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ASYMPTOTIC RESULTS ON TAIL MOMENT AND TAIL CENTRAL MOMENT FOR DEPENDENT RISKS

JINZHU LI,* *Nankai University*

Abstract

In this paper, we consider a financial or insurance system with a finite number of individual risks described by real-valued random variables. We focus on two kinds of risk measures, referred to as the tail moment (TM) and the tail central moment (TCM), which are defined as the conditional moment and conditional central moment of some individual risk in the event of system crisis. The first-order TM and the second-order TCM coincide with the popular risk measures called the marginal expected shortfall and the tail variance, respectively. We derive asymptotic expressions for the TM and TCM with any positive integer orders, when the individual risks are pairwise asymptotically independent and have distributions from certain classes that contain both light-tailed and heavy-tailed distributions. The formulas obtained possess concise forms unrelated to dependence structures, and hence enable us to estimate the TM and TCM efficiently. To demonstrate the wide application of our results, we revisit some issues related to premium principles and optimal capital allocation from the asymptotic point of view. We also give a numerical study on the relative errors of the asymptotic results obtained, under some specific scenarios when there are two individual risks in the system. The corresponding asymptotic properties of the degenerate univariate versions of the TM and TCM are discussed separately in an appendix at the end of the paper.

Keywords: Tail moment; tail central moment; risk measure; asymptotic independence; regular variation; Gumbel max-domain of attraction

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1. Introduction

Consider a financial or insurance system that contains a finite number of, say $d \geq 2$, individual components. Let $X_1, \ldots, X_d$ be d real-valued random variables describing the net losses (i.e., risks) of the individual components, and denote by $S_d = \sum_{i=1}^{d} X_i$ the overall risk of the system. All of $X_1, \ldots, X_d$ are assumed to be unbounded from above, since the ones with finite upper bounds will not pose substantial risks to the system. The tail moment (TM) of the kth individual risk for some $1 \leq k \leq d$ is defined as the conditional moment of X_k given that S_d exceeds a certain threshold. Specifically, for each positive integer $n \in \mathbf{N}^+$, each $1 \leq k \leq d$, and any $t > 0$, the nth-order TM of the kth individual risk is formulated as

$$\mathrm{TM}_k^{(n)}(t) = \mathbb{E}\left(X_k^n \,|S_d > t\right). \tag{1.1}$$

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* Postal address: School of Mathematical Science and LPMC, Nankai University, Tianjin 300071, P. R. China. Email address: lijinzhu@nankai.edu.cn

Here and hereafter, a mathematical expectation is assumed to exist by default whenever it appears. We further define the corresponding nth-order tail central moment (TCM) of the kth individual risk as

$$\mathrm{TCM}_k^{(n)}(t) = \mathbb{E}\left(\left(X_k - \mathrm{TM}_k^{(1)}(t) \right)^n \middle| \, S_d > t \right). \tag{1.2}$$

In practice, the threshold t is usually chosen to be the value at risk (VaR) of S_d under a confidence level $q \in (0, 1)$, i.e., $\mathrm{VaR}_q(S_d) = \inf\{x : \mathbb{P}(S_d \le x) \ge q\}$.

The TM and TCM with even orders, i.e., $\mathrm{TM}_k^{(2n)}(t)$ and $\mathrm{TCM}_k^{(2n)}(t)$ for $n \in \mathbf{N}^+$, are always non-negative for any $t > 0$. In addition, the following study indicates that under our models, $\mathrm{TM}_k^{(2n-1)}(t)$ will eventually be positive as t increases, which is not always the case for $\mathrm{TCM}_k^{(2n-1)}(t)$. Hence, under our models, $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(2n)}(t)$ can be regarded as standard measures with non-negativity when t is large or, correspondingly, when q is close to 1 if t is chosen to be $\mathrm{VaR}_q(S_d)$. It is worth noting that a risk measure is usually not required to be non-negative (see, e.g., McNeil *et al.* [36]), and then both $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(n)}(t)$ can be applied as risk measures. In fact, most of the time we are only interested in properties of $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(n)}(t)$ when t is very large, since only in such a case does the conditioning event $\{S_d > t\}$ mean an extreme system crisis that is of real concern. On the other hand, in general it is difficult or even impossible to derive exact closed-form expressions for $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(n)}(t)$ with respect to t, especially when there are various dependence structures among the individual risks. Thus, one of the main lines of study in this area is to seek effective and efficient estimates of $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(n)}(t)$ for large t, and this is also the main target of the present paper.

Remarkably, $\mathrm{TM}_k^{(1)}(t)$ is a very popular risk measure and has attracted much scholarly attention from both researchers and practitioners in recent decades. In the literature, it is usually called the marginal expected shortfall and is used to measure the contribution of some individual component to a system crisis. Additionally, it is also widely applied in capital allocation. Some regulatory environments (e.g., the Swiss Solvency Test) require that the total capital of the system equals the conditional tail expectation of S_d with respect to some threshold t, i.e., $\mathbb{E}(S_d \,|S_d > t)$. Then the most intuitive and commonly used capital allocation rule is the famous Euler one, which assigns the amount of $\mathrm{TM}_k^{(1)}(t)$ to the kth individual component; see Denault [13], Asimit and Li [3], and Baione *et al.* [4]. There have been many fruitful contributions to the study of $\mathrm{TM}_k^{(1)}(t)$. See Cai and Li [9], Furman and Landsman [20], Dhaene *et al.* [14], Bargès *et al.* [5], Vernic [43], Ignatieva and Landsman [25], and Marri and Moutanabbir [35] for works devoted to finding exact expressions for $\mathrm{TM}_k^{(1)}(t)$ when $(X_1, \ldots, X_d)$ follows some specific joint distributions. On the other hand, assuming that $X_1, \ldots, X_d$ are non-negative and have distributions from the Fréchet or Gumbel max-domain of attraction (MDA), Asimit *et al.* [2] obtained a series of asymptotic formulas for $\mathrm{TM}_k^{(1)}(t)$ as $t \to \infty$ under certain dependence structures, including both the asymptotic independence and asymptotic dependence cases. Some of the results in Asimit *et al.* [2] were extended to more general frameworks in the recent work of Li [34]. See Joe and Li [26], Hua and Joe [23], Zhu and Li [45], and Kley *et al.* [29] for related discussions under the assumption that $(X_1, \ldots, X_d)$ is of multivariate regular variation. Tang and Yuan [42] considered a variant of $\mathrm{TM}_k^{(1)}(t)$, in which $(X_1, \ldots, X_d)$ is modeled by a randomly weighted form $(\xi_1 Y_1, \ldots, \xi_d Y_d)$. They obtained asymptotic results under the assumptions that $Y_1, \ldots, Y_d$ are independent random variables with heavy-tailed distributions and that $\xi_1, \ldots, \xi_d$ satisfy certain moment conditions. Recently, Chen and Liu [11] extended the work of Tang and Yuan [42] to allow an asymptotic independence structure among $Y_1, \ldots, Y_d$. For related investigations from the statistical perspective, we refer the reader to El Methni *et al.* [17], Cai *et al.* [8], Acharya *et al.* [1], Hou and Wang [22], and Sun *et al.* [41].

Moreover, $\mathrm{TCM}_k^{(2)}(t)$ is a multivariate extension of the so-called tail variance (TV) risk measure proposed by Furman and Landsman [19], and it quantifies the degree of deviation between an individual risk and the corresponding marginal expected shortfall. Furman and Landsman [19] and Ignatieva and Landsman [24] derived explicit expressions for $\mathrm{TCM}_k^{(2)}(t)$ when $(X_1, \ldots, X_d)$ follows multivariate elliptical distributions. Other related studies have mainly concentrated on the TV, i.e., the degenerate univariate version of $\mathrm{TCM}_k^{(2)}(t)$ with $d = 1$, and most of the results obtained have been for the random risk with a distribution of elliptical type; see Kim [27] and Kim and Kim [28]. We can also find applications of $\mathrm{TM}_k^{(1)}(t)$ and $\mathrm{TCM}_k^{(2)}(t)$ in optimal capital allocation problems based on tail mean-variance models; see Landsman [30], Xu and Mao [44], Eini and Khaloozadeh [16], and Cai and Wang [10] for details. Nevertheless, few existing works have focused on $\mathrm{TM}_k^{(n)}(t)$ or $\mathrm{TCM}_k^{(n)}(t)$ with higher orders, which also have a wide range of applications in constructing insurance premium principles and other risk measures incorporating higher tail moments (e.g., tail skewness and tail kurtosis); see Ramsay [39] and Bawa and Lindenberg [6]. Among the few contributions, Kim [27] gave some explicit expressions for the degenerate univariate version of $\mathrm{TM}_k^{(n)}(t)$ with $d = 1$ when the risk has a distribution from the exponential family, and Landsman *et al.* [31] extended the work of Kim [27] to the elliptical and log-elliptical distribution classes.

In this paper, we study the asymptotic behavior of $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(n)}(t)$ as $t \to \infty$ under the framework in which $X_1, \ldots, X_d$ are pairwise asymptotically independent and possess distributions from the Fréchet or Gumbel MDA. Under our models, we will provide a uniform methodology by which asymptotic results on $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(n)}(t)$ can be obtained for any $n \in \mathbf{N}^+$. All of our results are in the concise form of some constant times t^n. The constants appearing in the formulas for $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(2n)}(t)$ are proved to be positive, and the constant corresponding to $\mathrm{TCM}_k^{(2n-1)}(t)$ is also nonzero for most of choices of the model parameters. Hence, most of our asymptotic results are precise ones which enable us to effectively estimate $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(n)}(t)$ for large t. Additionally, thanks to the assumption of asymptotic independence among $X_1, \ldots, X_d$, the results obtained depend only on information from the marginal distributions of $(X_1, \ldots, X_d)$, and hence can bring high efficiency to practical calculations. Another interesting finding observed from the derivations of our main results is that, although $X_1, \ldots, X_d$ are set to be real-valued, the left tails of $X_1, \ldots, X_d$ do not affect the asymptotic properties of $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(n)}(t)$ under our models, even when they are asymptotically comparable to the corresponding right tails.

The rest of this paper consists of four sections and an appendix. Section 2 introduces necessary preliminaries regarding some classes of distributions and asymptotic independence. Section 3 states the underlying assumptions of this work and presents our main asymptotic results for the TM and TCM. Section 4 gives a numerical study on the relative errors of the asymptotic results obtained when there are two components in the system. Section 5 proves our main results after some preparatory lemmas. The appendix is devoted especially to discussing the degenerate univariate versions of $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(n)}(t)$ with $d = 1$.

2. Preliminaries

In what follows, a distribution $V = 1 - \overline{V}$ is always assumed to have an infinite upper endpoint, i.e., $\overline{V}(x) > 0$ for any $x \in (-\infty, \infty)$. In extreme value theory, V is said to belong to the Fréchet MDA if there is some $\alpha \geq 0$ such that the relation

$$\lim_{t\to\infty} \frac{\overline{V}(tx)}{\overline{V}(t)} = x^{-\alpha} \tag{2.1}$$

holds for any $x > 0$. In this case, V is also said to be from the class of regular variation, and we express the regularity property in (2.1) as $V \in \mathcal{R}_{-\alpha}$, so that $\mathcal{R}$ is the union of all $\mathcal{R}_{-\alpha}$ over the range $\alpha \geq 0$. The class $\mathcal{R}$ is an important class of heavy-tailed distributions, and its main members include the Pareto distribution, Student's t-distribution, and the log-gamma distribution. See Bingham *et al.* [7] for a monograph on regular variation. By definition, V is said to belong to the Gumbel MDA (with an infinite upper endpoint) if there is some positive auxiliary function h such that the relation

$$\lim_{t\to\infty} \frac{\overline{V}(t + h(t)x)}{\overline{V}(t)} = \mathrm{e}^{-x} \tag{2.2}$$

holds for any $x \in (-\infty, \infty)$. We denote by $V \in \mathrm{GMDA}(h)$ the property stated in (2.2). It is known that the function h is unique up to asymptotic equivalence and satisfies $h(t) = o(t)$; see Chapter 1.1 of Resnick [40] or Chapter 3.3.3 of Embrechts *et al.* [18]. The Gumbel MDA contains both light-tailed distributions (e.g., the exponential and normal distributions) and heavy-tailed distributions (e.g., the log-normal distribution). It is easy to check that if $V \in \mathrm{GMDA}(h)$, then V belongs to the class of rapid variation, which is denoted by $\mathcal{R}_{-\infty}$ and is characterized by the following relation:

$$\lim_{t\to\infty} \frac{\overline{V}(tx)}{\overline{V}(t)} = 0, \qquad x > 1.$$

Clearly, the concepts of regular and rapid variation can be naturally extended to a general positive function g. Namely, for some $\beta \in [-\infty, \infty]$, we write $g \in \mathcal{R}_{\beta}$ if (2.1) holds with $\overline{V}$ and $-\alpha$ replaced by g and β, respectively. The well-known Karamata-type results hold for regularly and rapidly varying functions; i.e., if $g \in \mathcal{R}_{\beta}$ with $\beta \in (-\infty, -1)$, then

$$\lim_{t\to\infty} \frac{\int_t^{\infty} g(x)\mathrm{d}x}{tg(t)} = -\frac{1}{\beta + 1}, \tag{2.3}$$

while if $\beta = -\infty$ and g is non-increasing, then it holds for any $r \in (-\infty, \infty)$ that

$$\lim_{t\to\infty} \frac{\int_t^{\infty} x^r g(x)\mathrm{d}x}{t^{r+1} g(t)} = 0. \tag{2.4}$$

See, e.g., Appendix A3 of Embrechts *et al.* [18] for more details.

Given d real-valued random variables $Z_1, \ldots, Z_d$ without upper bounds, we say they are pairwise asymptotically independent if, for each pair $1 \leq i \neq j \leq d$,

$$\lim_{t\to\infty} \frac{\mathbb{P}(|Z_i| > t, Z_j > t)}{\mathbb{P}(Z_i > t) + \mathbb{P}(Z_j > t)} = 0; \tag{2.5}$$

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see, among many others, Chen and Yuen [12], Li [33], and Leipus *et al.* [32] for discussions and applications of this dependence structure. Note that the relation (2.5) will play an important role in dealing with joint probabilities related to the left or right tails of the random variables. If the right tails of $Z_1, \ldots, Z_d$ are asymptotically proportionally equivalent, i.e., $\lim_{t\to\infty} \mathbb{P}(Z_i > t)/\mathbb{P}(Z_1 > t) = c_i$ for each $1 \le i \le d$ and some $c_i > 0$, then the relation (2.5) is equivalent to

$$\lim_{t\to\infty} \frac{\mathbb{P}(|Z_i| > t, Z_j > t)}{\mathbb{P}(Z_1 > t)} = 0. \tag{2.6}$$

If, further, Z_1 has a regularly varying tail, then (2.6) implies that for any $a > 0$ and $b > 0$,

$$\lim_{x\to\infty} \frac{\mathbb{P}(|Z_i| > at, Z_j > bt)}{\mathbb{P}(Z_1 > t)} = 0. \tag{2.7}$$

To verify (2.7), we only need to note that

$$\limsup_{t\to\infty} \frac{\mathbb{P}(|Z_i| > at, Z_j > bt)}{\mathbb{P}(Z_1 > t)} \le \lim_{t\to\infty} \frac{\mathbb{P}\big(|Z_i| > \min\{a, b\}t, X_j > \min\{a, b\}t\big)}{\mathbb{P}(Z_1 > \min\{a, b\}t)} \frac{\mathbb{P}(Z_1 > \min\{a, b\}t)}{\mathbb{P}(Z_1 > t)} = 0.$$

Hereafter, unless otherwise stated, all limit relationships hold as $t \to \infty$. For two positive functions g_1 and g_2, we write $g_1(t) \lesssim g_2(t)$ or $g_2(t) \gtrsim g_1(t)$ if $\limsup g_1(t)/g_2(t) \le 1$; we write $g_1(t) \sim g_2(t)$ if $\lim g_1(t)/g_2(t) = 1$; and we write $g_1(t) \asymp g_2(t)$ if $0 < \liminf g_1(t)/g_2(t) \le \limsup g_1(t)/g_2(t) < \infty$. For a real number a, we write $a^+ = \max\{a, 0\}$ and $a^- = -\min\{a, 0\}$. As usual, $\mathbf{1}_{\{\cdot\}}$ stands for the indicator function.

3. Main results

In this section, we present our main asymptotic results for the TM and TCM defined by (1.1) and (1.2) with $n \in \mathbf{N}^+$. Denote by $F_1, \ldots, F_d$ the distributions of the individual risks $X_1, \ldots, X_d$. We conduct our study under the following two assumptions, respectively.

Assumption 3.1. *$F_1 \in \mathcal{R}_{-\alpha}$ for some $\alpha > n$, and $\overline{F}_i(t) \sim c_i\overline{F}_1(t)$ and $F_i(-t) = O(\overline{F}_1(t))$ for each $1 \le i \le d$ and some $c_i > 0$. Also, $X_1, \ldots, X_d$ are pairwise asymptotically independent.*

Assumption 3.2. *$F_1 \in \mathrm{GMDA}(h)$, and $\overline{F}_i(t) \sim c_i\overline{F}_1(t)$ and $F_i(-t) = O(\overline{F}_1(t))$ for each $1 \le i \le d$ and some $c_i > 0$. Also, for each pair $1 \le i \ne j \le d$,*

$$\lim_{t\to\infty} \frac{\mathbb{P}\big(|X_i| > \epsilon h(t), X_j > t\big)}{\overline{F}_1(t)} = 0, \quad \textit{for any } \epsilon > 0, \tag{3.1}$$

and

$$\lim_{t\to\infty} \frac{\mathbb{P}\big(X_i > Lh(t), X_j > Lh(t)\big)}{\overline{F}_1(t)} = 0, \quad \textit{for some } L > 0. \tag{3.2}$$

The conditions regarding the marginal distributions of $(X_1, \ldots, X_d)$ in Assumptions 3.1 and 3.2 guarantee the existence of the TM and TCM. The dependence structure defined by (3.1) and (3.2) was first proposed in Mitra and Resnick [37] and has been extensively studied and applied in risk theory; see Asimit *et al.* [2], Hashorva and Li [21], and Asimit and Li [3]. Since $h(t) = o(t)$, the relation (3.1) obviously implies pairwise asymptotic independence among $X_1, \ldots, X_d$.

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For brevity, in what follows we write

$$C_i = \frac{c_i}{\sum_{j=1}^{d} c_j} \in (0, 1) , \quad 1 \le i \le d, \tag{3.3}$$

where $c_1, \ldots, c_d$ are the positive constants from Assumption 3.1 or 3.2. Now we are ready to state our main results.

Theorem 3.1. *Consider the TM defined by* (1.1) *with* $n \in \mathbf{N}^+$.

(i) Under Assumption 3.1*, it holds for each* $1 \le k \le d$ *that*

$$\mathrm{TM}_k^{(n)}(t) \sim \frac{\alpha}{\alpha - n} C_k t^n. \tag{3.4}$$

(ii) Under Assumption 3.2*, it holds for each* $1 \le k \le d$ *that*

$$\mathrm{TM}_k^{(n)}(t) \sim C_k t^n. \tag{3.5}$$

Recall the TCM defined by (1.2). By the binomial expansion theorem, we have

$$\mathrm{TCM}_k^{(n)}(t) = \sum_{i=0}^{n-1} \binom{n}{i} (-1)^i \left(\mathrm{TM}_k^{(1)}(t)\right)^i \mathrm{TM}_k^{(n-i)}(t) + (-1)^n \left(\mathrm{TM}_k^{(1)}(t)\right)^n,$$

where $\binom{n}{i} = n!/\left(i!(n-i)!\right)$. Then, applying Theorem 3.1 immediately yields the following corollary for the TCM.

Corollary 3.1. *Consider the TCM defined by* (1.2) *with* $n \in \mathbf{N}^+$.

(i) Under Assumption 3.1*, it holds for each* $1 \le k \le d$ *that*

$$\mathrm{TCM}_k^{(n)}(t) = \left(A_{\alpha,n,k} + o(1)\right) t^n, \tag{3.6}$$

where

$$A_{\alpha,n,k} = \sum_{i=0}^{n-1} \binom{n}{i} (-1)^i \frac{\alpha^{i+1}}{(\alpha-1)^i (\alpha - n + i)} C_k^{i+1} + (-1)^n \left(\frac{\alpha}{\alpha-1}\right)^n C_k^n. \tag{3.7}$$

(ii) Under Assumption 3.2*, it holds for each* $1 \le k \le d$ *that*

$$\mathrm{TCM}_k^{(n)}(t) = \left(A_{n,k} + o(1)\right) t^n, \tag{3.8}$$

where

$$A_{n,k} = \lim_{\alpha \to \infty} A_{\alpha,n,k} = C_k (1 - C_k) \left((1 - C_k)^{n-1} + (-1)^n C_k^{n-1}\right). \tag{3.9}$$

Remark 3.1. The relations (3.4) and (3.5) imply that $\mathrm{TM}_k^{(n)}(t) \to \infty$ for any $n \in \mathbf{N}^+$ under our models. Hence, when t is large enough, $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(2n)}(t)$ define two standard measures with non-negativity. Moreover, it is easy to see from (3.9) that $A_{2n,k} > 0$, since $0 < C_k < 1$. Actually, we can also prove that

$$A_{\alpha,2n,k} > 0 \tag{3.10}$$

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under the conditions of Corollary 3.1(i); see Section 5. Therefore, (3.6) and (3.8) provide us with precise asymptotic estimates of $\mathrm{TCM}_k^{(2n)}(t)$. On the other hand, both $A_{\alpha,2n-1,k}$ and $A_{2n-1,k}$ may be less than or equal to 0 for certain choices of α and C_k. When $A_{\alpha,2n-1,k}<0$ or $A_{2n-1,k}<0$, (3.6) or (3.8) implies that $\mathrm{TCM}_k^{(2n-1)}(t)$ tends to $-\infty$ and hence is not a non-negative measure for large t. In case $A_{\alpha,2n-1,k}=0$ or $A_{2n-1,k}=0$, (3.6) or (3.8) fails to give a precise asymptotic result for $\mathrm{TCM}_k^{(2n-1)}(t)$. Seeking precise estimates of $\mathrm{TCM}_k^{(2n-1)}(t)$ in such a case requires more nuanced analysis, and we will not focus on it in the present paper.

Further consider the special case where, as mentioned before, the threshold t is chosen to be $\mathrm{VaR}_q(S_d)$ with $q\in(0,1)$. By Theorem 3.1 of Chen and Yuen [12] and Theorem 3.1 of Hashorva and Li [21], we have

$$\mathbb{P}(S_d>t)\sim\mathbb{P}\left(\sum_{i=1}^{d}X_i^+>t\right)\sim\sum_{i=1}^{d}\overline{F}_i(t)\sim\left(\sum_{i=1}^{d}c_i\right)\overline{F}_1(t) \tag{3.11}$$

under Assumption 3.1 or 3.2. Then, using Lemma 2.1 of Asimit *et al.* [2] gives that

$$\mathrm{VaR}_q(S_d)\sim\left(\sum_{i=1}^{d}c_i\right)^{1/\alpha}\mathrm{VaR}_q(X_1),\quad q\uparrow 1, \tag{3.12}$$

under Assumption 3.1, while using Lemma 2.4 and the analysis under Corollary 3.2 of Asimit *et al.* [2] gives that

$$\mathrm{VaR}_q(S_d)\sim\mathrm{VaR}_{1-(1-q)/\sum_{i=1}^{d}c_i}(X_1),\quad q\uparrow 1, \tag{3.13}$$

under Assumption 3.2. Hence, plugging (3.12) into (3.4) and (3.6) and plugging (3.13) into (3.5) and (3.8), we obtain the following asymptotic results for the corresponding TM and TCM as $q\uparrow 1$.

Corollary 3.2. *Consider the TM and TCM defined by* (1.1) *and* (1.2) *with* $n\in\mathbf{N}^+$ *and* $t=\mathrm{VaR}_q(S_d)$.

(i) Under Assumption 3.1*, we have, for each* $1\le k\le d$,

$$\mathrm{TM}_k^{(n)}\left(\mathrm{VaR}_q(S_d)\right)\sim\frac{\alpha}{\alpha-n}C_k\left(\sum_{i=1}^{d}c_i\right)^{n/\alpha}\left(\mathrm{VaR}_q(X_1)\right)^n,\quad q\uparrow 1,$$

and

$$\mathrm{TCM}_k^{(n)}\left(\mathrm{VaR}_q(S_d)\right)=\left(\left(\sum_{i=1}^{d}c_i\right)^{n/\alpha}A_{\alpha,n,k}+o(1)\right)\left(\mathrm{VaR}_q(X_1)\right)^n,\quad q\uparrow 1,$$

where $A_{\alpha,n,k}$ *is given by* (3.7).

(ii) Under Assumption 3.2*, we have, for each* $1\le k\le d$,

$$\mathrm{TM}_k^{(n)}\left(\mathrm{VaR}_q(S_d)\right)\sim C_k\left(\mathrm{VaR}_{1-(1-q)/\sum_{i=1}^{d}c_i}(X_1)\right)^n,\quad q\uparrow 1,$$

and

$$\mathrm{TCM}_k^{(n)}\left(\mathrm{VaR}_q(S_d)\right)=\left(A_{n,k}+o(1)\right)\left(\mathrm{VaR}_{1-(1-q)/\sum_{i=1}^{d}c_i}(X_1)\right)^n,\quad q\uparrow 1,$$

where $A_{n,k}$ *is given by* (3.9).

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Remark 3.2. Sine $X_1, \ldots, X_d$ are pairwise asymptotically independent, all of the asymptotic results obtained in Theorem 3.1, Corollary 3.1, and Corollary 3.2 involve only information from the marginal distributions of $(X_1, \ldots, X_d)$. This feature enables us to overcome the difficulties caused by dependence structures when estimating the values of the TM and TCM.

Remark 3.3. Furman and Landsman [19] proposed the tail variance premium (TVP) and tail standard deviation premium (TSDP) principles for individual insurance risks. In terms of our notation, they are formulated as

$$\mathrm{TVP}_k(t) = \mathrm{TM}_k^{(1)}(t) + w\mathrm{TCM}_k^{(2)}(t)$$

and

$$\mathrm{TSDP}_k(t) = \mathrm{TM}_k^{(1)}(t) + w\sqrt{\mathrm{TCM}_k^{(2)}(t)},$$

where w is a non-negative constant; see Definition 3 of Furman and Landsman [19]. Applying Theorem 3.1 and Corollary 3.1 immediately yields that

$$\mathrm{TVP}_k(t) \sim w\mathrm{TCM}_k^{(2)}(t) \sim w\left(\frac{\alpha}{\alpha-2}C_k - \frac{\alpha^2}{(\alpha-1)^2}C_k^2\right)t^2$$

and

$$\mathrm{TSDP}_k(t) \sim \left(\frac{\alpha}{\alpha-1}C_k + w\sqrt{\frac{\alpha}{\alpha-2}C_k - \frac{\alpha^2}{(\alpha-1)^2}C_k^2}\right)t$$

under Assumption 3.1, and that

$$\mathrm{TVP}_k(t) \sim w\mathrm{TCM}_k^{(2)}(t) \sim wC_k\,(1-C_k)\,t^2$$

and

$$\mathrm{TSDP}_k(t) \sim \left(C_k + w\sqrt{C_k\,(1-C_k)}\right)t$$

under Assumption 3.2.

Remark 3.4. Our results can also be applied to find asymptotic solutions for some optimal capital allocation problems based on tail moment models. Denote by p_k the capital allocated to X_k for each $1 \le k \le d$, and by $p = \sum_{i=1}^d p_i$ the total capital, which is a fixed number. Dhaene *et al.* [15] proposed a capital allocation criterion suggesting that the individual capital p_k be set as close as possible to X_k, in the sense of minimizing some distance measure. Xu and Mao [44] extended the idea of Dhaene *et al.* [15] to a tail mean-variance model. Here we choose a quadratic distance measure and consider the following reduced version of the optimization problem studied in Xu and Mao [44]:

$$\min_{p_1(t),\ldots,p_d(t)} \sum_{i=1}^d \mathbb{E}\left((X_i - p_i(t))^2 \,\middle|\, S_d > t\right), \quad \text{s.t.} \sum_{i=1}^d p_i(t) = p(t), \tag{3.14}$$

where $p_1(t), \ldots, p_d(t)$ and $p(t)$ are the individual capitals and total capital corresponding to the threshold t. By Theorem 2.2 of Xu and Mao [44], the optimal solution of (3.14) is

$$p_k^*(t) = \frac{p(t) - \sum_{i=1}^d \mathrm{TM}_i^{(1)}(t)}{d} + \mathrm{TM}_k^{(1)}(t), \quad 1 \le k \le d. \tag{3.15}$$

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Note that (3.15) holds for any $X_1, \ldots, X_d$ with finite second-order moments that make the problem (3.14) meaningful. Now, given the conditions of Theorem 3.1, the asymptotic estimate of $p_k^*(t)$ depends on the asymptotic behavior of $p(t)$. Assume that $p(t)$ is set to be asymptotically proportionally equivalent to the conditional tail expectation of S_d, i.e., for some $H \geq 1$,

$$p(t) \sim H\mathbb{E}(S_d \,|S_d > t) = H \sum_{i=1}^{d} \mathrm{TM}_i^{(1)}(t).$$

Then, applying Theorem 3.1 to (3.15) gives that

$$p_k^*(t) \sim \frac{\alpha}{\alpha - 1} \left(\frac{H-1}{d} + C_k \right) t$$

under Assumption 3.1, and that

$$p_k^*(t) \sim \left(\frac{H-1}{d} + C_k \right) t$$

under Assumption 3.2. When $H = 1$, so that $p(t) \sim \mathbb{E}(S_d \,|S_d > t)$, it holds under both Assumptions 3.1 and 3.2 that

$$p_k^*(t) \sim \mathrm{TM}_k^{(1)}(t),$$

which coincides with the Euler rule. Actually, in the case of $H = 1$, the asymptotic optimal solution $\left(p_1^*(t), \ldots, p_d^*(t)\right)$ minimizes each term in the summation of the optimization objective in (3.14). Moreover, if $p(t)$ is set to be a quantity such that $t = o(p(t))$, say, a linear combination of some higher-order TM and TCM, then it holds under both Assumptions 3.1 and 3.2 that

$$p_1^*(t) \sim \cdots \sim p_d^*(t) \sim \frac{1}{d} p(t).$$

In this case, the asymptotic optimal allocation rule is just to allocate the total capital equally to each individual risk.

4. Numerical study on relative errors

In this section, we study the relative errors between our main asymptotic results and the accurate values corresponding to them when there are two individual risks (i.e., $d = 2$) in the system. For this purpose, denote by $\widetilde{\mathrm{TM}}_k^{(n)}(t)$ and $\widetilde{\mathrm{TCM}}_k^{(n)}(t)$ with $k = 1, 2$ the asymptotic results obtained in Theorem 3.1 and Corollary 3.1 for $\mathrm{TM}_k^{(n)}(t)$ and $\mathrm{TCM}_k^{(n)}(t)$. Denote by $R_{\mathrm{TM},k}^{(n)}(t)$ and $R_{\mathrm{TCM},k}^{(n)}(t)$ the corresponding relative errors, i.e.,

$$R_{\mathrm{TM},k}^{(n)}(t) = \frac{\left| \mathrm{TM}_k^{(n)}(t) - \widetilde{\mathrm{TM}}_k^{(n)}(t) \right|}{\mathrm{TM}_k^{(n)}(t)} \tag{4.1}$$

and

$$R_{\mathrm{TCM},k}^{(n)}(t) = \frac{\left| \mathrm{TCM}_k^{(n)}(t) - \widetilde{\mathrm{TCM}}_k^{(n)}(t) \right|}{\mathrm{TCM}_k^{(n)}(t)}. \tag{4.2}$$

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For simplicity, let the random risks X_1 and X_2 be non-negative in this section. We assume that the joint distribution of (X_1, X_2) belongs to the Farlie–Gumbel–Morgenstern (FGM) family with a parameter $\theta \in [-1, 1]$, i.e.,

$$\mathbb{P}(X_1 \leq x, X_2 \leq y) = F_1(x)F_2(y)\left(1 + \theta \overline{F}_1(x)\overline{F}_2(y)\right). \tag{4.3}$$

It follows from (4.3) that

$$\mathbb{P}(X_1 > x, X_2 > y) = \overline{F}_1(x)\overline{F}_2(y)\,(1 + \theta F_1(x)F_2(y)), \tag{4.4}$$

which implies asymptotic independence between X_1 and X_2. Note that X_1 and X_2 are positively dependent if $\theta \geq 0$, and are negatively dependent if $\theta \leq 0$; see Definition 5.2.1 of Nelsen [38].

In what follows, we give some numerical analyses in two specific scenarios to illustrate the effects of different model parameters on the convergence rates of $R^{(n)}_{\mathrm{TM},1}(t)$ and $R^{(2)}_{\mathrm{TCM},1}(t)$.

Scenario 4.1. Let (X_1, X_2) have the joint distribution given by (4.3) with the marginal distributions

$$F_i(x) = 1 - \lambda_i^{\alpha}\,(x + \lambda_i)^{-\alpha}, \quad x \geq 0,\ \alpha > 0,\ i = 1, 2,$$

where λ_1 and λ_2 are set to be 100 and 120, respectively.

In this scenario, it is easy to check that Assumption 3.1 holds with $c_1 = 1$ and $c_2 = (\lambda_2/\lambda_1)^{\alpha}$. For choices of α and n such that $\alpha > n$, applying Theorem 3.1(i) gives that

$$\widetilde{\mathrm{TM}}^{(n)}_1(t) = \frac{\alpha}{(\alpha - n)\,(1 + (\lambda_2/\lambda_1)^{\alpha})} t^n. \tag{4.5}$$

On the other hand, denote by f_i the density of X_i for $i = 1, 2$, i.e.,

$$f_i(x) = \alpha\lambda_i^{\alpha}\,(x + \lambda_i)^{-\alpha-1}, \quad x \geq 0.$$

By (4.3), the joint density of (X_1, X_2) is

$$f_{X_1,X_2}(x, y) = f_1(x)f_2(y)\,(1 + \theta(1 - 2F_1(x))\,(1 - 2F_2(y))), \quad x, y \geq 0.$$

Then, the joint density of $(X_1, X_1 + X_2)$ can be calculated and has the form

$$f_{X_1,X_1+X_2}(x, y) = f_1(x)f_2(y - x)\,(1 + \theta(1 - 2F_1(x))\,(1 - 2F_2(y - x))), \quad 0 \leq x \leq y.$$

Further calculations yield that

$$\begin{aligned}\mathbb{P}(X_1 \in \mathrm{d}x, X_1 + X_2 > t) &= \int_t^{\infty} f_{X_1,X_1+X_2}(x, y)\mathrm{d}y\mathrm{d}x\\ &= f_1(x)\overline{F}_2(t - x)\,(1 - \theta(1 - 2F_1(x))\,F_2(t - x))\,\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(X_1 + X_2 > t) &= \int_0^{\infty} \mathbb{P}(X_1 \in \mathrm{d}x, X_1 + X_2 > t)\\ &= \int_0^{t} f_1(x)\overline{F}_2(t - x)\,(1 - \theta(1 - 2F_1(x))\,F_2(t - x))\,\mathrm{d}x + \overline{F}_1(t).\end{aligned}$$

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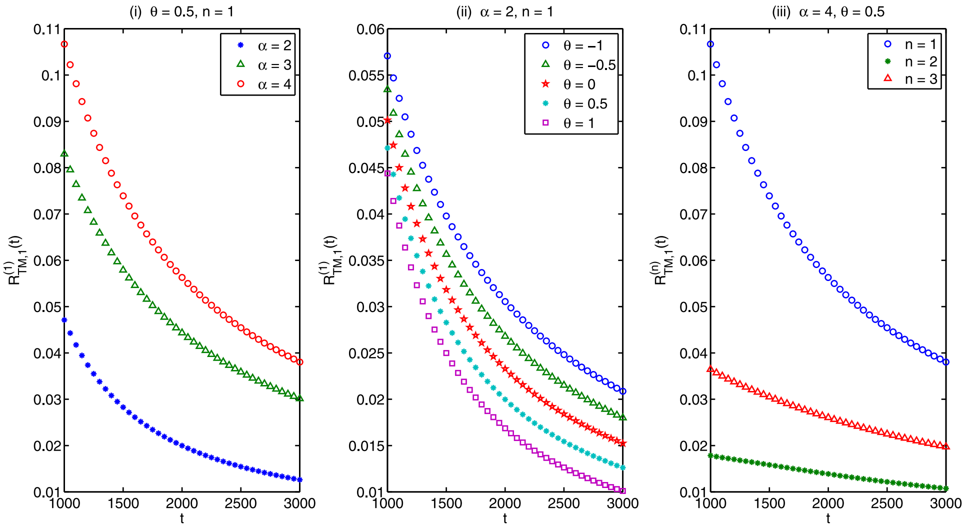

FIGURE 1. The graph of $R^{(n)}_{\mathrm{TM},1}(t)$ in Scenario 4.1 with different parameters.

Thus, we can obtain the exact value of $\mathrm{TM}^{(n)}_1(t)$ using

$$\mathrm{TM}^{(n)}_1(t) = \int_0^\infty x^n \mathbb{P}(X_1 \in \mathrm{d}x \mid X_1 + X_2 > t)$$
$$= \frac{\int_0^t x^n f_1(x)\overline{F}_2(t-x)\,(1-\theta(1-2F_1(x))\,F_2(t-x))\,\mathrm{d}x + \int_t^\infty x^n f_1(x)\mathrm{d}x}{\int_0^t f_1(x)\overline{F}_2(t-x)\,(1-\theta(1-2F_1(x))\,F_2(t-x))\,\mathrm{d}x + \overline{F}_1(t)}. \tag{4.6}$$

Then, plugging (4.5) and (4.6) into (4.1) gives $R^{(n)}_{\mathrm{TM},1}(t)$. We calculate the values of $R^{(n)}_{\mathrm{TM},1}(t)$ from $t = 1000$ to $t = 3000$ in steps of 50 for different choices of α, θ, and n, respectively, with the other parameters fixed. The corresponding numerical results are plotted in Figure 1. It is worth noting that the parameters α, θ, and n describe the tail behavior of the risks, the degree of the dependence, and the order of the TM, respectively. Thus, Figure 1(i) indicates that a heavier tail of X_1, i.e., a smaller value of α, implies a faster convergence rate for $R^{(1)}_{\mathrm{TM},1}(t)$. Additionally, Figure 1(ii) shows that the convergence rate of $R^{(1)}_{\mathrm{TM},1}(t)$ increases as the value of θ increases. In other words, the positive dependence between the risks may help to speed up the convergence of $R^{(1)}_{\mathrm{TM},1}(t)$, while the case of negative dependence is just the opposite. On the other hand, however, Figure 1(iii) seems inadequate to reveal a clear change law for $R^{(n)}_{\mathrm{TM},1}(t)$ with respect to the order parameter n.

For $R^{(n)}_{\mathrm{TCM},1}(t)$, we consider here only $R^{(2)}_{\mathrm{TCM},1}(t)$, which corresponds to the interesting TV risk measure. By Corollary 3.1(i), it holds for $\alpha > 2$ that

$$\widetilde{\mathrm{TCM}}^{(2)}_1(t) = \left(\frac{\alpha}{(\alpha-2)\,(1+(\lambda_2/\lambda_1)^\alpha)} - \frac{\alpha^2}{(\alpha-1)^2\,(1+(\lambda_2/\lambda_1)^\alpha)^2}\right) t^2. \tag{4.7}$$

The exact value of $\mathrm{TCM}^{(2)}_1(t)$ can be calculated by using (4.6) and the equality

$$\mathrm{TCM}^{(2)}_1(t) = \mathrm{TM}^{(2)}_1(t) - \left(\mathrm{TM}^{(1)}_1(t)\right)^2. \tag{4.8}$$

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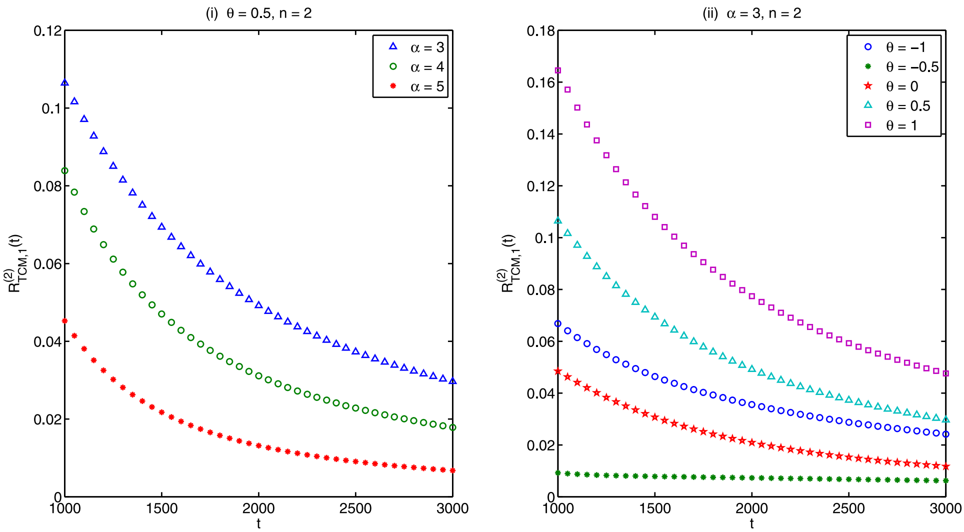

FIGURE 2. The graph of $R^{(n)}_{\mathrm{TCM},1}(t)$ in Scenario 4.1 with different parameters.

Then, $R^{(2)}_{\mathrm{TCM},1}(t)$ can be obtained by plugging (4.7) and (4.8) into (4.2). We calculate the values of $R^{(2)}_{\mathrm{TCM},1}(t)$ from $t = 1000$ to $t = 3000$ in steps of 50 for different choices of α and θ, respectively, and present the corresponding numerical results in Figure 2. In contrast to the phenomenon shown in Figure 1(i) for $R^{(1)}_{\mathrm{TM},1}(t)$, Figure 2(i) indicates that the lighter the tail of X_1 is, the faster $R^{(2)}_{\mathrm{TCM},1}(t)$ converges to 0. On the other hand, we cannot make a clear judgment on the change law of $R^{(2)}_{\mathrm{TCM},1}(t)$ with respect to the dependence parameter θ according to Figure 2(ii).

Scenario 4.2. Let (X_1, X_2) have the joint distribution given by (4.3) with the marginal distributions

$$F_1(x) = F_2(x) = 1 - \mathrm{e}^{-(\log(x+1))^{\gamma}}, \quad x \geq 0, \ \gamma > 1.$$

In this scenario, we have $F_1 \in \mathrm{GMDA}(h)$ with

$$h(t) \sim \frac{t}{\gamma(\log t)^{\gamma-1}}.$$

Since $h(t) \to \infty$, (3.1) follows from (4.4). It is not difficult to verify that

$$\overline{F}_1^2\,(h(t)) = o\big(\overline{F}_1\,(t)\big)\,,$$

which, combined with (4.4), implies (3.2). Thus, Assumption 3.2 holds with $c_1 = c_2 = 1$. Applying Theorem 3.1(ii) gives that

$$\widetilde{\mathrm{TM}}_1^{(n)}(t) = \frac{1}{2}t^n. \tag{4.9}$$

The exact value of $\mathrm{TM}_1^{(n)}(t)$ can be obtained using (4.6) with

$$f_1(x) = \frac{\gamma(\log\,(x+1))^{\gamma-1}\,\mathrm{e}^{-(\log(x+1))^{\gamma}}}{x+1}, \quad x \geq 0.$$

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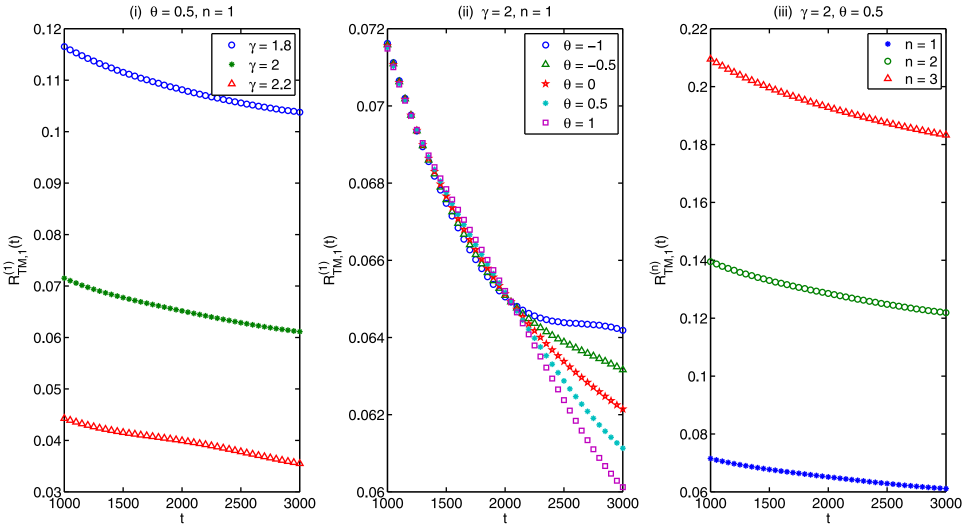

FIGURE 3. The graph of $R^{(n)}_{\mathrm{TM},1}(t)$ in Scenario 4.2 with different parameters.

Then, plugging (4.9) and (4.6) into (4.1) gives $R^{(n)}_{\mathrm{TM},1}(t)$. Additionally, by Corollary 3.1(ii), we have

$$\widetilde{\mathrm{TCM}}^{(2)}_1(t)=\frac{1}{4}t^2,$$

which, together with (4.8) and (4.2), gives $R^{(2)}_{\mathrm{TCM},1}(t)$. Similarly as in Scenario 4.1, we calculate the values of $R^{(n)}_{\mathrm{TM},1}(t)$ and $R^{(2)}_{\mathrm{TCM},1}(t)$ from $t=1000$ to $t=3000$ in steps of 50 for different choices of the model parameters; we present the corresponding numerical results in Figures 3 and 4. In this scenario, X_1 and X_2 have an identical rapidly varying tail, which is lighter than any regularly varying tails as considered in Scenario 4.1. The numerical results reflect different relationships between the model parameters and the convergence rates of $R^{(n)}_{\mathrm{TM},1}(t)$ and $R^{(2)}_{\mathrm{TCM},1}(t)$. Figures 3(i, ii) and 4(i, ii) indicate that a lighter tail of X_1 or a larger value of θ may lead to faster convergence rates for both $R^{(1)}_{\mathrm{TM},1}(t)$ and $R^{(2)}_{\mathrm{TCM},1}(t)$. Moreover, Figure 3(iii) shows that a smaller value of n tends to give a faster convergence rate for $R^{(n)}_{\mathrm{TM},1}(t)$.

It should be clarified that all the observations from the numerical study above are based on the model setups and value range of t that we considered in Scenarios 4.1 and 4.2. Rigorous investigations of the convergence properties of $R^{(n)}_{\mathrm{TM},k}(t)$ and $R^{(n)}_{\mathrm{TCM},k}(t)$ would require us to seek analytical asymptotic expressions for $R^{(n)}_{\mathrm{TM},k}(t)$ and $R^{(n)}_{\mathrm{TCM},k}(t)$ with respect to the model parameters. This task is essentially related to second-order asymptotic results on $\mathrm{TM}^{(n)}_k(t)$ and $\mathrm{TCM}^{(n)}_k(t)$, and we will not pursue it in this paper. The reader is referred to Section 5.1 of Li [34] for analytical asymptotic results on $R^{(1)}_{\mathrm{TM},1}(t)$ in some special cases of Scenario 4.1.

5. Proofs of the main results

We begin with two lemmas established under general frameworks, in which the distributions of the random variables are not assumed to be of any specific type.

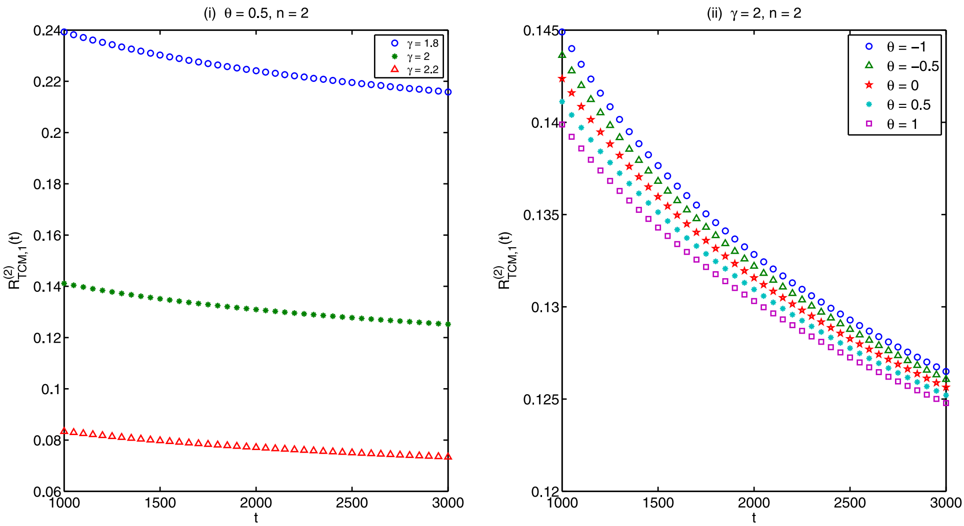

FIGURE 4. The graph of $R^{(n)}_{\mathrm{TCM},1}(t)$ in Scenario 4.2 with different parameters.

Lemma 5.1. *Let $Z_1, \ldots, Z_d$ be d real-valued random variables with distributions $V_1, \ldots, V_d$. Assume that the following conditions hold:*

(a) $\overline{V}_i(t) \asymp \overline{V}_1(t)$ *for each* $1 \le i \le d$.

(b) *For any* $\epsilon > 0$ *and each pair* $1 \le i \ne j \le d$,

$$\lim_{t\to\infty} \frac{\mathbb{P}\big(Z_i > \epsilon t, Z_j > t\big)}{\overline{V}_1(t)} = 0.$$

(c) $\mathbb{P}\Big(\sum_{i=1}^d Z_i > t\Big) \sim \mathbb{P}\Big(\sum_{i=1}^d Z_i^+ > t\Big) \sim \sum_{i=1}^d \overline{V}_i(t)$.

Then, for each $1 \le k \le d$, *we have the following assertions:*

(i) *For any* $0 < \varepsilon \le 1$, *it holds uniformly for* $y \in [\varepsilon, 1]$ *that*

$$\mathbb{P}\left(Z_k > yt, \sum_{i=1}^d Z_i > t\right) \sim \overline{V}_k(t).$$

(ii) *It holds uniformly for* $s \in [t, \infty)$ *that*

$$\mathbb{P}\left(Z_k > s, \sum_{i=1}^d Z_i > t\right) \sim \overline{V}_k(s).$$

Proof. **Part (i):** Clearly, for all $y \in [\varepsilon, 1]$,

$$\mathbb{P}\left(Z_k > t, \sum_{i=1}^d Z_i > t\right) \le \mathbb{P}\left(Z_k > yt, \sum_{i=1}^d Z_i > t\right) \le \mathbb{P}\left(Z_k > \varepsilon t, \sum_{i=1}^d Z_i > t\right).$$

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Thus, we only need to prove

$$\mathbb{P}\left(Z_k > \varepsilon t, \sum_{i=1}^{d} Z_i > t\right) \lesssim \overline{V}_k(t) \tag{5.1}$$

and

$$\mathbb{P}\left(Z_k > t, \sum_{i=1}^{d} Z_i > t\right) \gtrsim \overline{V}_k(t). \tag{5.2}$$

We have

$$\begin{aligned}\mathbb{P}\left(Z_k > \varepsilon t, \sum_{i=1}^{d} Z_i > t\right) &\le \mathbb{P}\left(Z_k > \varepsilon t, \sum_{i=1}^{d} Z_i^+ > t\right)\\ &= \mathbb{P}\left(\sum_{i=1}^{d} Z_i^+ > t\right) - \mathbb{P}\left(\sum_{i=1}^{d} Z_i^+ > t, Z_k \le \varepsilon t\right)\\ &=: \mathbb{P}\left(\sum_{i=1}^{d} Z_i^+ > t\right) - I(t). \end{aligned} \tag{5.3}$$

It holds that

$$\begin{aligned} I(t) &\ge \mathbb{P}\left(\bigcup_{i=1}^{d} \{Z_i > t\}, Z_k \le \varepsilon t\right)\\ &= \mathbb{P}\left(\bigcup_{i=1}^{d} \{Z_i > t\}\right) - \mathbb{P}\left(\bigcup_{i=1}^{d} \{Z_i > t\}, Z_k > \varepsilon t\right)\\ &\ge \sum_{\substack{i=1\\ i\ne k}}^{d} \mathbb{P}(Z_i > t) - \sum_{1\le i<j\le d} \mathbb{P}\big(Z_i > t, Z_j > t\big) - \sum_{\substack{i=1\\ i\ne k}}^{d} \mathbb{P}(Z_i > t, Z_k > \varepsilon t)\,, \end{aligned} \tag{5.4}$$

where in the last step we used the Bonferroni inequality and the fact that $\{Z_k > t, Z_k > \varepsilon t\} = \{Z_k > t\}$, since $0 < \varepsilon \le 1$. We then obtain (5.1) by plugging (5.4) into (5.3) and applying the conditions (a)–(c). On the other hand, we write

$$\begin{aligned}\mathbb{P}\left(Z_k > t, \sum_{i=1}^{d} Z_i > t\right) &= \mathbb{P}\left(\sum_{i=1}^{d} Z_i > t\right) - \mathbb{P}\left(\sum_{i=1}^{d} Z_i > t, Z_k \le t\right)\\ &=: \mathbb{P}\left(\sum_{i=1}^{d} Z_i > t\right) - J(t). \end{aligned} \tag{5.5}$$

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It holds that

$$
\begin{aligned}
J(t) &= \mathbb{P}\left(\sum_{i=1}^{d} Z_i > t,\, Z_k \le t,\, \bigcup_{\substack{i=1\\ i\neq k}}^{d} \{Z_i > t\}\right) + \mathbb{P}\left(\sum_{i=1}^{d} Z_i > t,\, \bigcap_{i=1}^{d} \{Z_i \le t\}\right) \\
&\le \sum_{\substack{i=1\\ i\neq k}}^{d} \mathbb{P}(Z_i > t) + \mathbb{P}\left(\sum_{i=1}^{d} Z_i^{+} > t,\, \bigcap_{i=1}^{d} \{Z_i \le t\}\right) \\
&= \sum_{\substack{i=1\\ i\neq k}}^{d} \mathbb{P}(Z_i > t) + \mathbb{P}\left(\sum_{i=1}^{d} Z_i^{+} > t\right) - \mathbb{P}\left(\bigcup_{i=1}^{d} \{Z_i > t\}\right) \\
&\le \mathbb{P}\left(\sum_{i=1}^{d} Z_i^{+} > t\right) - \mathbb{P}(Z_k > t) + \sum_{1\le i<j\le d} \mathbb{P}\big(Z_i > t,\, Z_j > t\big)\,.
\end{aligned} \tag{5.6}
$$

Plugging (5.6) into (5.5) and noting the conditions (a)–(c), we obtain (5.2) and complete the proof of the assertion (i).

Part (ii): We always have

$$\mathbb{P}\left(Z_k > s,\, \sum_{i=1}^{d} Z_i > t\right) \le \overline{V}_k(s).$$

By the assertion (i) with $\varepsilon = 1$, it holds uniformly for $s \in [t, \infty)$ that

$$\mathbb{P}\left(Z_k > s,\, \sum_{i=1}^{d} Z_i > t\right) \ge \mathbb{P}\left(Z_k > s,\, \sum_{i=1}^{d} Z_i > s\right) \sim \overline{V}_k(s).$$

Combining the two estimates above, we obtain the assertion (ii). □

Lemma 5.2. *Let $Z_1, \ldots, Z_d$ be d real-valued random variables with distributions $V_1, \ldots, V_d$. Assume that the condition (a) of Lemma 5.1 and the following conditions hold:*

(b') For any $\epsilon > 0$ and each pair $1 \le i \neq j \le d$,

$$\lim_{t\to\infty} \frac{\mathbb{P}\big(Z_i^{-} > \epsilon t,\, Z_j > t\big) + \mathbb{P}\big(Z_i > t,\, Z_j > t\big)}{\overline{V}_1(t)} = 0.$$

(c') $\mathbb{P}\Big(\sum_{i=1}^{d} Z_i^{+} > t\Big) \sim \sum_{i=1}^{d} \overline{V}_i(t)$.

Then, for each $1 \le k \le d$ and any $\varepsilon > 0$, it holds uniformly for $y \in [\varepsilon, \infty)$ that

$$\mathbb{P}\left(Z_k^{-} > yt,\, \sum_{i=1}^{d} Z_i > t\right) = o\big(\overline{V}_k(t)\big)\,.$$

Proof. It holds for all $y \in [\varepsilon, \infty)$ that

$$\mathbb{P}\left(Z_k^{-} > yt,\, \sum_{i=1}^{d} Z_i > t\right) \le \mathbb{P}\left(Z_k^{-} > \varepsilon t,\, \sum_{i=1}^{d} Z_i^{+} > t\right).$$

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Then, using steps similar to those shown in (5.3) and (5.4) and noting the fact that $\{Z_k > t, Z_k^- > \varepsilon t\} = \varnothing$, we have

$$\mathbb{P}\left(Z_k^- > \varepsilon t, \sum_{i=1}^d Z_i^+ > t\right) \le \mathbb{P}\left(\sum_{i=1}^d Z_i^+ > t\right) - \sum_{i=1}^d \mathbb{P}(Z_i > t) + \sum_{1\le i<j\le d} \mathbb{P}\big(Z_i > t, Z_j > t\big) + \sum_{\substack{i=1\\ i\neq k}}^d \mathbb{P}\big(Z_i > t, Z_k^- > \varepsilon t\big).$$

Applying the conditions (a), (b′), and (c′) to the above relation completes the proof. □

The next two lemmas can be regarded as a decomposition of the proof of Theorem 3.1. Through these two lemmas, we also find that the left tails of the individual risks do not affect the asymptotic properties of the TM and TCM under our models.

Lemma 5.3. *Let* $\mathrm{TM}_{k,+}^{(n)}(t) = \mathbb{E}\left(\left(X_k^+\right)^n | S_d > t\right)$ *for each* $1 \le k \le d$.

(i) If all the conditions, except $F_i(-t) = O(\overline{F}_1(t))$, *of Assumption 3.1 are satisfied, then*

$$\mathrm{TM}_{k,+}^{(n)}(t) \sim \frac{\alpha}{\alpha - n} C_k t^n.$$

(ii) If all the conditions, except $F_i(-t) = O(\overline{F}_1(t))$, *of Assumption 3.2 are satisfied, then*

$$\mathrm{TM}_{k,+}^{(n)}(t) \sim C_k t^n.$$

Proof. By Theorem 3.1 of Chen and Yuen [12] and Theorem 3.1 of Hashorva and Li [21], the equivalence relations shown in (3.11) hold under the conditions of this lemma. Recalling also (2.7) and (3.1), it is easy to check that all the conditions of Lemma 5.1 are satisfied by $(X_1, \ldots, X_d)$, and hence the assertions obtained in Lemma 5.1 hold for $(X_1, \ldots, X_d)$. For any $0 < \varepsilon < 1$, we write

$$\begin{aligned}\mathrm{TM}_{k,+}^{(n)}(t) &= \left(\int_0^{\varepsilon t^n} + \int_{\varepsilon t^n}^{t^n} + \int_{t^n}^{\infty}\right) \mathbb{P}\left(X_k > x^{1/n} | S_d > t\right) \mathrm{d}x \\ &=: I_1(t) + I_2(t) + I_3(t). \end{aligned} \tag{5.7}$$

Clearly,

$$0 \le I_1(t) \le \varepsilon t^n. \tag{5.8}$$

For $I_2(t)$, we first note that, by (3.11) with $\overline{F}_i(t) \sim c_i \overline{F}_1(t)$ for each $1 \le i \le d$,

$$\frac{\overline{F}_i(t)}{\mathbb{P}(S_d > t)} \to C_i, \quad 1 \le i \le d, \tag{5.9}$$

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where C_i is given by (3.3). Then, under Assumption 3.1 or 3.2, it holds that

$$\begin{aligned}
I_2(t) &= \frac{\int_{\varepsilon t^n}^{t^n} \mathbb{P}\left(X_k > x^{1/n}, S_d > t\right) \mathrm{d}x}{\mathbb{P}(S_d > t)} \\
&= \frac{\int_{\varepsilon}^{1} \mathbb{P}\left(X_k > ty^{1/n}, S_d > t\right) \mathrm{d}y}{\mathbb{P}(S_d > t)} t^n \\
&\sim (1-\varepsilon)\, \frac{\overline{F}_k(t)}{\mathbb{P}(S_d > t)} t^n \\
&\sim (1-\varepsilon)\, C_k t^n,
\end{aligned} \tag{5.10}$$

where in the second step we used a change of variables, in the third step we used Lemma 5.1(i), and in the last step we used (5.9). Finally, under Assumption 3.1 or 3.2, we have

$$\begin{aligned}
I_3(t) &= \frac{\int_{t^n}^{\infty} \mathbb{P}\left(X_k > x^{1/n}, S_d > t\right) \mathrm{d}x}{\mathbb{P}(S_d > t)} \\
&\sim \frac{\int_{t^n}^{\infty} \overline{F}_k(x^{1/n})\mathrm{d}x}{\mathbb{P}(S_d > t)} \\
&= \frac{n \int_{t}^{\infty} y^{n-1}\overline{F}_k(y)\mathrm{d}y}{t^n \overline{F}_k(t)} \frac{\overline{F}_k(t)}{\mathbb{P}(S_d > t)} t^n,
\end{aligned}$$

where in the second step we used Lemma 5.1(ii) and in the last step we used a change of variables. If Assumption 3.1 holds, then $t^{n-1}\overline{F}_k(t) \in \mathcal{R}_{-\alpha+n-1}$ with $-\alpha+n-1<-1$. Hence, it follows from (5.9) and (2.3) that

$$I_3(t) \sim \frac{n}{\alpha - n} C_k t^n. \tag{5.11}$$

If Assumption 3.2 holds, then using (5.9) and (2.4) gives that

$$I_3(t) = o\left(t^n\right). \tag{5.12}$$

Plugging (5.8), (5.10), and (5.11) or (5.12) into (5.7) and letting $\varepsilon \to 0$, we complete the proof. □

Lemma 5.4. *Let* $\mathrm{TM}_{k,-}^{(n)}(t) = \mathbb{E}\left(\left(X_k^-\right)^n | S_d > t\right)$ *for each* $1 \le k \le d$*. Under Assumption* 3.1 *or* 3.2*, we have*

$$\mathrm{TM}_{k,-}^{(n)}(t) = o\left(t^n\right).$$

Proof. In view of (3.11), it is easy to see that Lemma 5.2 is applicable for $(X_1, \ldots, X_d)$ under Assumption 3.1 or 3.2. For any $0<\varepsilon<1$, we write

$$\begin{aligned}
\mathrm{TM}_{k,-}^{(n)}(t) &= \left(\int_0^{\varepsilon t^n} + \int_{\varepsilon t^n}^{\varepsilon^{-1}t^n} + \int_{\varepsilon^{-1}t^n}^{\infty}\right) \mathbb{P}\left(X_k^- > x^{1/n} | S_d > t\right) \mathrm{d}x \\
&=: I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{5.13}$$

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Clearly, the relation (5.8) still holds for $I_1(t)$. In addition, under Assumption 3.1 or 3.2,

$$\begin{aligned} I_2(t) &= \frac{\int_{\varepsilon t^n}^{\varepsilon^{-1}t^n} \mathbb{P}\left(X_k^- > x^{1/n}, S_d > t\right) \mathrm{d}x}{\mathbb{P}(S_d > t)} \\ &\le \frac{\mathbb{P}\left(X_k^- > \varepsilon^{1/n} t, S_d > t\right)}{\overline{F}_k(t)} \frac{\overline{F}_k(t)}{\mathbb{P}(S_d > t)} \left(\varepsilon^{-1} - \varepsilon\right) t^n \\ &= o\left(t^n\right), \end{aligned} \tag{5.14}$$

where in the last step we used Lemma 5.2 and (5.9). Moreover, since $F_k(-t) = O(\overline{F}_1(t))$ under Assumption 3.1 or 3.2, there is some constant B such that $\mathbb{P}\left(X_k^- > t\right) \le B\overline{F}_1(t)$ for t large enough. Hence, we have

$$\begin{aligned} I_3(t) &= \frac{\int_{\varepsilon^{-1}t^n}^{\infty} \mathbb{P}\left(X_k^- > x^{1/n}, S_d > t\right) \mathrm{d}x}{\mathbb{P}(S_d > t)} \\ &\le \frac{\int_{\varepsilon^{-1}t^n}^{\infty} \mathbb{P}\left(X_k^- > x^{1/n}\right) \mathrm{d}x}{\mathbb{P}(S_d > t)} \\ &\lesssim \frac{B \int_{\varepsilon^{-1}t^n}^{\infty} \overline{F}_1\left(x^{1/n}\right) \mathrm{d}x}{\mathbb{P}(S_d > t)} \\ &= \frac{Bn \int_{\varepsilon^{-1/n}t}^{\infty} y^{n-1} \overline{F}_1(y) \mathrm{d}y}{\varepsilon^{-1} t^n \overline{F}_1\left(\varepsilon^{-1/n} t\right)} \frac{\overline{F}_1\left(\varepsilon^{-1/n} t\right)}{\mathbb{P}(S_d > t)} \varepsilon^{-1} t^n, \end{aligned}$$

where in the last step we used a change of variables. If Assumption 3.1 holds, then using (2.3), $F_1 \in \mathcal{R}_{-\alpha}$, and (5.9) yields that

$$I_3(t) \lesssim \frac{Bn}{\alpha - n} C_1 \varepsilon^{\alpha/n - 1} t^n. \tag{5.15}$$

If Assumption 3.2 holds, then it follows from (2.4) and (5.9) that

$$I_3(t) \lesssim \frac{Bn \int_{\varepsilon^{-1/n}t}^{\infty} y^{n-1} \overline{F}_1(y) \mathrm{d}y}{\varepsilon^{-1} t^n \overline{F}_1\left(\varepsilon^{-1/n} t\right)} \frac{\overline{F}_1(t)}{\mathbb{P}(S_d > t)} \varepsilon^{-1} t^n = o\left(t^n\right). \tag{5.16}$$

Plugging (5.8), (5.14), and (5.15) or (5.16) into (5.13) and letting $\varepsilon \to 0$, we complete the proof. □

Proof of Theorem 3.1. It is clear that

$$\begin{aligned} \mathrm{TM}_k^{(n)}(t) &= \mathbb{E}\left(\left(X_k^+ - X_k^-\right)^n | S_d > t\right) \\ &= \mathbb{E}\left(\left(X_k^+\right)^n + (-1)^n \left(X_k^-\right)^n | S_d > t\right) \\ &= \mathrm{TM}_{k,+}^{(n)}(t) + (-1)^n \mathrm{TM}_{k,-}^{(n)}(t). \end{aligned}$$

A combination of Lemmas 5.3 and 5.4 indicates that

$$\mathrm{TM}_{k,-}^{(n)}(t) = o(1) \mathrm{TM}_{k,+}^{(n)}(t)$$

under Assumption 3.1 or 3.2. Hence, we have

$$\mathrm{TM}_k^{(n)}(t) \sim \mathrm{TM}_{k,+}^{(n)}(t).$$

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Then the assertions (i) and (ii) of Lemma 5.3 imply the assertions (i) and (ii) of Theorem 3.1, respectively.

Proof of (3.10). Noting that the random variable $\left(X_k - TM_k^{(1)}(t)\right)^{2n} \Big| D$ is non-negative for any event D, we have

$$\begin{aligned}
t^{2n} &\le \mathbb{E}\left(\left(X_k - TM_k^{(1)}(t)\right)^{2n} \middle| X_k > t + TM_k^{(1)}(t), S_d > t \right) \\
&= \frac{\int_0^\infty \mathbb{P}\left(\left(X_k - TM_k^{(1)}(t)\right)^{2n} > x, X_k > t + TM_k^{(1)}(t), S_d > t \right) \mathrm{d}x}{\mathbb{P}\left(X_k > t + TM_k^{(1)}(t), S_d > t\right)} \\
&\le \frac{\int_0^\infty \mathbb{P}\left(\left(X_k - TM_k^{(1)}(t)\right)^{2n} > x, S_d > t \right) \mathrm{d}x}{\mathbb{P}(S_d > t)} \frac{\mathbb{P}(S_d > t)}{\mathbb{P}\left(X_k > t + TM_k^{(1)}(t), S_d > t\right)} \\
&= \mathrm{TCM}_k^{(2n)}(t) \frac{\mathbb{P}(S_d > t)}{\mathbb{P}\left(X_k > t + TM_k^{(1)}(t), S_d > t\right)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{TCM}_k^{(2n)}(t) &\ge \frac{\mathbb{P}\left(X_k > t + TM_k^{(1)}(t), S_d > t\right)}{\mathbb{P}(S_d > t)} t^{2n} \\
&\sim \frac{\overline{F}_k\left(t + TM_k^{(1)}(t)\right)}{\mathbb{P}(S_d > t)} t^{2n},
\end{aligned}$$

where the last step follows from Lemma 5.1(ii). By Theorem 3.1(i), it holds that

$$TM_k^{(1)}(t) \sim \frac{\alpha}{\alpha - 1} C_k t \le \frac{2\alpha}{\alpha - 1} C_k t.$$

Then,

$$\begin{aligned}
\mathrm{TCM}_k^{(2n)}(t) &\gtrsim \frac{\overline{F}_k\left(\left(1 + \frac{2\alpha}{\alpha-1} C_k\right) t\right)}{\mathbb{P}(S_d > t)} t^{2n} \\
&\sim C_k \left(1 + \frac{2\alpha}{\alpha - 1} C_k\right)^{-\alpha} t^{2n},
\end{aligned}$$

where in the last step we used $F_k \in \mathcal{R}_{-\alpha}$ and (5.9). Comparing the above estimate with (3.6) gives that

$$A_{\alpha,2n,k} \ge C_k \left(1 + \frac{2\alpha}{\alpha - 1} C_k\right)^{-\alpha} > 0.$$

This completes the proof of (3.10).

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Appendix A. Supplementary discussions on univariate cases

In this appendix, we turn to the degenerate versions of the TM and TCM defined by (1.1) and (1.2) with only one risk under consideration. Denoting by a real-valued random variable X the single risk, for $n \in \mathbf{N}^+$ and $t > 0$ we write

$$\mathrm{TM}^{(n)}(t) = \mathbb{E}\left(X^n \,|X > t\right) \tag{A.1}$$

and

$$\mathrm{TCM}^{(n)}(t) = \mathbb{E}\left(\left(X - \mathrm{TM}^{(1)}(t)\right)^n \middle| X > t\right). \tag{A.2}$$

Note that $X|X > t$ is a non-negative random variable with the same distribution as $X^+ \left|X^+ > t\right.$. Hence, the asymptotic behavior of $\mathrm{TM}^{(n)}(t)$ and $\mathrm{TCM}^{(n)}(t)$ has nothing to do with the left tail of X. Let F be the distribution of X. When $F \in \mathcal{R}$, $\mathrm{TM}^{(n)}(t)$ and $\mathrm{TCM}^{(n)}(t)$ possess asymptotic expansions similar to those obtained for their multivariate counterparts in Theorem 3.1(i) and Corollary 3.1(i).

Theorem A.1. *Consider* $\mathrm{TM}^{(n)}(t)$ *and* $\mathrm{TCM}^{(n)}(t)$ *defined by* (A.1) *and* (A.2) *with* $n \in \mathbf{N}^+$. *If* $F \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > n$, *then*

$$\mathrm{TM}^{(n)}(t) \sim \frac{\alpha}{\alpha - n} t^n \tag{A.3}$$

and

$$\mathrm{TCM}^{(n)}(t) = \left(A_{\alpha,n} + o(1)\right) t^n, \tag{A.4}$$

where

$$A_{\alpha,n} = \sum_{i=0}^{n-1} \binom{n}{i} (-1)^i \frac{\alpha^{i+1}}{(\alpha-1)^i\,(\alpha - n + i)} + (-1)^n \left(\frac{\alpha}{\alpha-1}\right)^n.$$

Proof. We have

$$\begin{aligned}
\mathrm{TM}^{(n)}(t) &= \left(\int_0^{t^n} + \int_{t^n}^{\infty}\right) \mathbb{P}\left(X > x^{1/n} \,|X > t\right) \mathrm{d}x \\
&= t^n + \frac{\int_{t^n}^{\infty} \overline{F}\left(x^{1/n}\right) \mathrm{d}x}{\overline{F}(t)} \\
&= t^n + \frac{n\int_t^{\infty} y^{n-1}\overline{F}(y)\,\mathrm{d}y}{t^n\overline{F}(t)} t^n,
\end{aligned} \tag{A.5}$$

where in the last step we used a change of variables. Since $t^{n-1}\overline{F}(t) \in \mathcal{R}_{-\alpha+n-1}$ with $-\alpha + n - 1 < -1$, applying (2.3) to the second term of (A.5) gives (A.3). Then, (A.4) follows from combining (A.3) with the equality

$$\mathrm{TCM}^{(n)}(t) = \sum_{i=0}^{n-1} \binom{n}{i} (-1)^i \left(\mathrm{TM}^{(1)}(t)\right)^i \mathrm{TM}^{(n-i)}(t) + (-1)^n \left(\mathrm{TM}^{(1)}(t)\right)^n. \tag{A.6}$$

This completes the proof. □

Not surprisingly, the right-hand sides of (A.3) and (A.4) are equal to those of (3.4) and (3.6), respectively, with $C_k = 1$. By an approach similar to that used in proving (3.10), we can verify that $A_{\alpha,n} > 0$ for any even n.

On the other hand, we can also obtain a precise asymptotic formula for $\mathrm{TM}^{(n)}(t)$ by similar treatment when F comes from the wide class of rapid variation.

Theorem A.2. *Consider* $\mathrm{TM}^{(n)}(t)$ *defined by* (A.1) *with* $n \in \mathbf{N}^+$. *If* $F \in \mathcal{R}_{-\infty}$ *then*

$$\mathrm{TM}^{(n)}(t) \sim t^n. \tag{A.7}$$

Proof. The relation (A.7) can be derived by following the same approach as the proof of Theorem A.1, but applying (2.4) instead of (2.3). □

In fact, it is easy to see from the proofs of Theorems A.1 and A.2 that (A.3) and (A.7) hold for any positive (not necessarily integer-valued) order satisfying the conditions of the theorems, because $X|X > t$ is a non-negative random variable as mentioned before.

Now, plugging (A.7) into (A.6) yields only a rough estimate of $\mathrm{TCM}^{(n)}(t)$, i.e., $\mathrm{TCM}^{(n)}(t) = o(t^n)$ for any $n \in \mathbf{N}^+$. Thus, more conditions are required to obtain a precise asymptotic result of $\mathrm{TCM}^{(n)}(t)$ when $F \in \mathcal{R}_{-\infty}$. Here we consider only $\mathrm{TCM}^{(2)}(t)$, i.e., the TV risk measure, which is the simplest but also the most interesting special case of $\mathrm{TCM}^{(n)}(t)$. For this purpose, we restrict F to being a von Mises function with an infinite upper endpoint. That is to say, there is some real number z such that

$$\overline{F}(x) = \delta \exp\left\{-\int_z^x \frac{1}{h(y)}\mathrm{d}y\right\}, \qquad x > z, \tag{A.8}$$

where δ is some positive constant and h is a positive and absolutely continuous function with density h' such that $h'(t) \to 0$. It is known that if F is a von Mises function with the representation (A.8), then $F \in \mathrm{GMDA}(h) \subset \mathcal{R}_{-\infty}$, and F is differentiable on (z, ∞) with positive density f such that

$$f(x) = \frac{\overline{F}(x)}{h(x)}; \tag{A.9}$$

see Chapter 1.1 of Resnick [40] or Chapter 3.3.3 of Embrechts *et al.* [18]. The class of von Mises functions is an important subclass of the Gumbel MDA, and it contains many commonly used distributions, including the exponential, Erlang, normal, and log-normal distributions.

Theorem A.3. *Consider* $\mathrm{TCM}^{(2)}(t)$ *defined by* (A.2) *with* $n = 2$. *Let* F *be a von Mises function with the representation* (A.8). *If* h *is differentiable on* (z, ∞) *and* $\lim th'(t)/h(t)$ *exists, then*

$$\mathrm{TCM}^{(2)}(t) \sim h^2(t). \tag{A.10}$$

Proof. Denote by v the value of $\lim th'(t)/h(t)$, i.e.,

$$\frac{th'(t)}{h(t)} \to v \in (-\infty, \infty). \tag{A.11}$$

Recalling (A.5), it holds that

$$\mathrm{TM}^{(1)}(t) = t + \frac{\int_t^\infty \overline{F}(x)\,\mathrm{d}x}{\overline{F}(t)}. \tag{A.12}$$

We write

$$I(t) = \frac{\int_t^\infty \overline{F}(x)\,\mathrm{d}x/\overline{F}(t) - h(t)}{h^2(t)/t} = \frac{t\int_t^\infty \overline{F}(x)\,\mathrm{d}x - t\overline{F}(t)\,h(t)}{\overline{F}(t)\,h^2(t)}. \tag{A.13}$$

Since $F\in \mathrm{GMDA}(h)$, Theorem 3.3.26 of Embrechts *et al.* [18] tells us that

$$\int_t^{\infty} \overline{F}(x)\,\mathrm{d}x \sim \overline{F}(t)\,h(t). \tag{A.14}$$

The fact that $F\in \mathcal{R}_{-\infty}$ implies that $t^K\overline{F}(t)\to 0$ for any $K>0$. Then, noting also $h(t)=o(t)$, we have

$$\lim_{t\to\infty} t\int_t^{\infty} \overline{F}(x)\,\mathrm{d}x = \lim_{t\to\infty} t\overline{F}(t)\,h(t) = \lim_{t\to\infty} \overline{F}(t)\,h^2(t) = 0.$$

Thus, applying L'Hospital's rule yields that

$$\begin{aligned}
\lim_{t\to\infty} I(t) &= \lim_{t\to\infty} \frac{\int_t^{\infty} \overline{F}(x)\,\mathrm{d}x - t\overline{F}(x) - \overline{F}(t)\,h(t) + tf(t)\,h(t) - t\overline{F}(t)\,h'(t)}{-f(t)\,h^2(t) + 2\overline{F}(t)\,h(t)h'(t)}\\
&= \lim_{t\to\infty} \frac{\int_t^{\infty} \overline{F}(x)\,\mathrm{d}x - \overline{F}(t)\,h(t) - t\overline{F}(t)\,h'(t)}{-\overline{F}(t)\,h(t) + 2\overline{F}(t)\,h(t)h'(t)}\\
&= \lim_{t\to\infty} \frac{\int_t^{\infty} \overline{F}(x)\,\mathrm{d}x/(\overline{F}(t)\,h(t)) - 1 - th'(t)/h(t)}{-1+2h'(t)}\\
&= v,
\end{aligned}$$

where in the second step we used (A.9) and in the last step we used (A.14), (A.11), and $h'(t)\to 0$. Recalling (A.13), we have

$$\frac{\int_t^{\infty} \overline{F}(x)\,\mathrm{d}x}{\overline{F}(t)} = h(t) + v\frac{h^2(t)}{t} + o(1)\frac{h^2(t)}{t}.$$

Plugging the above estimate into (A.12) gives that

$$\mathrm{TM}^{(1)}(t) = t + h(t) + v\frac{h^2(t)}{t} + o(1)\frac{h^2(t)}{t},$$

and hence

$$\left(\mathrm{TM}^{(1)}(t)\right)^2 = t^2 + 2th(t) + (2v+1)\,h^2(t) + o(1)h^2(t). \tag{A.15}$$

On the other hand, it follows from (A.5) that

$$\mathrm{TM}^{(2)}(t) = t^2 + 2\frac{\int_t^{\infty} x\overline{F}(x)\,\mathrm{d}x}{\overline{F}(t)}. \tag{A.16}$$

Let

$$J(t) = \frac{\int_t^{\infty} x\overline{F}(x)\,\mathrm{d}x/\overline{F}(t) - th(t)}{h^2(t)} = \frac{\int_t^{\infty} x\overline{F}(x)\,\mathrm{d}x - t\overline{F}(t)\,h(t)}{\overline{F}(t)\,h^2(t)}.$$

It is easy to check via (2.2) that $t\overline{F}(t)$ is a tail of a distribution from $\mathrm{GMDA}(h)$. Thus, in terms of $t\overline{F}(t)$, (A.14) says that

$$\int_t^{\infty} x\overline{F}(x)\,\mathrm{d}x \sim t\overline{F}(t)\,h(t) \to 0.$$

Applying L'Hospital's rule and the same arguments as used in deriving $\lim I(t)$, we can obtain that

$$J(t)\to v+1,$$

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which implies that

$$\frac{\int_t^\infty x\overline{F}(x)\,\mathrm{d}x}{\overline{F}(t)} = th(t) + (v+1)\,h^2(t) + o(1)h^2(t).$$

Plugging the above estimate into (A.16) gives that

$$\mathrm{TM}^{(2)}(t) = t^2 + 2th(t) + 2\,(v+1)\,h^2(t) + o(1)h^2(t). \tag{A.17}$$

A combination of (A.15) and (A.17) yields that

$$\mathrm{TCM}^{(2)}(t) = \mathrm{TM}^{(2)}(t) - \left(\mathrm{TM}^{(1)}(t)\right)^2 = h^2(t) + o(1)h^2(t),$$

which is equivalent to (A.10). □

Since the trend of the function h at infinity is quite flexible, the asymptotic properties of $\mathrm{TCM}^{(2)}(t)$ corresponding to different choices of h may also be dramatically different. The following examples fully demonstrate this fact.

Example A.1. Let X follow an exponential distribution with

$$\overline{F}(x) = \mathrm{e}^{-\rho x}\mathbf{1}_{\{x>0\}} + \mathbf{1}_{\{x\le 0\}}, \qquad \rho > 0.$$

Clearly, F is a von Mises function with

$$h(x) = \frac{\overline{F}(x)}{f(x)} = \frac{1}{\rho}.$$

Thus, $h'(t) = th'(t)/h(t) = 0$. Using Theorem A.3 gives that

$$\mathrm{TCM}^{(2)}(t) \to \frac{1}{\rho^2}.$$

Actually, in this case routine calculations via (A.5) and (A.6) indicate that, for any $t > 0$,

$$\mathrm{TCM}^{(2)}(t) = \frac{1}{\rho^2}.$$

Example A.2. Let X follow a normal distribution with

$$\overline{F}(x) = \overline{\Phi}\left(\frac{x-\mu}{\sigma}\right), \qquad \mu \in (-\infty, +\infty),\ \sigma > 0,$$

where Φ is the standard normal distribution.

Denote by $\varphi(x) = \Phi'(x)$ the density of the standard normal distribution. It is easy to check by Proposition 1.1(b) of Resnick [40] that F is a von Mises function with

$$h(x) = \frac{\overline{F}(x)}{f(x)} = \frac{\sigma\overline{\Phi}\left(\frac{x-\mu}{\sigma}\right)}{\varphi\left(\frac{x-\mu}{\sigma}\right)}.$$

Using L'Hospital's rule yields the well-known Mill's ratio, i.e.,

$$\frac{\overline{\Phi}\,(t)}{\varphi(t)} \sim \frac{1}{t}. \tag{A.18}$$

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Note also that

$$\varphi'(x) = -x\varphi(x). \tag{A.19}$$

We have

$$\begin{aligned}\frac{th'(t)}{h(t)} &= \frac{-tf^2(t) - t\overline{F}(t)f'(t)}{\overline{F}(t)f(t)} \\ &= \frac{-\frac{t}{\sigma^2}\varphi^2\left(\frac{t-\mu}{\sigma}\right) - \frac{t}{\sigma^2}\overline{\Phi}\left(\frac{t-\mu}{\sigma}\right)\varphi'\left(\frac{t-\mu}{\sigma}\right)}{\frac{1}{\sigma}\overline{\Phi}\left(\frac{t-\mu}{\sigma}\right)\varphi\left(\frac{t-\mu}{\sigma}\right)} \\ &= \frac{-\sigma t\varphi\left(\frac{t-\mu}{\sigma}\right) + t(t-\mu)\overline{\Phi}\left(\frac{t-\mu}{\sigma}\right)}{\sigma^2\overline{\Phi}\left(\frac{t-\mu}{\sigma}\right)},\end{aligned} \tag{A.20}$$

where in the last step we used (A.19). It is easy to see that both the numerator and denominator of the right-hand side of (A.20) tend to 0 as $t \to \infty$. Applying L'Hospital's rule gives that

$$\begin{aligned}\lim_{t\to\infty}\frac{th'(t)}{h(t)} &= \lim_{t\to\infty}\frac{-\sigma\varphi\left(\frac{t-\mu}{\sigma}\right) - t\varphi'\left(\frac{t-\mu}{\sigma}\right) + (2t-\mu)\overline{\Phi}\left(\frac{t-\mu}{\sigma}\right) - \frac{t(t-\mu)}{\sigma}\varphi\left(\frac{t-\mu}{\sigma}\right)}{-\sigma\varphi\left(\frac{t-\mu}{\sigma}\right)} \\ &= \lim_{t\to\infty}\frac{\varphi\left(\frac{t-\mu}{\sigma}\right) - 2\frac{t-\mu}{\sigma}\overline{\Phi}\left(\frac{t-\mu}{\sigma}\right) - \frac{\mu}{\sigma}\overline{\Phi}\left(\frac{t-\mu}{\sigma}\right)}{\varphi\left(\frac{t-\mu}{\sigma}\right)} \\ &= -1,\end{aligned}$$

where in the second step we used (A.19) and in the last step we used (A.18). Hence, by Theorem A.3 and (A.18), we have

$$\mathrm{TCM}^{(2)}(t) \sim \left(\frac{\sigma\overline{\Phi}\left(\frac{t-\mu}{\sigma}\right)}{\varphi\left(\frac{t-\mu}{\sigma}\right)}\right)^2 \sim \frac{\sigma^4}{t^2},$$

which implies that $\mathrm{TCM}^{(2)}(t) \to 0$ in this case.

Example A.3. Let X follow a log-normal distribution with

$$\overline{F}(x) = \overline{\Phi}\left(\frac{\log x - \mu}{\sigma}\right)\mathbf{1}_{\{x>0\}} + \mathbf{1}_{\{x\le 0\}},$$

for μ, σ, and Φ as specified in Example A.2.

Proposition 1.1 (b) of Resnick [40] tells us that F is a von Mises function with

$$h(x) = \frac{\overline{F}(x)}{f(x)} = \frac{\sigma x\overline{\Phi}\left(\frac{\log x-\mu}{\sigma}\right)}{\varphi\left(\frac{\log x-\mu}{\sigma}\right)}.$$

It follows from (A.19) that

$$\begin{aligned}f'(x) &= -\frac{1}{\sigma x^2}\varphi\left(\frac{\log x - \mu}{\sigma}\right) + \frac{1}{\sigma^2 x^2}\varphi'\left(\frac{\log x - \mu}{\sigma}\right) \\ &= -\frac{\log x - \mu + \sigma^2}{\sigma^3 x^2}\varphi\left(\frac{\log x - \mu}{\sigma}\right).\end{aligned}$$

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Then, by steps similar to those shown in (A.20), we have

$$\frac{th'(t)}{h(t)} = \frac{-\sigma\varphi\left(\frac{\log t-\mu}{\sigma}\right) + \left(\log t - \mu + \sigma^2\right)\overline{\Phi}\left(\frac{\log t-\mu}{\sigma}\right)}{\sigma^2\overline{\Phi}\left(\frac{\log t-\mu}{\sigma}\right)}.$$

Applying L'Hospital's rule gives that

$$\begin{aligned}
\lim_{t\to\infty}\frac{th'(t)}{h(t)} &= \lim_{t\to\infty}\frac{-\frac{1}{t}\varphi'\left(\frac{\log t-\mu}{\sigma}\right) + \frac{1}{t}\overline{\Phi}\left(\frac{\log t-\mu}{\sigma}\right) - \frac{\log t-\mu+\sigma^2}{\sigma t}\varphi\left(\frac{\log t-\mu}{\sigma}\right)}{-\frac{\sigma}{t}\varphi\left(\frac{\log t-\mu}{\sigma}\right)}\\
&= \lim_{t\to\infty}\frac{-\frac{1}{\sigma}\overline{\Phi}\left(\frac{\log t-\mu}{\sigma}\right) + \varphi\left(\frac{\log t-\mu}{\sigma}\right)}{\varphi\left(\frac{\log t-\mu}{\sigma}\right)}\\
&= 1,
\end{aligned}$$

where in the second step we used (A.19) and in the last step we used (A.18). Hence, by Theorem A.3 and (A.18), we obtain that

$$\mathrm{TCM}^{(2)}(t) \sim \left(\frac{\sigma t\overline{\Phi}\left(\frac{\log t-\mu}{\sigma}\right)}{\varphi\left(\frac{\log t-\mu}{\sigma}\right)}\right)^2 \sim \frac{\sigma^4 t^2}{(\log t)^2},$$

which implies that $\mathrm{TCM}^{(2)}(t) \to \infty$ in this case.

Finally, recall the TVP and TSDP premium principles mentioned in Remark 3.3. The univariate versions of these premium principles are

$$\mathrm{TVP}(t) = \mathrm{TM}^{(1)}(t) + w\mathrm{TCM}^{(2)}(t)$$

and

$$\mathrm{TSDP}(t) = \mathrm{TM}^{(1)}(t) + w\sqrt{\mathrm{TCM}^{(2)}(t)}.$$

Applying Theorems A.1–A.3 gives that

$$\mathrm{TVP}(t) \sim w\mathrm{TCM}^{(2)}(t) \sim \frac{w\alpha}{(\alpha-2)\,(\alpha-1)^2}t^2$$

and

$$\mathrm{TSDP}(t) \sim \frac{1}{\alpha-1}\left(\alpha + w\sqrt{\frac{\alpha}{\alpha-2}}\right)t$$

if $F \in \mathcal{R}_{-\alpha}$ with $\alpha > 2$, and that

$$\mathrm{TVP}(t) \sim t + wh^2(t)$$

and

$$\mathrm{TSDP}(t) \sim \mathrm{TM}^{(1)}(t) \sim t$$

under the conditions of Theorem A.3.

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