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ON GROUPS WITH SMALL ORDERS OF ELEMENTS

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To Bernhard Neumann on his 90th birthday

For a periodic group G, denote by $\omega(G)$ the set of orders of elements in G. We prove that if $\omega(G)$ is a proper subset of the set $\{1,2,3,4,5\}$ then either G is locally finite or G contains a nilpotent normal subgroup N such that G/N is a 5-group.

Let G be a periodic group. Denote by $\omega(G)$ the set of orders of elements in G. It is obvious that a group with $\omega(G) = \{1,2\}$ is elementary Abelian. Levi and van der Waerden [5] proved that if $\omega(G) = \{1,3\}$ then G is nilpotent of class at most 3. B.H. Neumann [6] described the groups with $\omega(G) = \{1,2,3\}$. Sanov [8] and M. Hall [1] stated that a group G with $\omega(G) \subseteq \{1,2,3,4\}$, respectively with $\omega(G) \subseteq \{1,2,3,6\}$, is locally finite. Nothing is known about local finiteness of groups of exponent 5, but it follows from [11] that every group G with $\omega(G) = \{1,2,3,5\}$ is isomorphic to the alternating group A_5.

In this direction, we prove the following results.

Theorem 1. *Let G be a group with $\omega(G) = \{1,3,5\}$. Then one of the following holds:*

(i) *$G = FT$ where F is a normal 5-subgroup which is nilpotent of class at most 2 and $|T| = 3$;*

(ii) *G contains a normal 3-subgroup T which is nilpotent of class at most 3 such that G/T is a 5-group.*

Theorem 2. *If $\omega(G) = \{1,2,5\}$ then G contains either an elementary Abelian 5-subgroup of index 2, or an elementary abelian normal Sylow 2-subgroup.*

Theorem 3. *If $\omega(G) = \{1,2,4,5\}$ then one of the following holds:*

(i) *$G = TD$ where T is a non-trivial elementary Abelian 2-group and D is a non-Abelian group of order 10;*

(ii) *$G = FT$ where F is an elementary Abelian normal 5-subgroup and T is isomorphic to a subgroup of a quaternion group of order 8.*

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(iii) *G contains a normal 2-subgroup T which is nilpotent of class at most 6 such that G/T is a 5-group.*

In a forthcoming paper, we prove that every group G with $\omega(G) = \{1,2,3,4,5\}$ is locally finite. In this connection, we propose a conjecture that every group in the conclusion of Theorems 1-3 is also locally finite. This is equivalent to the following

CONJECTURE 1. *Let A be an automorphism group of an elementary Abelian $\{2,3\}$-group G such that every non-trivial element of A fixes in G only the trivial element. If A is of exponent 5 then A is cyclic.*

Notation and preliminary results

If H is a subgroup of a group G, $x, y \in G$, X, Y are subsets of G then $x^y = y^{-1}xy$, $X^y = \{y^{-1}xy \mid x \in X\}$, $[x,y] = x^{-1}x^y$, $x^Y = \{x^y \mid y \in Y\}$, $X^Y = \{x^y \mid x \in X,\ y \in Y\}$, $N_H(X) = \{g \in H \mid X^g = X\}$, $\langle X \rangle$ is the subgroup generated by X, $[X,Y] = \Big\langle [x,y] \mid x \in X,\ y \in Y] \Big\rangle$, $C_H(X) = \Big\{h \in H \mid (\forall x \in X)[h,x] = 1\Big\}$, $Z(G) = C_G(G)$. For a prime p, $O_p(G)$ is the product of all normal p-subgroups of G, A_m and S_m denote, respectively, the alternating and symmetric group on m letters.

An automorphism group of a group is said to be *regular* if every non-trivial element of it is fixed-point-free.

LEMMA 1. *If $R = \langle r \rangle$ is a regular automorphism group of order 3 of a finite group H then, for every Abelian subgroup A of H, $\langle A^R \rangle = \langle A, A^r \rangle$ and $\langle A, A^r \rangle$ is Abelian.*

PROOF: Let HR be the natural semi-direct product of H and R. Then HR is a Frobenius group and hence $(hr^{-1})^3 = 1$ for every element $h \in H$. Since $(hr^{-1})^3 = hh^r h^{r^2}$, we have $h^{r^2} = (h^{-1})^r h^{-1}$. Therefore $A^{r^2} \leqslant \langle A, A^r \rangle$. Let $a, b \in A$. Then $1 = ab(ab)^r (ab)^{r^2} = ab(a^r b^r) a^{r^2} b^{r^2} = a(bb^r)\Big(a^r a^{r^2}\Big) b^{r^2} = a(b^{-1})^{r^2} a^{-1} b^{r^2} = [a^{-1}, b^{r^2}]$. Therefore $[a, b^{r^2}] = 1$ and $[a^r, b] = 1$. This means that A^r centralises A. The lemma is proved. ∎

An element of order 2 in a group is called an *involution*. The proof of the following well-known lemma is straightforward.

LEMMA 2. *Let i, j be involutions. Then the following hold:*

1. *For every $x \in \langle ij \rangle$, $x^i = x^j = x^{-1}$.*
2. *If the order of ij is finite and odd then i, j are conjugate by an element in $\langle ij \rangle$ and by an involution in $i\langle ij \rangle$.*
3. *If the order of ij is finite and even, then $\langle ij \rangle$ contains an involution which commutes with i and j.*

LEMMA 3. *Suppose that t is an involution in the automorphism group A of a periodic group G. If t is fixed-point-free then, for every $g \in G$, $g^t = g^{-1}$, G is Abelian and t lies in the centre of A.*

PROOF: Let $H = G\langle t\rangle$ be the natural semi-direct product and $g \in G$. Since $C_H(t) = \langle t\rangle$ and $t^g t = g^{-1}tgt = g^{-1}g^t \in G$, the order of $t^g t$ is odd, by part 3 of Lemma 2. By part 2 of Lemma 2 there exists an involution $i \in tG$ such that $t^{gi} = t$ and hence $gi = 1$ or $gi = t$. If $gi = 1$ then $g = i \in tG$ which is not the case. Thus $gi = t$, $g = ti$ and by part 1 of Lemma 2, $g^t = g^{-1}$. If $h \in G$ then $gh = (h^{-1}g^{-1})^{-1} = (h^t g^t)^t = hg$, and G is Abelian. If $a \in A$ then $g^{ta} = (g^{-1})^a = (g^a)^{-1} = g^{at}$ and hence $ta = at$. The lemma is proved. □

LEMMA 4. *Let A be a proper subgroup of a group H. If A contains no elements of order 3 and every element in $H \setminus A$ is a 3-element then A is normal in H and A is nilpotent of class at most 2.*

PROOF: It is obvious that A is normal in H. Let x be an element of order 3 in H. Then, for every $a \in A$, $ax^{-1} \notin A$, so $(ax^{-1})^3 = 1$ and hence $aa^x a^{x^2} = 1$. It was proved by B.H. Neumann [7] that, in this situation, every element of A commutes with each of its conjugates. By a result of Levi [4], A is nilpotent of class at most 3 and the third term T of the lower central series of A is of exponent 3. By assumption, $T = 1$. The lemma is proved. □

LEMMA 5. *Let G be a finite group with $\omega(G) = \{1,3,5\}$. Then G contains a normal Sylow subgroup of prime index with non-cyclic centre.*

PROOF: Since the order of G is divisible only by two distinct prime numbers, G is soluble and $O_p(G) \neq 1$ for $p = 3$ or $p = 5$. If $F/O_p(G)$ is a minimal normal subgroup of $G/O_p(G)$ then F is a Frobenius group and $Z\big(O_p(G)\big)$ is non-cyclic. By [3, Theorem V.8.15], $F/O_p(G)$ is cyclic and hence $F = G$. □

LEMMA 6. *Let $\omega(G) = \{1,2,4,5\}$. If $V = O_2(G) \neq 1$ and $G = VD$ where D is a dihedral group of order 10 generated by an involution t and an element r of order 5 then V is elementary Abelian, $[V,t] = C_V(t)$ and $\big|V : C_V(t)\big| > 2$.*

PROOF: Suppose that V is elementary Abelian. If $v \in C_V(t)$ then $W = \langle v^{\langle r\rangle}\rangle$ is a D-invariant subgroup of order 16, $v \in [W,t]$ and $2 < \big|W : C_W(t)\big| \leqslant \big|V : C_V(t)\big|$. Thus, it suffices to prove that V is elementary Abelian. Since G is locally finite, we can assume that G is finite and proceed by induction on $|G|$. Suppose that V is not elementary Abelian. Let Z be a minimal normal subgroup of G. Then $Z \leqslant Z(V)$, $|Z| = 16$ and V/Z is elementary Abelian. If $C/Z = C_{V/Z}(t)$ and C contains an element u of order 4 then $U = \langle u^{\langle r\rangle}\rangle Z$ is a D-invariant subgroup and all elements in $U\setminus Z$ are of order 4. But then G contains an element of order 8. Thus C is elementary Abelian. Let $v \in V$ be an element of order 4. Then, by induction, $V = \langle v^{\langle r\rangle}\rangle Z$, $|V| = 2^{12}$ and $|C| = 2^8$. If $c \in C \setminus Z$ then $U_c = \langle c^{\langle r\rangle}\rangle Z$ is an elementary Abelian group of order 2^8 and $U_{c_1} \cap U_{c_2} = Z$ if $c_1 Z \neq c_2 Z$. Thus $V = U_{c_1}U_{c_2}$ for some c_1, c_2 and there exists a uniquely defined D-homomorphism ϕ of the tensor product $X \otimes Y$ of a D-module $X = U_{c_1}/D$ and D-module $Y = U_{c_2}/Z$ into a D-module Z which extends the map $xZ \otimes yZ \to [x,y]$. Let F be a splitting field of D over a field F_2 of order 2, $\overline{X} = X \otimes_{F_2} F$, $\overline{Y} = Y \otimes_{F_2} F$, $\overline{Z} = Z \otimes_{F_2} F$. Then there exists a uniquely defined homomorphism $\overline{\phi}$ of $\overline{X} \otimes \overline{Y}$ into $\overline{Z}$ which extends ϕ. Let $1 \neq \lambda \in F$, $\lambda^5 = 1$. We can choose bases $\{x_i \mid i = 1, \dots, 4\}$, $\{y_i \mid i = 1, \dots, 4\}$ and $\{z_i \mid i = 1, \dots, 4\}$ of $\overline{X}, \overline{Y}, \overline{Z}$, respectively, such that $x_i^r = \lambda^i x_i$, $y_i^r = \lambda^i y_i$, $z_i^r = \lambda^i z_i$, $i = 1, \dots, 4$ and $x_1^t = x_4$, $x_2^t = x_3$, $y_1^t = y_4$, $y_2^t = y_3$, $z_1^t = z_4$, $z_2^t = z_3$. Denote $(x, y) = \overline{\phi}(x \otimes y)$. Since $(x_i, y_i)^r = (x_i^r, y_i^r) = (\lambda^i x_i, \lambda^j y_j) = \lambda^{i+j}(x_i, y_j)$, we see that

(1) $(x_i, y_j) = 0$ for $(i, j) \in \big\{(1,4), (2,3), (3,2), (4,1)\big\}$, and, for other pairs (i, j), $(x_i, x_j) = \alpha_{ij} z_k$ where $\alpha_{ij} \in F$ and k is defined by $\lambda^{i+j} = \lambda^k$, $1 \leqslant k \leqslant 4$.

Since C is Abelian,

$$\begin{aligned}(2) \quad (x_1 + x_4, y_1 + y_4) &= (x_1 + x_4, y_2 + y_3) = (x_2 + x_3, y_1 + y_4)\\ &= (x_2 + x_3, y_2 + y_3) = 0.\end{aligned}$$

By (1), (2) gives $0 = (x_1 + x_4, y_1 + y_4) = (x_1, y_1) + (x_4, y_4) = \alpha_{11} z_2 + \alpha_{44} z_3$ and hence

$$(x_1, y_1) = (x_4, y_4) = 0.$$

Similarly,

$$(x_1, y_2) = (x_4, y_2) = (x_1, y_3) = (x_4, y_3) = 0,$$

$$(x_2, y_1) = (x_3, y_1) = (x_2, y_4) = (x_3, y_4) = 0,$$

$$(x_2, y_2) = (x_3, y_3) = 0\ .$$

Thus $\overline{\phi}$ is the zero-homomorphism, $[x, y] = 1$ for $x \in U_{c_1}, y \in U_{c_2}$ and V is elementary Abelian. The lemma is proved. ∎

Lemma 7. *Let $\omega(G) \subseteq \{1, 2, 4, 5\}$. If G is locally finite then either G has a normal Sylow subgroup or $G = VD$ where $V = O_2(G)$ is a non-trivial elementary Abelian group and D is a dihedral group of order* 10.

Proof: Suppose first that G is finite and proceed by induction on G. If $V = O_2(G) \neq 1$ then, by induction, G/V contains a normal Sylow 5-subgroup P/V. If $P = G$ then the conclusion is true. If $P \neq G$, then $P \neq V$ and, by [**3**, Theorem V.8.15], $|P : V| = 5$, G/V is a Frobenius group of order 10 or 20. In particular, there exist involutions $x, y \in G \setminus P$ such that xy is not a 2-element and hence $D = \langle x, y \rangle$ is a dihedral group of order 10. Let $H = O_2(G)D$. By Lemma 6, $V = O_2(H)$ is elementary Abelian. If G/V contains an element of order 4 then G contains an element of order 8. Thus $H = G$ and the conclusion is true. If $O_2(G) = 1$ then $P = O_5(G) \neq 1$ and $C_G(P) \leqslant P$. By Lemma 3, $Z(G/P)$ contains an element of order 2 and hence G/P is a 2-group.

Suppose G is infinite. If the product of every two 2-elements (every two 5-elements) in G is a 2-element (respectively, 5-element) then a Sylow 2-subgroup (respectively, a Sylow 5-subgroup) of G is normal in G. Suppose that there exist elements x, y, z, t such that x, y are 2-elements, z, t are 5-elements, xy is not a 2-element and zt is not a 5-element. Then $H = \langle x, y, z, t\rangle$ is a finite group without non-trivial normal Sylow subgroups. Therefore, $V = O_2(H)$ is elementary Abelian and $H = VD$ where D is a dihedral group of order 10. Let $C = C_G(V), N = CD$. By Lemma 6, every element in C is of order 2. If $u \in G$ then $\langle u, H\rangle$ is finite and hence $H \leqslant CD = N$. Thus $N = G$ and the lemma is proved. □

The following four lemmas can be verified by the coset enumeration algorithm (see, for instance [**9**]):

LEMMA 8. *Let $A = \langle a, b \mid R\rangle$. Table 1 gives the order of A for some defining relations R.*

Table 1

R	$\|A\|$
$a^3, b^3, (ab)^3, (ab^{-1})^3$	27
$a^3, b^3, (ab)^3, (ab^{-1})^5$	75
$a^3, b^3, (ab)^5, (ab^{-1})^5, (aba^{-1}b)^3$	1
$a^3, b^3, (ab)^5, (ab^{-1})^5, (aba^{-1}b)^5$	62400

Table 2

R	$\|A\|$
$[a,b]^5, (ab^{-1}ab)^3$	5
$[a,b]^3, (ab^{-1}ab)^3$	$3^9 \cdot 5$
$[a,b]^3, (ab^{-1}ab)^5$	5

LEMMA 9. *Let $A = \left\langle a, b \mid a^3, b^5, (ab)^5, (ab^{-1})^5, (ab^2)^5, (ab^{-2})^5, R\right\rangle$. Table 2 gives the order of A for various values of R.*

LEMMA 10. *Let $A = \left\langle a, b, c \mid a^3, b^3, c^3, aba^{-1}b^{-1}, (ac)^5, (ac^{-1})^3, (bc)^3, R\right\rangle$. Table 3 gives the order of A for various values of R.*

LEMMA 11. *Let $A = \langle a, b \mid a^2, b^5, R\rangle$. Table 4 gives the order of A for various values of R.*

PROOFS OF MAIN RESULTS

Let G be a group with $\omega(G) = \{1, 3, 5\}$. Note, that if $x^3 = y^3 = 1$ for $x, y \in G$ then $(xy)^3 = 1$ or $(x^{-1}y)^3 = 1$. Indeed, $X = \langle x, y\rangle$ is finite by Lemma 8 and if $X \neq 1$ then, by Lemma 5, X contains a normal 5-subgroup Y of index 3, hence one of the elements $xy, x^{-1}y$ is not contained in Y. Since every element in $X \setminus Y$ is of order 3, the assertion follows.

If G contains a normal Sylow 5-subgroup P then, by Lemma 4, P is nilpotent of class at most 2 and hence G is locally finite. By Lemma 5, $|G/P| = 3$. Suppose that there exist elements $a, b \in G$ of order 5 such that ab is not a 5-element. Then the order of $c = ab$ is equal to 3 and $\langle a, b\rangle = \langle c, a\rangle$. If $(ca^i)^3 = 1$ for $i = 1, 2, 3$ or 4 then, for $d = ca^i$, $\langle a, b\rangle = \langle c, a\rangle = \langle c, d\rangle$ is finite by Lemma 8. Suppose that, for $i = 1, 2, 3, 4$, $(ca^i)^5 = 1$. Since $(c^{-1}c^a)^3 = 1$ or $(cc^a)^3 = 1$, by Lemma 9, $\langle a, b\rangle = \langle c, a\rangle$ is finite. By Lemma 5, $\langle a, b\rangle$ contains a subgroup T of order 9. Let H be a maximal 3-subgroup of G which contains T. Then H is nilpotent of class at most 3 by [**5**]. Suppose that H is not a normal

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subgroup of G. Then there exists an element $u \notin H$ of order 3 and an element $v \in H$ such that vu is not a 3-element. Then the order of vu is equal to 5. Since H is non-cyclic nilpotent, there exists an element $t \in H$ such that $\langle v, t\rangle$ is an elementary Abelian group of order 9. As above, $\langle t, u\rangle$ is finite and one of the elements $tu, t^{-1}u$ is of order 3. We can assume that $(tu)^3 = 1$. By Lemma 10, $\langle u, v, t\rangle$ is finite which is impossible by Lemma 5. Theorem 1 is proved. □

Table 3

R	$\lvert A\rvert$
$(bc^2)^3, (abc)^3, (abc^2)^3, (ab^2c)^3, (ba^2c)^3$	9
$(bc^2)^3, (abc)^3, (abc^2)^3, (ab^2c)^3, (ba^2c)^5$	3
$(bc^2)^3, (abc)^3, (abc^2)^3, (ab^2c)^5, (ba^2c)^3$	3
$(bc^2)^3, (abc)^3, (abc^2)^5, (ab^2c)^3, (ba^2c)^3$	3
$(bc^2)^3, (abc)^5, (abc^2)^3, (ab^2c)^3, (ba^2c)^3$	3
$(bc^2)^3, (abc)^3, (abc^2)^5, (ab^2c)^3, (ba^2c)^5$	1
$(bc^2)^3, (abc)^3, (abc^2)^5, (ab^2c)^5, (ba^2c)^3$	1
$(bc^2)^3, (abc)^5, (abc^2)^3, (ab^2c)^3, (ba^2c)^5$	1
$(bc^2)^3, (abc)^5, (abc^2)^3, (ab^2c)^5, (ba^2c)^3$	75
$(bc^2)^5, (abc)^3, (abc^2)^3, (ab^2c)^3, (ba^2c)^3$	3
$(bc^2)^5, (abc)^3, (abc^2)^3, (ab^2c)^3, (ba^2c)^5$	75
$(bc^2)^5, (abc)^3, (abc^2)^3, (ab^2c)^5, (ba^2c)^3$	1
$(bc^2)^5, (abc)^3, (abc^2)^5, (ab^2c)^3, (ba^2c)^3$	1
$(bc^2)^5, (abc)^3, (abc^2)^5, (ab^2c)^3, (ba^2c)^5$	1
$(bc^2)^5, (abc)^3, (abc^2)^5, (ab^2c)^5, (ba^2c)^3$	1
$(bc^2)^5, (abc)^5, (abc^2)^3, (ab^2c)^3, (ba^2c)^3$	1
$(bc^2)^5, (abc)^5, (abc^2)^3, (ab^2c)^3, (ba^2c)^5$	1
$(bc^2)^5, (abc)^5, (abc^2)^3, (ab^2c)^5, (ba^2c)^3$	1

Table 4

R	$\lvert A\rvert$
$(ab)^5, [a,b]^5, (bab)^5$	5
$(ab)^5, [a,b]^4, (bab)^5$	$2^9 \cdot 5$
$(ab)^5, [a,b]^5, (bab)^4$	360
$(ab)^5, [a,b]^4, (bab)^4$	1
$(ab)^4, [a,b]^5, (bab)^5$	1
$(ab)^4, [a,b]^4, (bab)^5$	360
$(ab)^4, [a,b]^5, (bab)^4$	160
$(ab)^4, [a,b]^4, (bab)^4$	2

Proof of Theorem 2: Let G be a group with $\omega(G) = \{1, 2, 5\}$. By Lemma 11, every subgroup H of G generated by an element of order 5 and an element of order 2 is finite. If H contains a normal Sylow 5-subgroup P then, by Lemma 3, P is elementary Abelian and hence H is a dihedral group of order 10. Suppose first that all subgroups of G generated by an element of order 5 and an element of order 2 are of this type. Then the product of every two 5-elements is a 5-element and hence $O_5(G) \neq 1$. By Lemma 3, $G/O_5(G)$ contains at most one involution and hence is a 2-group. Thus there exists an H containing a non-trivial Sylow 2-subgroup T, and hence G contains an elementary Abelian subgroup V of order 4. Let F be the subgroup of G generated by all involutions in G. If F is a 2-group then the conclusion of the theorem is true.

Suppose that F is not a 2-group. Then there exists an element $x \in G$ of order 5 such that $x = t_1 t_2 \cdots t_s$ where every t_i, $i = 1, 2, \ldots, s$ is an involution. Choose x such that s is minimal. Then $t_1 \cdots t_{s-1}$ is an involution and $X = \langle x, t_s \rangle$ is a dihedral group of order 10. Let $t = t_s$. If $C_G(t)$ contains only one involution then, by Lemma 2, every involution in G is a conjugate of t, t is contained in a subgroup which is a conjugate of V and hence $C_G(t)$ contains an involution $u \neq t$. If $(ux)^2 = 1$ then the involution ut centralises x which is impossible by assumption. Thus $\langle u, x \rangle$ is a finite subgroup which has a non-trivial normal Sylow 2-subgroup. This subgroup is t-invariant and hence $X = \langle u, x, t \rangle$ is a finite group which has no non-trivial normal Sylow subgroups. It is easy to see that H must contain an element of order 4 contrary to the assumption. Thus, G contains a non-trivial normal Sylow p-subgroup P. If $p = 5$ then, by Lemma 3, P is Abelian, G/P contains only one involution and (i) holds. If $p = 2$ then P is elementary Abelian. Theorem 2 is proved. ∎

Proof of Theorem 3: Let G be a group with $\omega(G) = \{1, 2, 4, 5\}$.

Lemma 12. *Suppose that every finite non-trivial subgroup of G contains a non-trivial normal Sylow subgroup. Then G contains a non-trivial normal Sylow subgroup and* (ii) *or* (iii) *in the conclusion of Theorem 3 holds.*

Proof: Let F be the subgroup of G generated by all involutions in G. If F is a 2-group then G/F does not contain an element of order 4 or 10. By Theorem 3, the conclusion of the theorem is true. Hence F is not a 2-group. It follows that there exists an element $x \in G$ of order 5 such that $x = t_1 t_2 \cdots t_s$ where every t_i, $i = 1, 2, \ldots, s$ is an involution. Choose x such that s is minimal. Then $t_1 \cdots t_{s-1}$ is a non-trivial 2-element and, by Lemma 11, $X = \langle x, t_s \rangle$ is a finite group which cannot contain a normal Sylow 2-subgroup. Thus X is a dihedral group of order 10. Let $t = t_s$. If $C_G(t)$ contains only one involution then $C_G(t)$ is a finite 2-group, by **[10]**, G is locally finite and hence contains, by Lemma 7, a non-trivial normal Sylow 5-subgroup.

Suppose that $C_G(t)$ contains an involution $u \neq t$. If $(ux)^2 = 1$ then the involution ut centralises x which is impossible by assumption. Thus $\langle u, x \rangle$ has a normal Sylow 2-subgroup and hence $(ux)^5 = 1$. Thus $\langle u, x \rangle$ is a finite subgroup which has a non-trivial normal Sylow 2-subgroup. This subgroup is t-invariant and hence $\langle u, x, t \rangle$ is a finite group which has no non-trivial normal Sylow subgroups. This contradicts the assumption. Therefore, G contains a non-trivial normal Sylow p-subgroup P. If $p = 5$ then, by Lemma 3, P is Abelian, G/P contains only one involution and, by [**3**, Theorem V.8.15], (ii) holds. If $p = 2$ then P is locally finite. Let $x_1, \ldots, x_7 \in P$, $y \in G \setminus P$. Then the order of y is 5 and $Y = \langle x_1, \ldots, x_7, y \rangle$ is a finite group with a normal Sylow 2-subgroup $Z = P \cap Y$. By assumption, $\langle y \rangle$ acts regularly on Z and Z is nilpotent of class at most 6 by **[2]**. In particular, $[[\ldots[[x_1, x_2], x_3], \ldots], x_7] = 1$. This means that P is nilpotent of class at most 6 and (iii) holds. The lemma is proved. ∎

Suppose that G does not contain a non-trivial normal Sylow subgroup.

Lemma 13. *There exists a non-trivial elementary Abelian subgroup V in G such* *that $N = N_G(V) = VD$ where D is generated by an element r of order 5 and an involution t with $r^t = r^{-1}$. Furthermore, if V_0 is a non-trivial normal subgroup of N which is contained in V then $N_G(V_0) = N$.*

PROOF: By Lemmas 7 and 12, there exists a finite subgroup H of G such that $U = O_2(H)$ is a non-trivial elementary Abelian group and $H = UD$ where D is generated by an element r of order 5 and an involution t with $r^t = r^{-1}$. Let $V = C_G(U)$. Then V is a locally finite 2-group and $N = VD$ is also locally finite. By Lemma 7, V is elementary Abelian and, since $U \leqslant V$, $C_G(V) = V$. Therefore $O_2\big(N_G(V)\big) = O_2\big(N_G(V)\big)D$ is also elementary Abelian and hence $O_2\big(N_G(V)\big) = V$. If $N_G(V)/V$ contains an invariant Sylow 5-subgroup then, by Lemma 12, $N_G(V)$ is locally finite. By Lemma 7, $N_G(V)/V \simeq D$.

If $N_G(V)/V$ does not contain an invariant Sylow 5-subgroup then, by Lemma 7, $N_G(V)/V$ contains a subgroup S such that $O_2(S) \neq 1$ and $S/O_2(S) \simeq D$. The full preimage U of $O_2(S)$ in G is again elementary Abelian and hence $U = V$ contrary to the choice of S. Thus $N_G(V) = N$.

Suppose that $1 \neq V_0 \leqslant V$ and V_0 is normal in N. Then $V \leqslant C_G(V_0)$, $C_G(V_0)$ is a N-invariant 2-group and $C_G(V_0)D$ is locally finite. Again, by Lemma 7, $C_G(V_0)$ is elementary Abelian and $C_G(V_0) \leqslant C_G(V) = V$. Thus $C_G(V) = V$ and $N_G(V_0) \leqslant N_G\big(C_G(V_0)\big) = N_G(V) = N$. The lemma is proved. ∎

Throughout the rest of the proof, N, V, D, r, t are the subgroups and elements of G defined in Lemma 13.

LEMMA 14. *If v is an involution in V then $C_G(v) \leqslant N$.*

PROOF: Suppose that there exists $x \in C_G(v) \setminus N$. Then $V_0 = \langle v^D, x\rangle$ is a finite 2-subgroup in $C_G(v)$ and $V_0 \nleqslant N$. Let $V_1 = V \cap V_0$. Then $V \leqslant C_G(V_1) \leqslant C_G\big(\langle v^D\rangle\big) \leqslant V$. If V_1 is normal in V_0 then $V_0 \leqslant N_G(V_1) \leqslant N_G\big(C_G\big(V_1\big)\big) = N_G(V) = V$ contrary to the choice of V_0. Thus $V_2 = N_{V_0}(V_1) \neq V_3 = N_{V_0}(V_2)$. Let $y \in V_3 \setminus V_2$. Then $V_1^y \neq V_1, V_1^y \neq V, V_1^y \leqslant V_2 \leqslant N$ and $|V_1V_1^y : V_1| = 2$. But then $\big|V_1 : C_{V_1}(V_1^y)\big| = 2$, contradicting Lemma 6. ∎

LEMMA 15. *If v is an involution in $N \setminus V$ then $C_G(v) \leqslant N$.*

PROOF: Since $\langle v, v^r\rangle/V \simeq D$, vv^r is of order 5 and hence $v \in N_N(R)$ for some Sylow 5-subgroup R of N. Therefore v is a conjugate of t in N and we can assume that $v = t$. Let V_0 be a subgroup of order 4 in $C_v(t)$. Then $V_1 = \langle V_0, t\rangle$ is an elementary Abelian group of order 8 in $C_G(t)$. Let x be an element in $C_G(t) \setminus N$. Then $V_2 = \langle V_1, x\rangle$ is a finite subgroup in $C_G(t)$. Let $V_3 = V_2 \cap N$. Then $V_3 = V_4 \times \langle t\rangle$ where $V_4 \leqslant V$ and there exists $y \in N_{V_2}(V_3) \setminus N$. Since $|V_3 : V_4| = 2 < |V_4|$, $C_{V_4}(y) \neq 1$. This contradicts Lemma 14. ∎

LEMMA 16. $N = G$.

PROOF: There exists an involution $v \in N$ such that $C_N(v)$ is non-Abelian. Since, By Lemma 15, $C_G(t)$ is Abelian, not all involutions of N are conjugate in G. If all involutions of G are contained in N then N is normal in G and hence $G \leqslant N_G(V) = N$. Suppose that $N \neq G$ and let x be an involution in $G \setminus N$. Then there exists an involution $y \in N$ which is not a conjugate of x in G. By Lemma 2, there exists an involution $z \in Z(\langle x, y \rangle)$. By Lemmas 14 and 15, $z \in C_G(y) \leqslant N$ and $x \in C_G(z) \leqslant N$ contrary to the choice of x. The lemma and Theorem 3 are proved. ∎

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