2.5}$$
2. By [5, Theorem 3.19], the map $t : \mathcal{K}_{0,n}\big(\sqrt[r]{L/X}, d\big)^{\vec{g}} \to P_n^{\vec{g}}$ is a μ_r-gerbe. Hence, by [11, Proposition 1.9], we have
$$t_*\left[\mathcal{O}^{vir}_{\mathcal{K}_{0,n}\left(\sqrt[r]{L/X},d\right)^{\vec{g}}}\right] = \left[\mathcal{O}^{vir}_{P_n^{\vec{g}}}\right]. \tag{2.6}$$

2.4. Invariants

The evaluation maps on $\mathcal{K}_{0,n}(X,d)$ and $\mathcal{K}_{0,n}\big(\sqrt[r]{L/X}, d\big)^{\vec{g}}$ fit into the following commutative diagram:

$$\begin{tikzcd}
\mathcal{K}_{0,n}(\sqrt[r]{L/X}, d)^{\vec{g}} \arrow[r, "ev_i"] \arrow[d, "p"] & \bar{I}(\sqrt[r]{L/X})_{g_i} \arrow[d, "\bar{I}\rho"] \\
\mathcal{K}_{0,n}(X,d) \arrow[r, "ev_i"] & \bar{I}X = X.
\end{tikzcd}$$

Consider the descendant line bundles $L_1, \ldots, L_n \to \mathcal{K}_{0,n}(X,d)$ associated to the marked points. The following is the main result of this note:

Proposition 2.2. *For $\alpha_1, \ldots, \alpha_n \in K^*(X)$ and $k_1, \ldots, k_n \in \mathbb{Z}$, we have*
$$\chi\left(\mathcal{K}_{0,n}\left(\sqrt[r]{L/X}, d\right)^{\vec{g}}, \mathcal{O}^{vir}_{\mathcal{K}_{0,n}\left(\sqrt[r]{L/X},d\right)^{\vec{g}}} \otimes \bigotimes_{i=1}^n \left((p^*L_i)^{\otimes k_i} \otimes ev_i^*\left(\left(\bar{I}\rho\right)^* \alpha_i\right)\right)\right)$$
$$= \chi\left(\mathcal{K}_{0,n}(X,d), \mathcal{O}^{vir}_{\mathcal{K}_{0,n}(X,d)} \otimes \bigotimes_{i=1}^n \left(L_i^{\otimes k_i} \otimes ev_i^*(\alpha_i)\right)\right).$$

Proof. Since $\bar{I}\rho \circ ev_i = ev_i \circ p$, projection formula gives
$$p_*\left(\mathcal{O}^{vir}_{\mathcal{K}_{0,n}\left(\sqrt[r]{L/X},d\right)^{\vec{g}}} \otimes \bigotimes_{i=1}^n \left((p^*L_i)^{\otimes k_i} \otimes ev_i^*\left(\left(\bar{I}\rho\right)^* \alpha_i\right)\right)\right) = p_*\left(\mathcal{O}^{vir}_{\mathcal{K}_{0,n}\left(\sqrt[r]{L/X},d\right)^{\vec{g}}}\right) \otimes \bigotimes_{i=1}^n \left(L_i^{\otimes k_i} \otimes ev_i^*(\alpha_i)\right).$$
Since $p = r \circ r' \circ t$, the result follows from (2.5) and (2.6). □

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2.5. Banded abelian gerbes

Suppose G is a finite abelian group. Suppose $\mathcal{G} \to X$ is a gerbe banded by G. Then the isomorphism class of $\mathcal{G} \to X$ is classified by the cohomology group $H^2(X, G)$, where G is viewed as a constant sheaf on X. We say that $\mathcal{G} \to X$ is *essentially trivial* if the image of its class is trivial for maps $H^2(X, G) \to H^2(X, \mathbb{C}^*)$ induced by group homomorphisms $G \to \mathbb{C}^*$. Examples of essentially trivial gerbes include toric gerbes [25].

Let $\mathcal{G} \to X$ be an essentially trivial gerbe over X. Then by [5, Lemma A.2], $\mathcal{G}$ is of the form

$$\mathcal{G} \simeq \sqrt[r_1]{L_1/X} \times_X \sqrt[r_1]{L_2/X} \times_X \ldots \times_X \sqrt[r_k]{L_k/X} \tag{2.7}$$

where $L_1, \ldots, L_k$ are line bundles over X and $r_1, \ldots, r_k$ are natural numbers.

Consider the morphism (2.2) in this case:

$$\mathcal{K}_{0,n}(\mathcal{G}, d) \to \mathcal{K}_{0,n}(X, d). \tag{2.8}$$

By the alaysis of [5, Appendix A], (2.8) also fits into diagram like (2.4), with a factorization

$$\mathcal{K}_{0,n}(\mathcal{G}, d)^{\vec{g}} \to P_n^{\vec{g}} \to \mathcal{K}_{0,n}(X, d). \tag{2.9}$$

Here, $\vec{g}$ is defined in [5, Definition A.5].

The map $P_n^{\vec{g}} \to \mathcal{K}_{0,n}(X, d)$ is by construction virtually birational, hence we can apply [11, Theorem 1.12] to it. By [5, Theorem A.6], the map $\mathcal{K}_{0,n}(\mathcal{G}, d)^{\vec{g}} \to P_n^{\vec{g}}$ is also a gerbe, so we can apply [11, Proposition 1.9] to it. Therefore, we may repeat the arguments in Section 2.4 to extend Proposition 2.2 to essentially trivial banded abelian gerbes $\mathcal{G} \to X$.

2.6. Root stacks

Let $D \subset X$ be a smooth irreducible divisor. For an integer $r > 0$, one can construct the stack $X_{D,r}$ of r-th roots of X along D, see [9] and [3, Appendix B]. The natural map

$$X_{D,r} \to X \tag{2.10}$$

is an isomorphism over $X \setminus D$ and is a μ_r-gerbe over D. Denote by $D_r \subset X_{D,r}$ the inverse image of D under (2.10).

It is shown in ref. [1] that genus 0 relative Gromov-Witten invariants of the pair (X, D) are the same as Gromov-Witten invariants of $X_{D,r}$ for r sufficiently large. Here we explain how their method can be adapted to K-theoretic Gromov-Witten theory.

By [2], there is an isomorphism between moduli spaces[1] of stable relative maps,

$$\Psi : \overline{M}_{0,n}(X_{D,r}, D_r) \to \overline{M}_{0,n}(X, D),$$

see also [1, Theorem 2.1]. This implies an identification of virtual structure sheaves,

$$\Psi_*\big[\mathcal{O}^{vir}_{\overline{M}_{0,n}(X_{D,r},D_r)}\big] = \big[\mathcal{O}^{vir}_{\overline{M}_{0,n}(X,D)}\big]. \tag{2.11}$$

There is a natural map that forgets the relative structure

$$\Phi : \overline{M}_{0,n}(X_{D,r}, D_r) \to \overline{M}_{0,n}(X_{D,r}).$$

Assume that r is sufficiently large. The proof of [1, Theorem 2.2] implies that Φ is virtually birational. Hence, by [11, Theorem 1.12], we have

$$\Phi_*\big[\mathcal{O}^{vir}_{\overline{M}_{0,n}(X_{D,r},D_r)}\big] = \big[\mathcal{O}^{vir}_{\overline{M}_{0,n}(X_{D,r})}\big]. \tag{2.12}$$

[1] We omit curve classes from notations.

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Evaluation maps of these moduli spaces are compatible with Ψ and Φ, see [1, Section 2.2]. It follows from (2.11) and (2.12) that for $\alpha_1, \ldots, \alpha_k \in K^*(X)$, $\gamma_1, \ldots, \gamma_l \in K^*(D)$, and r sufficiently large, we have

$$\chi\left(\overline{M}_{0,n}(X_{D,r}), \mathcal{O}^{vir}_{\overline{M}_{0,n}(X_{D,r})} \otimes \bigotimes_{i=1}^{k} L_i^{k_i} \otimes ev_i^*(\alpha_i) \otimes \bigotimes_{j=1}^{l} L_j^{m_j} \otimes ev_j^*(\beta_j)\right)$$
$$= \chi\left(\overline{M}_{0,n}(X, D), \mathcal{O}^{vir}_{\overline{M}_{0,n}(X,D)} \otimes \bigotimes_{i=1}^{k} L_i^{k_i} \otimes ev_i^*(\alpha_i) \otimes \bigotimes_{j=1}^{l} L_j^{m_j} \otimes ev_j^*(\beta_j)\right). \tag{2.13}$$

We view (2.13) as a correspondence between genus 0 K-theoretic Gromov-Witten invariants of (X, D) and $X_{D,r}$.

3. Comments

3.1. On higher genus

3.1.1. Root gerbes

For $h > 0$, the genus-h version of the morphism (2.3),

$$\mathcal{K}_{h,n}\left(\sqrt[r]{L/X}, d\right)^{\vec{g}} \to \mathcal{K}_{h,n}(X, d), \tag{3.1}$$

has been studied in ref. [6]. The map (3.1) is understood well enough so that a result on the pushforward of virtual fundamental *classes* is proven in ref. [6]. However, pushforward of virtual structure *sheaves* under (3.1) appears to be difficult. The key issue is that, in order to apply [11, Proposition 1.9, Theorem 1.12], we need (3.1) to be factored into virtual birational maps and gerbes. A factorization of (3.1) was obtained for more general banded gerbes in [6, Diagram (41)]. In our setting, this gives

$$\mathcal{K}_{h,n}\left(\sqrt[r]{L/X}, d\right)^{\vec{g}} \to P^{\vec{g}}_{h,n} \to \mathcal{K}_{h,n}(X, d). \tag{3.2}$$

For a root gerbe $\sqrt[r]{L/X} \to X$, one can check that $P^{\vec{g}}_{h,n} \to \mathcal{K}_{h,n}(X, d)$ is also virtually birational. However, by the discussion of [6, Section 6.2], the map $\mathcal{K}_{h,n}(\sqrt[r]{L/X}, d)^{\vec{g}} \to P^{\vec{g}}_{h,n}$ is a composition of two maps, one has degree $1/r$ and the other had degree $r^{2h} > 1$. The degree r^{2h}-map cannot possibly be a gerbe. Hence, [11, Proposition 1.9] is not applicable to $\mathcal{K}_{h,n}(\sqrt[r]{L/X}, d)^{\vec{g}} \to P^{\vec{g}}_{h,n}$. This prevents us from obtaining genus-h version of Proposition 2.2.

3.1.2. Root stacks

The relative/orbifold correspondence in cohomological Gromov-Witten theory has been extended to higher genus in ref. [27]. A K-theoretic relative/orbifold correspondence in higher genus is an interesting question. It is unlikely that virtual pushforwards used in genus 0 will be enough in higher genus. Some foundational work in K-theoretic Gromov-Witten theory is required in order to follow the arguments in ref. [27].

3.2. On virtual pushforward

There are many situations in cohomological Gromov-Witten theory in which "virtually birational" maps occur, see [18] for a detailed list. In addition, we note that the morphism u in [12, Lemma 4.16] is virtually birational. Hence, we can apply [11, Theorem 1.12] to obtain a calculation of the K-theoretic J-function of weighted projective spaces. Since such a result is a special case of the work [28] on quantum K-theory of toric stacks, we do not pursue it in detail.

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3.3. On decomposition conjecture

Consider an étale gerbe $\mathcal{G} \to X$. As defined in (2.1), K-theoretic Gromov-Witten invariants of $\mathcal{G}$ have insertions coming from the K-theory $K^*(\bar{I}\mathcal{G})$ of the rigidified inertia stack $\bar{I}\mathcal{G}$. Since $\mathcal{G} \subset \bar{I}\mathcal{G}$ is a connected component, the K-theory $K^*(\mathcal{G})$ of $\mathcal{G}$ is a direct summand of $K^*(\bar{I}\mathcal{G})$. The proof of Proposition 2.2 only allows classes in $K^*(\bar{I}\sqrt[r]{L/X})$ pulled back from X. Studying K-theoretic Gromov-Witten invariants of $\sqrt[r]{L/X}$ with other kinds of insertions requires new ideas.

For root gerbes $\mathcal{G} \to X$ arising in toric geometry, for example, weighted projective spaces and more general toric gerbes, it may be possible to study the decomposition conjecture by analyzing the K-theoretic I-functions calculated in ref. [28] in a manner similar to [25]. An additive decomposition of the K-theory $K(\bar{I}\mathcal{G})$ is a basic question.

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