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A SELF-DUAL EQUATIONAL BASIS FOR BOOLEAN ALGEBRAS

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ABSTRACT. The principle of duality for Boolean algebra states that if an identity $f = g$ is valid in every Boolean algebra and if we transform $f = g$ into a new identity by interchanging (i) the two lattice operations and (ii) the two lattice bound elements 0 and 1, then the resulting identity $f = g$ is also valid in every Boolean algebra. Also, the equational theory of Boolean algebras is finitely based. Believing in the cosmic order of mathematics, it is only natural to ask whether the equational theory of Boolean algebras can be generated by a finite irredundant set of identities which is already closed for the duality mapping. Here we provide one such equational basis.

The class of all Boolean algebras is well-known to be self-dual. In other-words, the set of all equational identities valid in a Boolean algebra is closed with respect to the duality transformation obtained by interchanging the two binary operations of meet and join and the two bound elements 0 and 1. Thus it is natural to ask whether this class of identities can be generated by an irredundant class of identities which is closed under the duality transformation. In spite of an abundance of different axiomatic approaches to the subject (see e.g. the references in [5]), no independent equational basis for Boolean algebras which is also self-dual is known to exist. The earliest attempt to provide one such basis was by B. A. Bernstein [1], but, as proved by Montague and Tarski [3], that basis turned out to be redundant. In 1964, M. F. Sioson [6] made a complete analysis of the usual equational laws of Boolean algebras (including the Bernstein set) and proved that no self-dual subset of it will be an independent basis for Boolean algebras. In 1971, G. Grätzer raised this as a problem (Problem #29) in his book [2] on Lattice theory. In this note we prove one such basis for Boolean algebras, treated as a variety of type $\langle 2, 2, 1, 0, 0\rangle$ and it can be easily modified to be minimal self-dual basis of type $\langle 2, 2, 1\rangle$ free from the requirement of special elements which was Bernstein's aim in [1]. Incidentally, as a by-product, we do get a minimal self-dual basis for

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distributive lattices. Neither the existence of such a basis is known, nor—and this is strange—this question was never even asked for the equally beautiful self-dual class of modular lattice identities. Here also the answer turns out to be in the affirmative.

1. The dual f of a Boolean polynomial f is obtained from f by interchanging the two binary operation symbols $+$ and $\cdot$ and the two nullary operation symbols 0 and 1. The dual of an identity $f=g$ is the identity $f=g$. A set Σ of identities (of type $\langle 2,2,1,0,0\rangle$ or $\langle 2,2,1\rangle$ is said to be self-dual if whenever $f=g\in\Sigma$ then $f=g\in\Sigma$. Let Σ_1 be the set of identities:

(1) $(x+y)y=y$	$(\underline{1})$ $xy+y=y$
(2) $x(y+z)=yx+zx$	$(\underline{2})$ $x+yz=(y+x)(z+x)$
(3) $xx'=0$	$(\underline{3})$ $x+x'=1$

Let $\Sigma_2=\{(1), (\underline{1}), (2), (\underline{2}), (4), (\underline{4})\}$ where (4) is the identity $xx'=yy'$ and $(\underline{4})$ is, of course, $x+x'=y+y'$.

THEOREM 1. *The set Σ_1 is an independent self-dual set of identities characterizing Boolean algebras.*

Proof.

$$\begin{aligned} yy &= (xy+y)y && \text{by } (\underline{1}) \\ &= y && \text{by } (1) \end{aligned}$$

Thus $\Sigma_1 \models yy=y$ and $y+y=y$. From now on we will use both the idempotent laws without further comment. Now $x+y=x+yy=(y+x)(y+x)$ by $(\underline{2})$ and hence $\Sigma_1\models x+y=y+x$ and $xy=yx$. So we have the two commutative laws and hence, by (1) and $(\underline{1})$, all the absorption laws. Let us define a binary relation $a\le b$ by the familiar condition $ab=a$, or equivalently, $a+b=b$. Clearly $\le$ is both reflexive and antisymmetric. Let $a\le b$ and $b\le c$. Then

$$\begin{aligned} a+c &= ab+c && \text{since } ab=a \\ &= (a+c)(b+c) && \text{by } (\underline{2}) \\ &= (a+c)c && \text{since } b+c=c \\ &= c && \text{by } (1) \end{aligned}$$

and thus the relation $\le$ is a partial order relation. Since $ab+a=a$ and $ab+b=b$, the element ab is a lower bound of a and b. Now let $x\le a, b$. Then $x+ab=(x+a)(x+b)=ab$ and hence ab is the greatest lower bound of a and b. Dually, $a+b$ is the least upper bound of a and b. Thus both $+$ and $\cdot$ are associative operations and hence any algebra $\mathfrak{A}=\langle A;+,\cdot\rangle$ satisying (1), $(\underline{1})$, (2), $(\underline{2})$ is a distributive lattice. By (3), $0=xx'\le x$ and hence 0 is the least element of the lattice while dually $1=x+x'\ge x$ is the largest element. Since (3) and $(\underline{3})$ simply claim that each element x has at least one complement x', any algebra

$\mathfrak{A}=\langle A;+,\cdot,',0,1\rangle$ satisfying Σ_1 is obviously a Boolean algebra. From the above proof it is clear that an algebra $\mathfrak{A}=\langle A;+,\cdot,'\rangle$ of type $\langle 2,2,1\rangle$ is a Boolean algebra iff it satisfies the self-dual set of identities Σ_2.

2. Proof of the independence of the system

(i) Let $A=\{0,1\}$; $x+y=1\ \forall x,y\in A$; $x'=0\ \forall x\in A$. Define $\cdot: A\times A\to A$ by the rule $x\cdot y=0$ if $y=0$ and $=1$ otherwise. Then $1\cdot 0+0=1\neq 0$ and hence ($\underline{1}$) fails. All the remaining five identities are valid here.

(ii) Let A be any finite set with $|A|\geq 2$ and define $x+y=y$ identically and let $\langle A;\cdot,0\rangle$ be a semilattice with the least element 0. Define $x'=0$ and $\mathbf{0}=\mathbf{1}=0$. Then the algebra $\mathfrak{A}=\langle A;+,\cdot,',\mathbf{0},\mathbf{1}\rangle$ satisfies all the identities except (**2**): $x+yz=yz$ while $(y+x)(z+x)=xx=x$ and, of course, $yz\neq x$.

(iii) Let $\mathfrak{A}=\langle A;+,\cdot\rangle$ be any finite distributive lattice with $0,1$ $|A|\geq 2$, and define $x'=0$ identically. Then ($\underline{3}$) fails but all the other remaining identities are valid here.

Theorem 2. *The self-dual class* **M** *of all modular lattices is defined by the minimal self-dual set of identities* $\Sigma_3=\{(1),(\underline{1}),(5),(\underline{5}),(6)\}$ *where* (5) *is the identity* $(xy)z=(yz)x$ *and* (6) *is the self-dual identity* $(x+yz)(y+z)=x(y+z)+yz$.

Proof. As in the proof of Theorem 1, $\Sigma_3\models xx+x$ and $x+x=x$ and hence, by [4], $\Sigma_3\models$ all lattice identities. It is clear that a lattice is modular iff it satisfies the identity (6).

Example 1. Let $|A|\geq 2$, $x+y=1\in A\ \forall x,y\in A$ and let $\langle A,\cdot\rangle$ be a semilattice operation. Then the identity ($\underline{1}$) is not valid in $\mathfrak{A}=\langle A;+,\cdot\rangle$ while the rest of Σ_3 are automatically valid.

Example 2. Let $|A|\geq 2$, $x+y=y\ \forall x,y\in A$ and let $\langle A;\cdot\rangle$ be a semilattice operation. Then the identity ($\underline{5}$) clearly fails in $\mathfrak{A}=\langle A;+,\cdot\rangle$ while the rest of Σ_3 are valid.

Example 3. The familiar non-modular lattice N_5 (see, e.g. figure 2.2, page 13 of [2]) satisfies (1), ($\underline{1}$), (5) and ($\underline{5}$) but not the modular law (6).

The identity (6) is due to Barry Wolk.

I would like to conclude this note with the obvious open problem:

Let **K** be an equational class of lattices such that whenever $L\in\mathbf{K}$, its dual $L\in\mathbf{K}$. Can **K** be defined by a minimal self-dual set of identities?

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