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# THE SET-THEORETIC MULTIVERSE

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**Abstract.** The multiverse view in set theory, introduced and argued for in this article, is the view that there are many distinct concepts of set, each instantiated in a corresponding set-theoretic universe. The universe view, in contrast, asserts that there is an absolute background set concept, with a corresponding absolute set-theoretic universe in which every set-theoretic question has a definite answer. The multiverse position, I argue, explains our experience with the enormous range of set-theoretic possibilities, a phenomenon that challenges the universe view. In particular, I argue that the continuum hypothesis is settled on the multiverse view by our extensive knowledge about how it behaves in the multiverse, and as a result it can no longer be settled in the manner formerly hoped for.

**§1. Introduction.** Set theorists commonly take their subject as constituting an ontological foundation for the rest of mathematics, in the sense that abstract mathematical objects can be construed fundamentally as sets, and because of this, they regard set theory as the domain of all mathematics. Mathematical objects *are* sets for them, and being precise in mathematics amounts to specifying an object in set theory. These sets accumulate transfinitely to form the cumulative universe of all sets, and the task of set theory is to discover its fundamental truths.

The *universe view* is the commonly held philosophical position that there is a unique absolute background concept of set, instantiated in the corresponding absolute set-theoretic universe, the cumulative universe of all sets, in which every set-theoretic assertion has a definite truth-value. On this view, interesting set-theoretic questions, such as the continuum hypothesis and others, have definitive final answers. Adherents of the universe view often point to the increasingly stable consequences of the large cardinal hierarchy, particularly in the realm of projective sets of reals with its attractive determinacy and regularity features, as well as the forcing absoluteness properties for $L(\mathbb{R})$, as evidence that we are on the right track towards the final answers to these set theoretical questions. Adherents of the universe view therefore also commonly affirm these large cardinal and regularity features in the absolute set-theoretic universe. The pervasive independence phenomenon in set theory is described on this view as a distraction, a side discussion about provability rather than truth—about the weakness of our theories in finding the truth, rather than about the truth itself—for the independence of a set-theoretic assertion from ZFC tells us little about whether it holds or not in the universe.

In this article, I shall argue for a contrary position, the *multiverse view*, which holds that there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths. Each such universe exists

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independently in the same Platonic sense that proponents of the universe view regard their universe to exist. Many of these universes have been already named and intensely studied in set theory, such as the classical models $L$ and HOD, the increasingly sophisticated definable inner models of large cardinals and the vast diversity of forcing extensions, although it is better to understand these descriptions as relative construction methods, since the resulting universe described depends on the initial universe in which the constructions are undertaken. Often the clearest way to refer to a set concept is to describe the universe of sets in which it is instantiated, and in this article I shall simply identify a set concept with the model of set theory to which it gives rise. By adopting a particular concept of set, we in effect adopt that universe as our current mathematical universe; we jump inside and explore the nature of set theory offered by that universe. In this sense, the multiverse view does not undermine the claim that set theory serves an ontological foundation for mathematics, since one expects to find all the familiar classical mathematical objects and structures inside any one of the universes in the multiverse, but rather it is directed at the claim that there is a unique absolute background concept of set, whose set-theoretic truths are immutable.

In particular, I shall argue in Section §7 that the question of the continuum hypothesis is settled on the multiverse view by our extensive, detailed knowledge of how it behaves in the multiverse. As a result, I argue, the continuum hypothesis can no longer be settled in the manner formerly hoped for, namely, by the introduction of a new natural axiom candidate that decides it. Such a dream solution template, I argue, is impossible because of our extensive experience in the CH and ¬CH worlds.

The multiverse view is one of higher-order realism—Platonism about universes—and I defend it as a realist position asserting actual existence of the alternative set-theoretic universes into which our mathematical tools have allowed us to glimpse. The multiverse view, therefore, does not reduce via proof to a brand of formalism. In particular, we may prefer some of the universes in the multiverse to others, and there is no obligation to consider them all as somehow equal.

The assertion that there are diverse concepts of set is a metamathematical as opposed to a mathematical claim, and one does not expect the properties of the multiverse to be available when undertaking an internal construction within a universe. That is, we do not expect to see the whole multiverse from within any particular universe. Nevertheless, set theory does have a remarkable ability to refer internally to many alternative set concepts, as when we consider definable inner models or various outer models to which we have access. In this way, set theory allows us to mount largely mathematical explorations of questions having a deeply philosophical nature, perhaps providing mathematical footholds for further philosophical inquiry. In the appendix of this article I describe two examples of such work, the modal logic of forcing and set-theoretic geology, which investigate the features of the set-theoretic universe in the context of all its forcing extensions and grounds.

**§2. The challenge of diverse set-theoretic possibilities.** Imagine briefly that set theory had followed an alternative history, that as the theory developed, theorems were increasingly settled in the base theory; that the independence phenomenon was limited to paradoxical-seeming metalogic statements; that the few true independence results occurring were settled by missing natural self-evident set principles; and that the basic structure of the set-theoretic universe became increasingly stable and agreed upon. Such developments would have constituted evidence for the universe view. But the actual history is not like this.

Instead, the most prominent phenomenon in set theory has been the discovery of a shocking diversity of set-theoretic possibilities. Our most powerful set-theoretic tools, such as forcing, ultrapowers, and canonical inner models, are most naturally and directly understood as methods of constructing alternative set-theoretic universes. A large part of set theory over the past half-century has been about constructing as many different models of set theory as possible, often to exhibit precise features or to have specific relationships with other models. Would you like to live in a universe where CH holds, but $\diamondsuit$ fails? Or where $2^{\aleph_n} = \aleph_{n+2}$ for every natural number $n$? Would you like to have rigid Suslin trees? Would you like every Aronszajn tree to be special? Do you want a weakly compact cardinal $\kappa$ for which $\diamondsuit_\kappa(\text{REG})$ fails? Set theorists build models to order.

As a result, the fundamental objects of study in set theory have become the models of set theory, and set theorists move with agility from one model to another. While group theorists study groups, ring theorists study rings and topologists study topological spaces, set theorists study the models of set theory. There is the constructible universe $L$ and its forcing extensions $L[G]$ and nonforcing extensions $L[0^\sharp]$; there are increasingly sophisticated definable inner models with large cardinals $L[\mu]$, $L[\vec{E}]$ and so on; there are models $V$ with much larger large cardinals and corresponding forcing extensions $V[G]$, ultrapowers $M$, cut-off universes $L_\delta$, $V_\alpha$, $H_\kappa$, universes $L(\mathbb{R})$, HOD, generic ultrapowers, boolean ultrapowers and on and on and on. As for forcing extensions, there are those obtained by adding one Cohen real, or many, or by other c.c.c. or proper (or semiproper) forcing, or by long iterations of these, or by the Lévy collapse, or by the Laver preparation or by self-encoding forcing, and on and on and on. Set theory appears to have discovered an entire cosmos of set-theoretic universes, revealing a category-theoretic nature for the subject, in which the universes are connected by the forcing relation or by large cardinal embeddings in complex commutative diagrams, like constellations filling a dark night sky.[1]

This abundance of set-theoretic possibilities poses a serious difficulty for the universe view, for if one holds that there is a single absolute background concept of set, then one must explain or explain away as imaginary all of the alternative universes that set theorists seem to have constructed. This seems a difficult task, for we have a robust experience in those worlds, and they appear fully set theoretic to us. The multiverse view, in contrast, explains this experience by embracing them as real, filling out the vision hinted at in our mathematical experience, that there is an abundance of set-theoretic worlds into which our mathematical tools have allowed us to glimpse.

The case of the definable inner models, such as $L(\mathbb{R})$ and HOD, may seem at first to be unproblematic for the universe view, as they are directly accessible from their containing universes. I counter this attitude, however, by pointing out that much of our knowledge of these inner models has actually arisen by considering them inside various outer models. We understand the coquettish nature of HOD, for example, by observing it to embrace an entire forcing extension, where sets have been made definable, before relaxing again in a subsequent extension, where they are no longer definable. In the case of the axiom of choice, Cohen first moves to an outer model before constructing the desired inner model of $\neg$AC. In this sense, the universe view lacks a full account of the definable inner models.

[1] This situation was anticipated by Mostowski, who in a talk recorded in Lakatos (1967) asserted that there are "several essentially different notions of set which are equally admissible as an intuitive basis for set theory," and Kalmár agrees with him about future practice (in 1965) by saying "I guess that in the future we shall say as naturally "let us take a set theory," as we take now a group G or a field F."

But it is the outer models, of course, such as the diverse forcing extensions, that most directly challenge the universe view. We do have a measure of access into the forcing extensions via names and the forcing relation, allowing us to understand the objects and truths of the forcing extension while remaining in the ground model. The multiverse view explains our mathematical experience with these models by positing that, indeed, these alternative universes exist, just as they seem to exist, with a full mathematical existence, fully as real as the universe under the universe view. Thus, in our mathematical experience the classical set concept splinters into a diverse array of parallel set concepts, and there seems little reason to think that we have discovered more than a tiny part of the multiverse.

The multiverse view does not abandon the goal of using set theory as an epistemological and ontological foundation for mathematics, for we expect to find all our familiar mathematical objects, such as the integer ring, the real field and our favorite topological spaces, inside any one of the universes of the multiverse. On the multiverse view, set theory remains a foundation for the classical mathematical enterprise. The difference is that when a mathematical issue is revealed to have a set-theoretic dependence, as in the independence results, then the multiverse response is a careful explanation that the mathematical fact of the matter depends on which concept of set is used, and this is almost always a very interesting situation, in which one may weigh the desirability of various set-theoretic hypotheses with their mathematical consequences. The universe view, in contrast, insists on one true answer to the independent question, although we seldom know which it is.

The beginning of a multiverse perspective appears already with von Neumann in his paper "An axiomatization of set theory," (1925, available in van Heijenoort, 1967, p. 393–413), where he considers (p. 412−413) the situation where one model of set theory can be a set inside another model of set theory and notes that a set that is "finite" in the former may be infinite in the latter; and similarly, a well-ordering in the former model may be ill-founded in the latter. He concludes "we have one more reason to entertain reservations about set theory and that for the time being no way of rehabilitating this theory is known." Solovay (2010), regarding this reaction as "rather hysterical," nevertheless points out that, "[von Neumann's] paper, as a whole, was an important way station in the evolution of formal set theory. (The system of Gödel's orange monograph is a direct descendant.)"

**§3. The ontology of forcing.** A central question in the dispute between the universe view and the multiverse view is whether there are universes outside $V$, the universe taken under the universe view to be the absolute background universe of all sets. A special case of this question, of course, concerns the status of forcing extensions of $V$, and this special case appears in many ways to capture the full debate. On the universe view, of course, forcing extensions of $V$ are deemed illusory, for $V$ is already everything, while the multiverse perspective regards $V$ as a relative concept, referring to whichever universe is currently under consideration, without there being any absolute background universe. On the multiverse view, the use of the symbol $V$ to mean "the universe" is something like an introduced constant that might refer to any of the universes in the multiverse, and for each of these the corresponding forcing extensions $V[G]$ are fully real. Thus, I find the following question, concerning the ontological status of forcing extensions $V[G]$ of the universe $V$, to be at the heart of the matter.

**Question 1.** *Do forcing extensions of the universe exist?*

More concretely, if $\mathbb{P} \in V$ is a nontrivial forcing notion, is there a $V$-generic filter $G$ over $\mathbb{P}$? Of course, all set theorists agree that there can be no such filter *in* $V$. Proponents of the universe view, which take $V$ to be everything, therefore answer negatively, and the assertion one sometimes hears

*"There are no $V$-generic filters"*

is a catechism for the universe view. Such an answer shares a parallel with the assertion

*"There is no square root of* $-1$."

Of course, $\sqrt{-1}$ does not exist in the real field $\mathbb{R}$. One must go to the field extension, the complex numbers, to find it. Similarly, one must go to the forcing extension $V[G]$ to find $G$. Historically, $\sqrt{-1}$ was viewed with suspicion, and existence deemed imaginary, but useful. Nevertheless, early mathematicians manipulated expressions like $2 + \sqrt{-5}$ and thereby caught a glimpse into the richer mathematical world of complex numbers. Similarly, set theorists now manipulate names and the forcing relation, thereby catching a glimpse into the forcing extensions $V[G]$. Eventually, of course, mathematicians realized how to simulate the complex numbers $a+bi \in \mathbb{C}$ concretely inside the real numbers, representing them as pairs $(a, b)$ with a peculiar multiplication $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. This way, one gains some access to the complex numbers, or a simulation of them, from a world having only real numbers, and full acceptance of complex numbers was on its way. The case of forcing has some similarities. Although there is no generic filter $G$ inside $V$, there are various ways of simulating the forcing extension $V[G]$ inside $V$, using the forcing relation, or using the Boolean-valued structure $V^{\mathbb{B}}$, or by using the Naturalist account of forcing. None of these methods provides a full isomorphic copy of the forcing extension inside the ground model (as the complex numbers are simulated in the reals), and indeed they provably cannot—it is simply too much to ask—but nevertheless some of the methods come maddeningly close to this. In any case it is true that with forcing, one has a high degree of access from the ground model to the forcing extension.

I would like to explain some of the details of this aspect of the various approaches to forcing, and so let us briefly review them. The forcing method was introduced by Cohen (1963, 1964, 1966), and was used initially to prove the independence of the axiom of choice and the continuum hypothesis. It was followed by an explosion of applications that produced an enormous variety of models of set theory. Forcing revealed the huge extent of the independence phenomenon, for which numerous set-theoretic assertions are independent of the ZFC axioms of set theory and others. With forcing, one begins with a ground model model $V$ of set theory and a partial order $\mathbb{P}$ in $V$. Supposing that $G \subseteq \mathbb{P}$ is a $V$-generic filter, meaning that $G$ contains members from every dense subset of $\mathbb{P}$ in $V$, one proceeds to build the forcing extension $V[G]$ by closing under elementary set-building operations.

$$V \subseteq V[G]$$

In effect, the forcing extension has adjoined the "ideal" object $G$ to $V$, in much the same way that one might build a field extension such as $\mathbb{Q}[\sqrt{2}]$. In particular, every object in $V[G]$ has a name in $V$ and is constructed algebraically from its name and $G$. Remarkably, the forcing extension $V[G]$ is always a model of ZFC. But it can exhibit different set-theoretic truths in a way that can be precisely controlled by the choice of $\mathbb{P}$. The ground model $V$ has a surprising degree of access to the objects and truths of $V[G]$. The method has by now produced a staggering collection of models of set theory.

Some accounts of forcing proceed by defining the *forcing relation* $p \Vdash \varphi$, which holds whenever every $V$-generic filter $G$ containing the condition $p$ has $V[G] \models \varphi$. The fundamental facts of forcing are that: (1) the forcing extension $V[G]$ satisfies ZFC. (2) Every statement $\varphi$ that holds in $V[G]$ is forced by some condition $p$ in $G$, and (3) the forcing relation $p \Vdash \varphi$ is definable in the ground model (for fixed $\varphi$, or for $\varphi$ of fixed complexity).

In order to construct the forcing extension, it appears that we need to find a suitably generic filter, and how do we do this? The difficulty of finding a generic filter is, of course, the crux. One traditional resolution of the difficulty is to force over a countable transitive ground model $M$. I shall call this the *countable transitive ground model* method of forcing, and this is also the initial instance of what I shall later refer to as the *toy model* method of formalization. One starts with such a countable transitive model $M$. Because it is countable, it has only countably many dense subsets of $\mathbb{P}$, which we may enumerate externally as $D_0$, $D_1$, $D_2$, and so on, and then proceed to pick any condition $p_0 \in D_0$, and then $p_1$ in $D_1$ below $p_0$, and so on. In this way, we build by diagonalization a descending sequence $p_0 \geq p_1 \geq p_2 \geq \cdots$, such that $p_n \in D_n$. It follows that the filter $G$ generated by this sequence is $M$-generic, and we may proceed to construct the forcing extension $M[G]$. Thus, with the countable transitive model approach, one arrives at the forcing extensions in a very concrete manner. Indeed, the elementary diagram of $M[G]$ is Turing computable from the diagram of $M$ and vice versa.

There are a number of drawbacks, however, to the countable transitive ground model approach to forcing. The first drawback is that it provides an understanding of forcing over only some models of set theory, whereas other accounts of forcing allow one to make sense of forcing over any model of set theory. With the countable transitive model approach to forcing, for example, the question "Is $\varphi$ forceable?" appears sensible only when asked in connection with a countable transitive model $M$, and this is an impoverishment of the method. A second drawback concerns metamathematical issues surrounding the existence of countable transitive models of ZFC: the basic problem is that we cannot prove that there are any such models, because by Gödel's Incompleteness Theorem, if ZFC is consistent then it cannot prove that there are any models of ZFC at all. Even if we were to assume Con(ZFC), then we still can't prove that there is a *transitive* model of ZFC, since the existence of such a model implies Con(ZFC + Con(ZFC)), and the consistency of this, and so on transfinitely. In this sense, the existence of a transitive model of ZFC is something like a very weak large cardinal axiom, which, although weaker than the assertion that there is an inaccessible cardinal, nevertheless transcends ZFC in consistency strength. The problem recurs with any stronger theory, such as ZFC plus large cardinals, since no theory (nor the consistency of the theory) can prove the existence of a transitive model of that same theory. Thus, it is no help to work in a background theory with many large cardinals, if one also wants to do forcing over such models. As a result, this approach to forcing seems to require one to pay a sort of tax just to implement the forcing method, starting with a stronger hypothesis than one ends up with just in order to carry out the argument. One way to avoid this problem, which is rarely implemented even though it answers this objection completely, is simply to drop the transitivity part of the hypothesis; the transitivity of the ground model is not actually needed in the construction, provided one is not squeamish about ill-founded models, for the countability of $M$ is sufficient to produce the $M$-generic filter, and with care one can still build the forcing extension $M[G]$ even when $M$ is ill-founded. This solution may be rare precisely because set theorists favor the well-founded models. More commonly, the defect is addressed by working not with countable transitive models of full ZFC, but rather with countable transitive models of some finite fragment of

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ZFC, whose existence is a consequence of the Reflection Theorem. Such a fragment ZFC$^*$ is usually unspecified, but deemed 'sufficiently large' to carry out all the usual set-theoretic constructions, and set theorists habitually take ZFC$^*$ to be essentially a substitute for ZFC. To prove an independence result using this method, for a set-theoretic assertion $\varphi$, one shows that every countable transitive model $M$ of a sufficient finite fragment ZFC$^{**}$ has a forcing extension $M[G]$ satisfying ZFC$^* + \varphi$, and concludes as a result that ZFC cannot prove $\neg\phi$, for if there were such a proof, then it would use only finitely many of the ZFC axioms, and we could therefore put those axioms in a finite fragment ZFC$^*$, find a model $M \models$ ZFC$^{**}$ and build the corresponding $M[G] \models$ ZFC$^* + \varphi$, a contradiction. Such an argument establishes Con(ZFC) $\implies$ Con(ZFC $+ \varphi$). Although this move successfully avoids the extra assumption of the existence of countable transitive models, it presents drawbacks of its own, the principal one being that it does not actually produce a model of ZFC $+ \varphi$, but only shows that this theory is consistent, leaving one ostensibly to produce the model by the Henkin construction or some other such method. Another serious drawback is that this approach to forcing pushes much of the technique inappropriately into the metatheory, for at the step where one says that the proof of $\neg\varphi$ involves only finitely many axioms and then appeals to the Reflection Theorem to get the model $M$, one must have the specific list of axioms used, for we have the Reflection Theorem only as a theorem scheme, and not as a single assertion "For every finite list of formulas, there is a $V_\alpha$ modeling them..." Thus, because the method mixes together the internal notion of finiteness with that of the metatheory, the result is that important parts of the proof take place in the metatheory rather than in ZFC, and this is metamathematically unsatisfying in comparison with the approaches that formalize forcing entirely as an internal construction.

Other accounts of forcing, in contrast, can be carried out entirely within ZFC, allowing one to force over any model of ZFC, without requiring one to consider any countable transitive models or to make any significant moves external to the model or in the metatheory. One such method is what I call the Naturalist account of forcing, which I shall shortly explain. This method is closely related to (but not the same as) the Boolean-valued model approach to forcing, another traditional approach to forcing, which I shall explain first. The concept of a Boolean-valued model has nothing especially to do with set theory, and one can have Boolean-valued partial orders, graphs, groups, rings, and so on, for any first-order theory, including set theory. A Boolean-valued structure consists of a complete Boolean algebra $\mathbb{B}$, together with a collection of objects, called the *names*, and an assignment of the atomic formulas $\sigma = \tau$ and $R(\sigma, \tau)$ to elements of $\mathbb{B}$, their Boolean values, denoted $[\![\, \sigma = \tau \,]\!]$ and $[\![\, R(\sigma, \tau) \,]\!]$, in such a way that the axioms of equality are obeyed (a technical requirement). One then proceeds to extend the Boolean values to all formulas inductively, by defining $[\![\, \varphi \wedge \psi \,]\!] = [\![\, \varphi \,]\!] \wedge [\![\, \psi \,]\!]$ and $[\![\, \neg\varphi \,]\!] = \neg[\![\, \varphi \,]\!]$, using the algebraic Boolean operations in $\mathbb{B}$, and $[\![\, \exists x\, \varphi(x) \,]\!] = \bigvee_\tau [\![\, \varphi(\tau) \,]\!]$, ranging over all names $\tau$, using the fact that $\mathbb{B}$ is complete. The Boolean-valued structure is said to be *full* if the Boolean value of every existential sentence is realized by some particular name, so that $[\![\, \exists x\, \varphi(x) \,]\!] = [\![\, \varphi(\tau) \,]\!]$ for some particular $\tau$. So to specify a Boolean-valued model, one needs a collection of objects, called the names, and a definition of the Boolean value on the atomic relations on them. The Boolean values of more complex assertions follow by recursion.

To carry this out in the case of set theory, one begins in a universe $V$, with a complete Boolean algebra $\mathbb{B}$ in $V$, and defines the class of names inductively, so that $\tau$ is a $\mathbb{B}$-name if it consists of pairs $\langle \sigma, b \rangle$, where $\sigma$ is a (previously constructed) $\mathbb{B}$-name and $b \in \mathbb{B}$. The idea is that this name promises to put the set named by $\sigma$ into the set named by $\tau$ with Boolean value at least $b$. The atomic values are defined to implement this idea, while also obeying the laws of equality, and one therefore eventually arrives at $V^{\mathbb{B}}$ as a Boolean-valued structure. The remarkable fact is that every axiom of ZFC holds with Boolean value 1 in $V^{\mathbb{B}}$. Furthermore, for a fixed formula $\varphi$, the map $\tau \mapsto [\![\, \varphi(\tau) \,]\!]$ is definable in $V$, since $V$ can carry out the recursive definition, and the structure $V^{\mathbb{B}}$ is full. Since Boolean values respect deduction, it follows that for any $b \neq 0$, the collection of $\varphi$ with $b \leq [\![\, \varphi \,]\!]$ is closed under deduction and contains no contradictions (since these get Boolean value 0). Thus, to use this method to prove relative consistency results, one must merely find a complete Boolean algebra $\mathbb{B}$ in a model of ZFC such that $[\![\, \varphi \,]\!] \neq 0$ for the desired statement $\varphi$. Since the ZFC axioms all get Boolean value 1, it follows that ZFC $+\ \varphi$ is also consistent. It is interesting to note that one can give a complete development of the Boolean-valued model $V^{\mathbb{B}}$ without ever talking about genericity, generic filters or dense sets.

Let me turn now to what I call the Naturalist account of forcing, which seeks to legitimize the actual practice of forcing, as it is used by set theorists. In any set-theoretic argument, a set theorist is operating in a particular universe $V$, conceived as the (current) universe of all sets, and whenever it is convenient he or she asserts "let $G$ be $V$-generic for the forcing notion $\mathbb{P}$," and then proceeds to make an argument in $V[G]$, while retaining everything that was previously known about $V$ and basic facts about how $V$ sits inside $V[G]$. This is the established pattern of how forcing is used in the literature, and it is proved legitimate by the following theorem.

THEOREM 3.1 (Naturalist Account of Forcing). *If $V$ is a (the) universe of set theory and $\mathbb{P}$ is a notion of forcing, then there is in $V$ a class model of the theory expressing what it means to be a forcing extension of $V$. Specifically, in the language with $\in$, constant symbols for every element of $V$, a predicate for $V$, and constant symbol $G$, the theory asserts:*

*(1) The full elementary diagram of $V$, relativized to the predicate for $V$.*
*(2) The assertion that $V$ is a transitive proper class in the (new) universe.*
*(3) The assertion that $G$ is a $V$-generic ultrafilter on $\mathbb{P}$.*
*(4) The assertion that the (new) universe is $V[G]$, and* ZFC *holds there.*

This is really a theorem scheme, since $V$ does not have full uniform access to its own elementary diagram, by Tarski's theorem on the nondefinability of truth. Rather, the theorem identifies a particular definable class model, and then asserts as a scheme that it satisfies all the desired properties. Another way to describe the method is the following:

THEOREM 3.2. *For any forcing notion $\mathbb{P}$, there is an elementary embedding*

$$V \precsim \overline{V} \subseteq \overline{V}[G]$$

*of the universe $V$ into a class model $\overline{V}$ for which there is a $\overline{V}$-generic filter $G \subseteq \bar{\mathbb{P}}$. In particular, $\overline{V}[G]$ is a forcing extension of $\overline{V}$, and the entire extension $\overline{V}[G]$, including the embedding of $V$ into $\overline{V}$, are definable classes in $V$, and $G \in V$.*

The point here is that $V$ has full access to the model $\overline{V}[G]$, including the object $G$ and the way that $V$ sits inside $\overline{V}$, for they are all definable classes in $V$. The structure $\overline{V}[G]$ is the model asserted to exist in Theorem 3.1, and $\overline{V}$ is the interpretation of the predicate for the ground model $V$ in that theory. Theorem 3.2 is also a theorem scheme, since the elementarity of the embedding is made as a separate assertion of elementarity for each formula. The models $\overline{V}$ and $\overline{V}[G]$ are not necessarily transitive or well-founded, and one should view them as structures $\langle \overline{V}, \bar{\in} \rangle$, $\langle \overline{V}[G], \bar{\in} \rangle$ with their own set membership relation $\bar{\in}$. Under general conditions connected with large cardinals, however, there is a substantial

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class of cases for which one can arrange that they are transitive (this is the main theme of Hamkins & Seabold, in preparation), and in this case, the embedding $V \precsim \overline{V}$ is, of course, a large cardinal embedding.

The connection between the Naturalist Account of forcing and the Boolean-valued model approach to forcing is that the easiest way to prove Theorems 3.1 and 3.2 is by means of Boolean-valued models and more precisely, the Boolean ultrapower, a concept going back to Vopenka in the 1960s. (See Hamkins & Seabold, in preparation, for a full account of the Boolean ultrapower.) If $\mathbb{B}$ is any complete Boolean algebra, then one can introduce a predicate $\check{V}$ for the ground model into the forcing language, defining $[\![\, \tau \in \check{V} \,]\!] = \bigvee_{x \in V} [\![\, \tau = \check{x} \,]\!]$, where $\check{x} = \{ \langle \check{y}, 1 \rangle \mid y \in x \}$ is the canonical name for $x$. If $\dot{G} = \{ \langle \check{b}, b \rangle \mid b \in \mathbb{B} \}$ is the canonical name for the generic filter, then the theory of Theorem 3.1 has Boolean value 1. The last claim in that theory amounts to the technical assertion $[\![\, \tau = \mathrm{val}(\check{\tau}, \dot{G}) \,]\!] = 1$, expressing the fact that the structure $V^{\mathbb{B}}$ knows with Boolean value 1 that the set named by $\tau$ is indeed named by $\tau$ via the generic object.

Let me give a few more details. Any Boolean-valued structure, whether it is a Boolean-valued graph, a Boolean-valued partial order, a Boolean-valued group or a Boolean-valued model of set theory, can be transformed into a classical two-valued first-order structure simply by taking the quotient by an ultrafilter. Specifically, let $U \subseteq \mathbb{B}$ be any ultrafilter on $\mathbb{B}$. There is no need for $U$ to be generic in any sense, and $U \in V$ is completely fine. Define an equivalence relation on the names by $\sigma \equiv_U \tau \iff [\![\, \sigma = \tau \,]\!] \in U$. A subtle and sometimes misunderstood point is that this is not necessarily the same as $\mathrm{val}(\sigma, U) = \mathrm{val}(\tau, U)$, when $U$ is not $V$-generic. Nevertheless, the relation $\sigma \in_U \tau \iff [\![\, \sigma \in \tau \,]\!]$ is well-defined on the equivalence classes, and one can form the quotient structure $V^{\mathbb{B}}/U$, using Scott's trick concerning reduced equivalence classes. The quotient $V^{\mathbb{B}}/U$ is now a classical 2-valued structure, and one proves the Łos theorem that $V^{\mathbb{B}}/U \models \varphi(\tau) \iff [\![\, \varphi(\tau) \,]\!] \in U$. The collection $\overline{V}$ of (equivalence classes of) names $\tau$ with $[\![\, \tau \in \check{V} \,]\!] \in U$ serves as the ground model of $V^{\mathbb{B}}/U$, and one proves that $V^{\mathbb{B}}/U$ is precisely $\overline{V}[G]$, where $G$ is the equivalence class $[\dot{G}]_U$. That is, the object $G$ is merely the equivalence class of the canonical name for the generic object, and this is a set that exists already in $V$. The entire structure $\overline{V}[G]$ therefore exists as a class in $V$, if the ultrafilter $U$ is in $V$. The map $x \mapsto [\check{x}]_U$ is an elementary embedding from $V$ to $\overline{V}$, the map mentioned in Theorem 3.2. This map is known as the Boolean ultrapower map, and it provides a way to generalize the ultrapower concept from ultrapowers by ultrafilters on power sets to ultrapowers by ultrafilters on any complete Boolean algebra. One can equivalently formulate the Boolean ultrapower map as the direct limit of the system of classical ultrapowers, obtained via the ultrafilters generated by $U$ on the power sets of all the various maximal antichains in $\mathbb{B}$, ordering the antichains under refinement. These induced ultrapower embeddings form a large commutative diagram, of which the Boolean ultrapower is the direct limit.

Part of the attraction of the Naturalist Account of forcing as developed in Theorem 3.1 is that one may invoke the theorem without paying attention to the manner in which it was proved. In such an application, one exists inside a universe $V$, currently thought of as the universe of all sets, and then, invoking Theorem 3.1 via the assertion

"Let $G \subseteq \mathbb{B}$ be $V$-generic. Argue in $V[G]\ldots$"

one adopts the new theory of Theorem 3.1. The theory explicitly stated in Theorem 3.1 allows one to keep all the previous knowledge about $V$, relativized to a predicate for $V$, but adopt the new (now current) universe $V[G]$, a forcing extension of $V$. Thus, although the proof did not provide an actual $V$-generic filter, the effect of the new theory is entirely as if it had. This method of application, therefore, implements in effect the content of the multiverse view. That is, whether or not the forcing extensions of $V$ actually exist, we are able to behave via the naturalist account of forcing entirely as if they do. In any set-theoretic context, whatever the current set-theoretic background universe $V$, one may at any time use forcing to jump to a universe $V[G]$ having a $V$-generic filter $G$, and this jump corresponds to an invocation of Theorem 3.1.

Of course, one might on the universe view simply use the naturalist account of forcing as the means of explaining the illusion: the forcing extensions don't really exist, but the naturalist account merely makes it seem as though they do. The multiverse view, however, takes this use of forcing at face value, as evidence that there actually are $V$-generic filters and the corresponding universes $V[G]$ to which they give rise, existing outside the universe. This is a claim that we cannot prove within set theory, of course, but the philosophical position makes sense of our experience—in a way that the universe view does not—simply by filling in the gaps, by positing as a philosophical claim the actual existence of the generic objects which forcing comes so close to grasping, without actually grasping. With forcing, we seem to have discovered the existence of other mathematical universes, outside our own universe, and the multiverse view asserts that yes, indeed, this is the case. We have access to these extensions via names and the forcing relation, even though this access is imperfect. Like Galileo, peering through his telescope at the moons of Jupiter and inferring the existence of other worlds, catching a glimpse of what it would be like to live on them, set theorists have seen via forcing that divergent concepts of set lead to new set-theoretic worlds, extending our previous universe, and many are now busy studying what it would be like to live in them.

An interesting change in the use of forcing has occurred in the history of set theory. In the earlier days of forcing, the method was used principally to prove independence results, with theorems usually having the form $\mathrm{Con}(\mathrm{ZFC} + \varphi) \implies \mathrm{Con}(\mathrm{ZFC} + \psi)$, proved by starting with a model of $\varphi$ and providing $\psi$ in a forcing extension. Contemporary work would state the theorem as: *If $\varphi$, then there is a forcing extension with $\psi$.* Furthermore, one would go on to explain what kind of forcing was involved, whether it was cardinal-preserving or c.c.c. or proper and so on. Such a description of the situation retains important information connecting the two models, and by emphasizing the relation between the two models, this manner of presentation conforms with the multiverse view.

**§4. The analogy between set theory and geometry.** There is a very strong analogy between the multiverse view in set theory and the most commonly held views about the nature of geometry. For 2000 years, mathematicians studied geometry, proving theorems about and making constructions in what seemed to be the unique background geometrical universe. In the late nineteenth century, however, geometers were shocked to discover non-Euclidean geometries. At first, these alternative geometries were presented merely as simulations within Euclidean geometry, as a kind of playful or temporary reinterpretation of the basic geometric concepts. For example, by temporarily regarding 'line' to mean a great circle on the unit sphere, one arrives at spherical geometry, where all lines intersect; by next regarding 'line' to mean a circle perpendicular to the unit circle, one arrives at one of the hyperbolic geometries, where there are many parallels to a given line through a given point. At first, these alternative geometries were considered as curiosities, useful perhaps for independence results, for with them one can prove that the parallel postulate is not provable from the other axioms. In time, however, geometers gained experience in the alternative geometries, developing intuitions about what it is like to live in them, and

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gradually they accepted the alternatives as geometrically meaningful. Today, geometers have a deep understanding of the alternative geometries, which are regarded as fully real and geometrical.

The situation with set theory is the same. The initial concept of set put forth by Cantor and developed in the early days of set theory seemed to be about a unique concept of set, with set-theoretic arguments and constructions seeming to take place in a unique background set-theoretic universe. Beginning with Gödel's constructible universe $L$ and particularly with the rise of forcing, however, alternative set-theoretic universes became known, and today set theory is saturated with them. Like the initial reactions to non-Euclidean geometry, the universe view regards these alternative universes as not fully real, while granting their usefulness for proving independence results. Meanwhile, set theorists continued, like the geometers a century ago, to gain experience living in the alternative set-theoretic worlds, and the multiverse view now makes the same step in set theory that geometers ultimately made long ago, namely, to accept the alternative worlds as fully real.

The analogy between geometry and set theory extends to the mathematical details about how one reasons about the alternative geometric and set-theoretic worlds. Geometers study the alternative geometries in several ways. First, they study a geometry by means of a simulation of it within Euclidean space, as in the presentations of spherical and hyperbolic geometry I mentioned above. This is studying a geometrical universe from the perspective of another geometrical universe that has some access to it or to a simulation of it. Second, they can in a sense jump inside the alternative geometry, for example, by adopting particular negations of the parallel postulate and reasoning totally within that new geometrical system. Finally, third, in a sophisticated contemporary understanding, they can reason abstractly by using the group of isometries that defines a particular geometry. The case of forcing offers these same three modes of reasoning. First, set theorists may reason about a forcing extension from the perspective of the ground model via names and the forcing relation. Second, they may reason about the forcing extension by jumping into it and reasoning as though they were living in that extension. Finally, third, they may reason about the forcing extension abstractly, by examining the Boolean algebra and its automorphism group, considering homogeneity properties, in order to make conclusions about the forcing extension.

A stubborn geometer might insist—like an exotic-travelogue writer who never actually ventures west of seventh avenue—that only Euclidean geometry is real and that all the various non-Euclidean geometries are merely curious simulations within it. Such a position is self-consistent, although stifling, for it appears to miss out on the geometrical insights that can arise from the other modes of reasoning. Similarly, a set theorist with the universe view can insist on an absolute background universe $V$, regarding all forcing extensions and other models as curious complex simulations within it. (I have personally witnessed the necessary contortions for class forcing.) Such a perspective may be entirely self-consistent, and I am not arguing that the universe view is incoherent, but rather, my point is that if one regards all outer models of the universe as merely simulated inside it via complex formalisms, one may miss out on insights that could arise from the simpler philosophical attitude taking them as fully real.

The history of mathematics provides numerous examples where initially puzzling imaginary objects become accepted as real. Irrational numbers, such as $\sqrt{2}$, became accepted; zero became a number; negative numbers and then imaginary and complex numbers were accepted; and then non-Euclidean geometries. Now is the time for $V$-generic filters.

**§5. Multiverse response to the categoricity arguments.** There is a rich tradition of categoricity arguments in set theory, going back to Peano's (1889) (van Heijenoort, 1967, pp. 83–97) proof that his second-order axioms characterize the unique structure of the natural numbers and to Zermelo's (1930) second-order categoricity proof for set theory, establishing the possible universes as $V_\kappa$ where $\kappa$ is inaccessible.[2] Continuing the categoricity tradition, Martin (2001) argues, to explain it very briefly, that the set-theoretic universe is unique, because any two set-theoretic concepts $V$ and $V'$ can be compared level-by-level through the ordinals, and at each stage, if they agree on $V_\alpha = V'_\alpha$ and each is claiming to be all the sets, then they will agree on $V_{\alpha+1} = V'_{\alpha+1}$, and so ultimately $V = V'$. The categoricity arguments, of course, tend to support the universe view.

The multiversist objects to Martin's presumption that we are able to compare the two set concepts in a coherent way. Which set concept are we using when undertaking the comparison? Martin's argument employs a background concept of 'property', which amounts to a common set-theoretic context in which we may simultaneously refer to both set concepts when performing the inductive comparison. Perhaps one would want to use either of the set concepts as the background context for the comparison, but it seems unwarranted to presume that either of the set concepts is able to refer to the other internally, and the ability to make external set (or property) concepts internal is the key to the success of the induction.

If we make explicit the role of the background set-theoretic context, then the argument appears to reduce to the claim that *within any fixed set-theoretic background concept*, any set concept that has all the sets agrees with that background concept; and hence any two of them agree with each other. But such a claim seems far from categoricity, should one entertain the idea that there can be different incompatible set-theoretic backgrounds.

Another more specific issue with Martin's comparison is that it requires that the two set concepts agree on the notion of ordinal sufficiently that one can carry out the transfinite recursive comparison. To be sure, Martin does explicitly assume in his argument that the concept of natural number is sharp in his sense, and he adopts a sharp account of the concept of wellordering, while admitting that "it is of course possible to have doubts about the sharpness of the concept of wellordering." Lacking a fixed background concept of set or of well-order, it seems that the comparison may become eventually incoherent. For example, perhaps each set concept individually provides an internally coherent account of set and of well-order, but in any common background, both are revealed as inadequate.

There seems little reason why two different concepts of set need to agree even on the concept of the natural numbers. Although we conventionally describe the natural numbers as 1, 2, 3, …, and so on, why are we so confident that this ellipses is meaningful as an absolute characterization? Peano's categoricity proof is a second-order proof that is sensible only in the context of a fixed concept of subsets of $\mathbb{N}$, and so this ellipses carries the baggage of a set-theoretic ontology. Our initial confidence that our "and so on" describes

[2] In what appears to be an interesting case of convergent evolution in the foundations of mathematics, this latter universe concept coincides almost completely with the concept of Grothendieck universe, now pervasively used in category theory Krömer (2001). The only difference is that the category theorists also view the empty set and $V_\omega$ as a Grothendieck universe, which amounts to considering $\aleph_0$ as an incipient inaccessible cardinal. Surely the rise of Grothedieck universes in category theory shares strong affinities with the multiverse view in set theory, although most set theorists find Grothendieck universes clumsy in comparison with the more flexible concept of a (transitive) model of set theory; nevertheless, the category theorists will point to the multiverse concepts present in the theory of topuses as more general still (see Blass, 1984).

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a unique structure of natural numbers should be tempered by the comparatively opaque task of grasping all sets of natural numbers, whose existence and nature supports and is required for the categoricity proof. On the multiverse view, the possibility that differing set concepts may lead to different and perhaps incomparable concepts of natural number suggests that Martin's comparison process may eventually become incoherent.

To illustrate, consider the situation of Peano's categoricity result for the natural numbers. Set theory surely provides a natural context in which to carry out Peano's second-order argument that all models of second-order arithmetic are canonically isomorphic. Indeed, one may prove in ZFC that there is a unique (second-order) inductive structure of the natural numbers. So once we fix a background concept of set satisfying (a small fragment of) ZFC, we achieve Peano's categoricity result for the natural numbers. Nevertheless, we also know quite well that different models of set theory can provide different and quite incompatible background concepts of set, and different models of ZFC can have quite different incomparable versions of their respective standard $n$s. Even though we prove in ZFC that $n$ is unique, the situation is that not all models of ZFC have the same $n$ or even the same arithmetic truths. Victoria Gitman and I have defined that a model of arithmetic is *a* standard model of arithmetic, as opposed to *the* standard model of arithmetic, if it is the $n$ of a model of ZFC, and Ali Enayat has characterized the nonstandard countable cofinality instances as precisely the computably saturated models of the arithmetic consequences of ZFC, and indeed, these are axiomatized by PA plus the assertions $\varphi \to \mathrm{Con}(\mathrm{ZFC}_n + \varphi^n)$, where $\mathrm{ZFC}_n$ is the $\Sigma_n$ fragment of ZFC.

The point is that a second-order categoricity argument, even just for the natural numbers, requires one to operate in a context with a background concept of set. And so although it may seem that saying "1, 2, 3, . . . and so on," has to do only with a highly absolute concept of finite number, the fact that the structure of the finite numbers is uniquely determined depends on our much murkier understanding of which subsets of the natural numbers exist. So why are mathematicians so confident that there is an absolute concept of finite natural number, independent of any set-theoretic concerns, when all of our categoricity arguments are explicitly set-theoretic and require one to commit to a background concept of set? My long-term expectation is that technical developments will eventually arise that provide a forcing analogue for arithmetic, allowing us to modify diverse models of arithmetic in a fundamental and flexible way, just as we now modify models of set theory by forcing, and this development will challenge our confidence in the uniqueness of the natural number structure, just as set-theoretic forcing has challenged our confidence in a unique absolute set-theoretic universe.

**§6. The multiverse provides a context for universe adjudication.** From the multiverse perspective, the advocates of the universe view want in effect to fix a particular set concept, with associated particular universe $V$, and declare it as the absolute background concept of set. In this way, the multiverse view can simulate the universe view by restricting attention to the lower cone in the multiverse consisting of this particular universe $V$ and the universes below it. Inside this cone, the universe $V$ provides in effect absolute background notions of countability, well-foundedness, and so on, and of course there are no nontrivial $V$-generic filters in this restricted multiverse. Any multiverse set theorist can pretend to be a universe set theorist simply by jumping into a specific $V$ and temporarily forgetting about the worlds outside $V$.

Meanwhile, there is understandably a level of disagreement within the universe community about precisely which universe $V$ to fix, about the features and truths of the *real* universe. The various arguments about the final answers to set-theoretic questions—is the continuum hypothesis *really* true or not?—amount to questions about which or what kind of $V$ we shall fix as the absolute background. The point here is that this is a debate that naturally takes place within the multiverse arena. It is explicitly a multiverse task to compare differing set concepts or universes or to entertain the possibility that there are fundamentally differing but legitimate concepts of set. It is only in a multiverse context that we may sensibly compare competing proposals for the unique absolute background universe. In this sense, the multiverse view provides a natural forum in which to adjudicate differences of opinion arising within the universe view.

**§7. Case study: multiverse view on the continuum hypothesis.** Let me discuss, as a kind of case study, how the multiverse and universe views treat one of the most important problems in set theory, the continuum hypothesis (CH). This is the famous assertion that every set of reals is either countable or equinumerous with $\mathbb{R}$, and it was a major open question from the time of Cantor, appearing at the top of Hilbert's well-known list of open problems in 1900. The continuum hypothesis is now known to be neither provable nor refutable from the usual ZFC axioms of set theory, if these axioms are consistent. Specifically, Gödel proved that ZFC + CH holds in the constructible universe $L$ of any model of ZFC, and so CH is not refutable in ZFC. In contrast, Cohen proved that $L$ has a forcing extension $L[G]$ satisfying ZFC + ¬CH, and so CH is not provable in ZFC. The generic filter directly adds any number of new real numbers, so that there could be $\aleph_2$ of them or more in $L[G]$, violating CH.

Going well beyond mere independence, however, it turns out that both CH and ¬CH are forceable over any model of set theory.

THEOREM 7.1. *The universe $V$ has forcing extensions*

*(1) $V[G]$, collapsing no cardinals, such that $V[G] \models \neg\mathrm{CH}$.*
*(2) $V[H]$, adding no new reals, such that $V[H] \models \mathrm{CH}$.*

In this sense, every model of set theory is very close to models with the opposite answer to CH. Since the CH and ¬CH are easily forceable, the continuum hypothesis is something like a light switch, which can be turned on and off by moving to ever larger forcing extensions. In each case the forcing is relatively mild, with the new universes, for example, having all the same large cardinals as the original universe. After decades of experience and study, set-theorists now have a profound understanding of how to achieve the continuum hypothesis or its negation in diverse models of set theory—forcing it or its negation in innumerable ways, while simultaneously controlling other set-theoretic properties—and have therefore come to a deep knowledge of the extent of the continuum hypothesis and its negation in the multiverse.

On the multiverse view, consequently, the continuum hypothesis is a settled question; it is incorrect to describe the CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse, about how to achieve it or its negation in combination with other diverse set-theoretic properties. Of course, there are and will always remain questions about whether one can achieve CH or its negation with this or that hypothesis, but the point is that the most important and essential facts about CH are deeply understood, and these facts constitute the answer to the CH question.

To buttress this claim, let me offer a brief philosophical argument that the CH can no longer be settled in the manner that set theorists formerly hoped that it might be. Set theorists traditionally hoped to settle CH according to the following template, which I shall now refer to as the *dream solution* template for CH:

> **Step 1.** Produce a set-theoretic assertion $\Phi$ expressing a natural 'obviously true' set-theoretic principle.
>
> **Step 2.** Prove that $\Phi$ determines CH.
> That is, prove that $\Phi \implies$ CH, or prove that $\Phi \implies \neg$CH.

In step 1, the assertion $\Phi$ should be obviously true in the same sense that many set-theorists find the axiom of choice and other set-theoretic axioms, such as the axiom of replacement, to be obviously true, namely, the statement should express a set-theoretic principle that we agree should be true in the intended intended interpretation, the prereflective set theory of our imagination. Succeeding in the dream solution would settle the CH, of course, because everyone would accept $\Phi$ and its consequences, and these consequences would include either CH or $\neg$CH.

I claim now that this dream solution has become impossible. It will never be realized. The reason has to do with our rich experience in set-theoretic worlds having CH and others having $\neg$CH. Our situation, after all, is not merely that CH is formally independent and we have no additional knowledge about whether it is true or not. Rather, we have an informed, deep understanding of the CH and $\neg$CH worlds and of how to build them from each other. Set theorists today grew up in these worlds, and they have flicked the CH light switch many times in order to achieve various set-theoretic effects. Consequently, if you were to present a principle $\Phi$ and prove that it implies $\neg$CH, say, then we can no longer see $\Phi$ as obviously true, since to do so would negate our experiences in the set-theoretic worlds having CH. Similarly, if $\Phi$ were proved to imply CH, then we would not accept it as obviously true, since this would negate our experiences in the worlds having $\neg$CH. The situation would be like having a purported 'obviously true' principle that implied that midtown Manhattan doesn't exist. But I know it exists; I live there. Please come visit! Similarly, both the CH and $\neg$CH worlds in which we have lived and worked seem perfectly legitimate and fully set-theoretic to us, and because of this, either implication in step 2 immediately casts doubt to us on the naturality of $\Phi$. Once we learn that a principle fulfills step 2, we can no longer accept it as fulfilling step 1, even if previously we might have thought it did. My predicted response to any attempt to carry out the dream solution is that claims of mathematical naturality (in step 1) will be met by objections arising from deep mathematical experience of the contrary situations (in step 2).

Allow me to give a few examples that illustrate this prediction. The first is the response to Freiling's (1986) delightful axiom of symmetry, which he presents as part of "a simple philosophical 'proof' of the negation of Cantor's continuum hypothesis." Freiling spends the first several pages of his article "subject[ing] the continuum to certain thought experiments involving random darts," before ultimately landing at his axiom of symmetry, which he presents as an "intuitively clear axiom." The axiom itself asserts that for any function $f$ mapping reals to countable sets of reals, there must reals $x$ and $y$ such that $y \notin f(x)$ and $x \notin f(y)$. Freiling argues at some length for the natural appeal of his axiom, asking us to imagine throwing two darts at a dart board, and having thrown the first dart, which lands at some position $x$, observing then that because $f(x)$ is a countable set, we should expect almost surely that the second dart will land at a point $y$ not in $f(x)$. And since the order in which we consider the darts shouldn't seem to matter, we conclude by symmetry that almost surely $x \notin f(y)$ as well. We therefore have natural reason, he argues, to expect not only that there is a pair $(x, y)$ with the desired property, but that intuitively almost all pairs have the desired property. Indeed, he asserts that, "actually [the axiom], being weaker than our intuition, does not say that the two darts have to do anything. All it claims is that what heuristically will happen every time, can happen." In this way, Freiling appears directly to be carrying out step 1 in the dream solution template. And sure enough, he proceeds next to carry out step 2, by proving in ZFC that the axiom of symmetry is exactly equivalent to $\neg$CH. The forward implication is easy, for if there is a well-ordering of $\mathbb{R}$ in order type $\omega_1$, then we may consider the function $f$ mapping every real $x$ to the initial segment of the order up to $x$, which is a countable set, and observe by linearity that for any pair of numbers either $x$ precedes $y$ or conversely, and so either $x \in f(y)$ or $y \in f(x)$, contrary to the axiom of symmetry, which establishes AS $\implies$ $\neg$CH. Conversely, if CH fails, then for any choice of $\omega_1$ many distinct $x_\alpha$, for $\alpha < \omega_1$, there must by $\neg$CH be a real $y \notin \bigcup_\alpha f(x_\alpha)$, since this union has size $\omega_1$, but $f(y)$ can only contain countably many $x_\alpha$, and so some late enough $x_\alpha$ and $y$ are as desired. In summary, AS $\iff$ $\neg$CH, and Freiling has exactly carried out the dream solution template for CH.

But was his argument received as a solution of CH? No. Many mathematicians objected that Freiling was implicitly assuming for a given function $f$ that various sets were measurable, including importantly the set $\{ (x, y) \mid y \in f(x) \}$. Freiling clearly anticipated this objection, making the counterargument in his paper that he was justifying his axioms prior to any mathematical development of measure, on the same philosophical or prereflective ideas that are used to justify our mathematical requirements for measure in the first place. Thus, he argues, we would seem to have as much intuitive support for the axiom of symmetry directly as we have for the idea that our measure should be countably additive, for example, or that it should satisfy the other basic properties we demand of it.

My point is not to defend Freiling specifically, but rather to observe that mathematicians objected to Freiling's argument largely from a perspective of deep experience and familiarity with nonmeasurable sets and functions, including extreme violations of the Fubini property, and for mathematicians with this experience and familiarity, the prereflective arguments simply fell flat. We are skeptical of any intuitive or naive use of measure precisely because we know so much now about the various mathematical pitfalls that can arise, about how complicated and badly behaved functions and sets of reals can be in terms of their measure-theoretic properties. We know that any naive account of measure will have a fundamental problem dealing with subsets of the plane all of whose horizontal sections are countable and all of whose vertical sections are cocountable, for example, precisely because the sets looks very small from one direction and very large from another direction, while we expect that rotating a set should not change its size. Indeed, one might imagine a variant of the Freiling argument, proceeding like this: intuitively all sets are measurable, and also rotating sets in the plane preserves measure; but if CH holds, then there are sets, such as the graph of a well-ordering of $\mathbb{R}$ in order type $\omega_1$, that look very small from one direction and very large from another. Hence, CH fails. Nevertheless, our response to this argument is similar to our response to the original Freiling argument, namely, because of our detailed experience with badly behaved sets and functions with respect to their measure properties, we simply cannot accept the naive expectations of the modified argument, and in the original case, we simply are not convinced by Freiling's argument that AS is intuitively true, even if he is using the same intuitions that guided us to the basic principles of measure in the first place. In an extreme instance of this, inverting Freiling's argument, set theorists sometimes reject the principle as a fundamental axiom precisely because of the counterexamples to it that one can produce under CH.

In the end, Freiling's argument is not generally accepted as a solution to CH, and his axiom instead is most often described as an intriguing equivalent formulation of ¬CH, which is interesting to consider and which reveals at its heart the need for us to take care with issues of nonmeasurability. In this way, Freiling's argument is turned on its head, as a warning about the error that may arise from a naive treatment of measure concepts. In summary, the episode exhibits my predicted response for any attempted answer following the dream solution template: rejection of the new axiom from a perspective of deep mathematical experience with the contrary.

Let me turn now to a second example. Consider the set-theoretic principle that I shall call the *powerset size axiom* PSA, which asserts in brief that strictly larger sets have strictly more subsets:

$$\forall x, y \qquad |x| < |y| \implies |P(x)| < |P(y)|.$$

Although set-theorists understand the situation of this axiom very well, as I will shortly explain, allow me first to discuss how it is received in nonlogic and non-set-theoretic mathematical circles, which is: extremely well! An enormous number of mathematicians, including many very good ones, view the axiom as extremely natural or even obviously true in the same way that various formulations of the axiom of choice or the other basic principles of set theory are obviously true. The principle, for example, is currently the top-rated answer among dozens to a popular mathoverflow question seeking examples of reasonable-sounding statements that are nevertheless independent of the axioms of set theory (Hamkins, 2009a).[3] In my experience, a brief talk with mathematicians at your favorite math tea has a good chance to turn up additional instances of mathematicians who find the assertion to express a basic fact about sets.

Meanwhile, set theorists almost never agree with this assessment, for they know that one can achieve all kinds of crazy patterns for the continuum function $\kappa \mapsto 2^\kappa$ via Easton's theorem, and even Cohen's original model of ZFC + ¬CH had $2^\omega = 2^{\omega_1}$, the assertion known as Luzin's hypothesis (Luzin, 1935), which had been proposed as an alternative to the continuum hypothesis. Furthermore, Martin's axiom implies $2^\omega = 2^\kappa$ for all $\kappa < 2^\omega$, which can mean additional violations of PSA when CH fails badly. So not only do set-theorists know that PSA can fail, but also they know that PSA must fail in models of the axioms, such as the proper forcing axiom PFA or Martin's maximum MM, that are often favored particularly by set-theorists with a universe view.

Thus, the situation is that a set-theoretic principle that many mathematicians find to be obviously true and which surely expresses an intuitively clear prereflective principle about the concept of size, and which furthermore is known by set-theorists to be perfectly safe in the sense that it is relatively consistent with the other axioms of ZFC and in fact a consequence of the generalized continuum hypothesis, is nevertheless almost universally rejected by set-theorists when it is proposed as a fundamental axiom. This rejection follows my predicted pattern: we simply know too much about the various ways that the principle can be violated and have too much experience working in models of set theory where the principle fails to accept it as a fundamental axiom.

Imagine briefly that the history of set theory had proceeded differently; imagine, for example, that the powerset size axiom had been considered at the very beginning of set theory—perhaps it was used in the proof of a critical theorem settling a big open question at

[3] There are, in addition, at least three other mathoverflow questions posted by mathematicians simply asking naively whether the PSA is true, or how to prove it, or indeed whether it is true that the assertion is not provable, incredible as that might seem.

the time—and was subsequently added to the standard list of axioms. In this case, perhaps we would now look upon models of ¬PSA as strange in some fundamental way, violating a basic intuitive principle of sets concerning the relative sizes of power sets; perhaps our reaction to these models would be like the current reaction some mathematicians (not all) have to models of $ZF + \neg AC$ or to models of Aczel's antifoundation axiom AFA, namely, the view that the models may be interesting mathematically and useful for a purpose, but ultimately they violate a basic principle of sets.

There have been other more sophisticated attempts to settle CH, which do not rely on the dream solution template. Woodin, for example, has advanced an argument for ¬CH based on Ω-logic, appealing to desirable structural properties of this logic, and more recently has proposed to settle CH positively in light of his proposed construction of *ultimate-L*. In both cases, these arguments attempt to settle CH by pointing out (in both cases under conjectured technical assumptions) that in order for the absolute background universe to be regular or desirable in a certain very precise, technical way, the favored solution of CH must hold. To my way of thinking, however, a similar objection still applies to these arguments. What the multiversist desires in such a line of reasoning is an explicit explanation of how our experience in the CH worlds or in the ¬CH worlds was somehow illusory, as it seems it must have been, if the argument is to succeed. Since we have an informed, deep understanding of how it could be that CH holds or fails, even in worlds very close to any given world, it will be difficult to regard these worlds as imaginary. To sloganize the dispute, the universists present a shining city on a hill; and indeed it is lovely, but the multiversists are widely traveled and know it is not the only one.

**§8. Case study: multiverse view on the axiom of constructibility.** Let me turn now, for another small case study, to the axiom of constructibility $V = L$ and how it is treated under the universe and multiverse positions. The constructible universe $L$, introduced by Gödel, is the result of the familiar proper class length transfinite recursive procedure

$$L = \bigcup_{\alpha \in ORD} L_\alpha,$$

where each $L_\alpha$ consists of the sets explicitly definable from parameters over the previous levels. The objects in the constructible universe would seem to be among the minimal collection of objects that one would be committed to, once one is living in a universe with those particular ordinals. The remarkable fact, proved by Gödel, is that because of its uniform structure, $L$ satisfies all of the ZF axioms of set theory, as well as the axiom of choice and the generalized continuum hypothesis; indeed, the fine structure and condensation properties of $L$ lead to numerous structural features, such as the $\diamondsuit$ principle and many others, and as a result the $V = L$ hypothesis settles many questions in set theory. For many of these questions, this was how set-theorists first knew that they were relatively consistent, and in this sense, the axiom $V = L$ has very high explanatory force.

Nevertheless, most set theorists reject $V = L$ as a fundamental axiom. It is viewed as limiting: why should we have only constructible sets? Furthermore, although the $V = L$ hypothesis settles many set-theoretic questions, it turns out too often to settle them in the "wrong" way, in the way of counterexamples and obstacles, rather than in a progressively unifying theory. One might compare the cumbersome descriptive set theory of $L$ to the smooth uniformity that is provable from projective determinacy. A significantly related observation is that $L$ can have none of the larger large cardinals.

Several set theorists have emphasized that even under $V \neq L$, we may still analyze what life would be like in $L$ by means of the the translation $\varphi \mapsto \varphi^L$, that is, by asking in $V$ whether a given statement $\varphi$ is true in the $L$ of $V$. Steel makes the point as follows in slides from his talk (Steel, 2004) as follows (see my criticism below):

> $V = L$ is restrictive, in that adopting it limits the interpretative power of our language.

1. The language of set theory as used by the $V = L$ believer can be translated into the language of set theory as used by the "$\exists$ measurable cardinal" believer:
$$\varphi \mapsto \varphi^L.$$
2. There is no translation in the other direction.
3. Proving $\varphi$ in ZFC$+V = L$ is equivalent to proving $\varphi^L$ in ZFC. Thus adding $V = L$ settles no questions not already settled by ZFC. It just prevents us from asking as many questions!

Despite the near universal rejection of $V = L$ as a fundamental axiom, the universe $L$ is intensively studied, along with many other $L$-like universes accommodating larger large cardinals. This practice exactly illustrates the multiverse perspective, for set-theorists study $L$ as one of many many possible set-theoretic universes, and our knowledge of $L$ and the other canonical inner models is particularly interesting when it sheds light on $V$. At the same time, the multiverse perspective on $L$ takes note of several observations against the view of $L$ as fundamentally limiting, for in several surprising ways, the constructible universe $L$ is not as impoverished as it is commonly described to be.

To begin, of course, there is not just one $L$. Each universe of set theory has its own constructible universe, and these can vary widely: different ordinals, different large cardinals, different reals, and even different arithmetic truths. On the multiverse view, one should understand talk of "the" constructible universe as referring to a rough grouping of very different models, all satisfying $V = L$.

One eye-opening observation is that $V$ and $L$ have transitive models of exactly the same (constructible) theories. For example, there is a transitive model of the theory $ZFC+$"there is a proper class of supercompact cardinals" if and only if there is such a transitive model inside $L$. The reason is that the assertion that a given theory $T$ has a transitive model has complexity $\Sigma^1_2(T)$, in the form "there is a real coding a well founded structure satisfying $T$," and so it is absolute between $V$ and $L$ by the Shoenfield absoluteness theorem, provided the theory itself is in $L$. Consequently, one can have transitive models of extremely strong theories, including huge cardinals and more, without ever leaving $L$.

Using this phenomenon, one may attack Steel's point 2 by proposing the translation $\varphi \mapsto$"$\varphi$ holds in a transitive model of set theory." The believer in large cardinals is usually happy to consider those cardinals and other set-theoretic properties inside a transitive model, and surely understanding how a particular set-theoretic concept behaves inside a transitive model of set theory is nearly the same as understanding how it behaves in $V$. But the point is that with such a relativization to transitive models, we get the same answer in $L$ as in $V$, and so the assumption of $V = L$ did not prevent us from asking about $\varphi$ in this way.

The large cardinal skeptics will object to Steel's point 1 on the grounds that the translation $\varphi \mapsto \varphi^L$, if undertaken in the large cardinal context in which Steel appears to intend it, carries with it all the large cardinal baggage from such a large cardinal $V$ to

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its corresponding $L$. After all, the $L$ of a model with supercompact cardinals, for example, will necessarily satisfy Con(ZFC + Woodin cardinals) and much more, and so the strategy of working in $V$ with large cardinals does not really offer a translation for the believers in $V = L$ who are skeptical about those large cardinals. Similarly, the relevant comparison in point 3 would seem instead to be ZFC+$V = L$ versus $ZFC +$ large cardinals, and here the simple equivalence breaks down, precisely because the large cardinals have consequences in $L$, such as the existence of the corresponding shadow large cardinals in $L$, as well as consistency statements that will be revealed in the arithmetic truths of $L$. If ZFC is consistent, then it is of course incorrect to say that ZFC + $V = L \vdash \varphi$ if and only if ZFC + large cardinals $\vdash \varphi^L$.

Pushing a bit harder on the examples above reveals that every real of $V$ exists in a model $M$ of any desired fixed finite fragment ZFC* of ZFC, plus $V = L$, which furthermore is well-founded to any desired countable ordinal height. That is, for every real $x$ and countable ordinal $\alpha$, there is a model $M \models$ ZFC* + $V = L$, whose well-founded initial segment exceeds $\alpha$ and such that $x$ is among the reals of $M$. The proof of this is that the statement is true inside $L$, and it has complexity $\Pi^1_2$, adjoining a code for $\alpha$ as a parameter if necessary, and hence true by the Shoenfield absoluteness theorem. One can achieve full ZFC in the model $M$ if $L$ has ZFC models $L_\alpha$ unbounded in $\omega_1^L$, which would hold if it had any uncountable transitive model of ZFC. Results in Hamkins *et al*. (submitted) show moreover that one can even arrange that the model $M$ is pointwise definable, meaning that every set in $M$ is definable there without parameters.

A striking example of the previous observation arises if one supposes that $0^\sharp$ exists. Considering it as a real, the previous remarks show that $0^\sharp$ exists inside a pointwise definable model of ZFC + $V = L$, well-founded far beyond $\omega_1^L$. Thus, true $0^\sharp$ sits unrecognized, yet definable, inside a model of $V = L$ which is well-founded a long way. For a second striking example, force to collapse $\omega_1$ to $\omega$ by a generic filter $g$, which can be coded as a real. In $V[g]$, there is $M \models$ ZFC + $V = L$ with $g \in M$ and $M$ well-founded beyond $\omega_1^V$. So $M$ thinks that the generic real $g$ is constructible, constructed at some (necessarily nonstandard) stage beyond $\omega_1^V$.

Another sense in which the axiom of constructibility is not inherently limiting is provided by the observation that every transitive model of set theory can in principle be continued upward to a model of $V = L$. For a countable model $M$, this follows from the previous observations about reals, since we simply code $M$ with a real and then place that real, and hence also $M$, into an $\omega$-model of ZFC + $V = L$, which in consequence will be well-founded beyond the height of $M$. More generally, for any transitive model, we may first collapse it to be countable by forcing, and then carry out the previous argument in the forcing extension. In this way, any transitive set $M$ can in principle exist as a transitive set inside a model of ZFC + $V = L$. For all we know, our entire current universe, large cardinals and all, is a countable transitive model inside a much larger model of $V = L$.

In consequence, the axiom of constructibility $V = L$ begins to shed its reputation as a limiting principle; perhaps it will gain the reputation instead as a *rewarder of the patient*, in the sense that if one waits long enough, then one's sets become constructible. Much of the view of $V = L$ as inherently limiting stems from an implicit commitment to the idea of an absolute background concept of ordinal and the assumption that our current universe $V$ has all the ordinals. That is, if we regard the current $L$ as already containing all the ordinals, then the universe $V$ can only grow sideways out of $L$, and with further sideways growth, one never recovers $V = L$. But on the multiverse view, one loses confidence in the idea of an absolute background concept of the class of all ordinals, and we saw above how easily the $V = L$ hypothesis can be recovered from any model of set theory by growing upward as well as outward, rather than only outward. As more ordinals arrive, we begin to see new constructible sets that were formerly thought nonconstructible, and the constructible universe $L$ can grow to encompass any given universe. In light of these considerations, the models of $V = L$ do not seem to be particularly impoverished.

**§9. Multiverse axioms.** Let me turn now to a more speculative enterprise by proposing a number of principles that express a fuller vision of the multiverse perspective. These principles, some of them provocative, should express the universe existence properties we might expect to find in the multiverse.

Before describing these principles, however, let me briefly remark on the issue of formalism. Of course we do not expect to be able to express universe existence principles in the first-order language of set theory or even in the second-order language of Gödel–Bernays or Kelly–Morse set theory, since the entire point of the multiverse perspective is that there may be other universes outside a given one. In any of the usual formalizations of set theory, we are of course at a loss to refer to these other universes or to the objects in them. Thus, our foundational ideas in the mathematics and philosophy of set theory outstrip our formalism, mathematical issues become philosophical, and set theory increasingly finds itself in need of philosophical assistance. The difficulty is that this philosophical need is arising in connection with some of the most highly technical parts of the subject, such as forcing, large cardinals, and canonical inner models. So the subject requires both technically minded philosophers and philosophically minded mathematicians.

It turns out that many interesting questions about the multiverse, such as those arising in the modal logic of forcing or in set-theoretic geology (two topics explained in the appendix of this article), can be formalized entirely in the usual first-order language of set theory. And for the issues in this realm, the work can be understood as purely mathematical. I find it a fascinating situation, where what might otherwise have been a purely philosophical enterprise, the consideration and analysis of alternative mathematical universes, becomes a fully rigorous mathematical one. But for many other natural questions about the multiverse, we seem unable adequately to formalize them, despite our interest in the true second-order or higher-order issues. One fallback method is the *toy model perspective*, discussed at length in Fuchs *et al*. (submitted), where one considers a multiverse merely as a set of models in the context of a larger ZFC background. This approach is adopted in Gitman & Hamkins (2010) and also in Friedman (2006); Friedman *et al*. (2008) in the context of the inner model hypothesis. It is precisely the toy model perspective that underlies the traditional countable transitive model approach to forcing. The toy model perspective can ultimately be unsatisfying, however, since it is of course in each case not the toy model in which we are interested, but rather the fully grown-up universe. Allow me, then, to state the multiverse axioms as unformalized universe existence assertions about what we expect of the genuine full multiverse.

The background idea of the multiverse, of course, is that there should be a large collection of universes, each a model of (some kind of) set theory. There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, $\mathrm{ZF}^-$, KP, and so on, perhaps even down to second-order number theory, as this is set-theoretic in a sense. At the same time, there is no reason to consider all universes in the multiverse equally, and we may simply be more interested in the parts of the multiverse consisting of universes satisfying very strong theories, such as ZFC plus large cardinals. The point is that there is little need to draw sharp boundaries as to what counts as a set-theoretic universe, and we may easily regard some universes as more set-theoretic than others.

Nevertheless, we have an expansive vision for the multiverse. Explaining set-theorist's resistance to $V = L$ as an axiom, Maddy (1998) has pointed to the MAXIMIZE maxim—asserting that there should be no undue limitations on the sets that might exist—for it would inflict an artificial limitation to insist that every set is constructible from ordinals. Here, we may follow a multiverse analogue of MAXIMIZE, by placing no undue limitations on what universes might exist in the multiverse. This is simply a higher-order analogue of the same motivation underlying set-theorists' ever more expansive vision of set theory. We want to consider that the multiverse is as big as we can imagine. At any time, we are living inside one of the universes, referred to as $V$ and thought of as the current universe, but by various means, sometimes metamathematical, we may be able to move around in the multiverse. We do not expect individual worlds to access the whole multiverse. Thus, we do not want to allow set construction principles in any one universe that involve quantifying over all universes, unless this quantification can be shown to reduce to first order quantification over sets in that universe.

Let me begin with the most basic universe existence principles. It goes without saying that the view holds first, that there is a universe. Going beyond this, the following principle asserts that inner models of a given universe, and more generally interpreted models, are also universes in their own right.

**Realizability Principle.** For any universe $V$, if $W$ is a model of set theory and definable or interpreted in $V$, then $W$ is a universe.

The principle expresses the multiverse capacity to regard constructed universes as real. In algebra, when one group is constructed from another or when a field is constructed as a quotient of a ring, these new structures are taken to be perfectly good groups and fields. There is no objection in such a case announcing "*but that isn't the real* $+$," analogous to what one might hear about a nonstandard or disfavored set membership relation: "*that isn't the real* $\in$." Such remarks in set theory are especially curious given that the nature and even the existence of the intended absolute background notion is so murky. On the multiverse view, if someone constructs a model of set theory, then we simply wonder, "what would it be like to live in that universe?" Jumping inside, we want to regard it as the real universe. The realizability principle allows this, and encompasses all the usual definable inner models, such as $L$, HOD, $L(\mathbb{R})$, as well as ultrapowers and many others, including models that are sets in $V$, such as $J_\beta$, $V_\alpha$, and $H_\kappa$, and so on. In at least a weak sense, the realizability principle should be unproblematic even for those with the universe view, since even if you have an absolute background universe, then the inner models have a real existence in at least some sense.

The next principle, in contrast, posits the existence of certain outer models and therefore engages with the main disagreement between the universe and multiverse views. I regard the following principle as a minimal foundational commitment for the multiverse view.

**Forcing Extension Principle.** For any universe $V$ and any forcing notion $\mathbb{P}$ in $V$, there is a forcing extension $V[G]$, where $G \subseteq \mathbb{P}$ is $V$-generic.

Naturally, we also expect many nonforcing extensions as well.

Next I consider a multiverse principle I refer to as the *reflection axiom*. Every model of set theory imagines itself as very tall, containing all the ordinals. But of course, a model contains only all of its *own* ordinals, and we know very well that it can be mistaken about how high these reach, since other models may have many more ordinals. We have a rich experience of this phenomenon with models of set theory, and many arguments rely precisely on this point, of a model thinking it has all the ordinals, when actually it is very small, even countable. Thus, on the multiverse perspective, we should expect that every universe can be extended to a much taller universe. In addition, many philosophers have emphasized the importance of reflection principles in set theory, using them to justify increasingly strong large cardinal axioms and to fulfill the ideas at the dawn of set theory of the unknowability of the full universe of sets. Thus, I propose the following multiverse principle:

**Reflection Axiom.** For every universe $V$, there is a much taller universe $W$ with an ordinal $\theta$ for which $V \precsim W_\theta \prec W$.

The toy model version of this axioms is easily seen to be a consequence of the compactness theorem, applied to the elementary diagram of $V$ and its relativization to $W_\theta$, using a new ordinal constant $\theta$. But here, in the full-blown multiverse context, the principle asserts that no universe is correct about the height of the ordinals, and every universe looks like an initial segment of a much taller universe having the same truths.

Next, consider a somewhat more provocative principle. Every model of set theory imagines itself as enormous, containing the reals $\mathbb{R}$, all the ordinals, and so on, and it certainly believes itself to be uncountable. But of course, we know that models can be seriously mistaken about this. The Lowenheim–Skolem theorem, of course, explains how there can be countable models of set theory, and these models do not realize there is a much larger universe surrounding them. Countability after all is a notion that depends on the set-theoretic background, since the very same set can be seen either as countable or uncountable in different set-theoretic contexts. Indeed, an elementary forcing argument shows that any set in any model of set theory can be made countable in a suitable forcing extension of that universe. On the multiverse perspective, the notion of an absolute standard of countability falls away. I therefore propose the following principle:

**Countability Principle.** Every universe $V$ is countable from the perspective of another, better universe $W$.

If one merely asserts that $V$ is an element of another universe $W$, then by the Forcing Extension principle it will be countable in the forcing extension $W[G]$ that collapses suitable cardinals to $\omega$, and so there is a measure of redundancy in these multiverse axioms, here and in several other instances.

Let me consider next what may be the most provocative principle. Set-theorists have traditionally emphasized the well-founded models of set theory, and the Mostowski collapses of these models are precisely the transitive models of set theory, universally agreed to be highly desirable models. The concept of well-foundedness, however, depends on the set-theoretic background, for different models of set theory can disagree on whether a structure is well-founded. Every model of set theory finds its own ordinals to be well-founded, but we know that models can be incorrect about this, and one model $W$ can be realized as ill-founded by another, better model. This issue arises, as I emphasized earlier, even with the natural numbers. Every model of set theory finds its own natural numbers $n$ to be 'the standard' model of arithmetic, but different models of set theory can have different (nonisomorphic) 'standard' models of arithmetic. The basic situation is that the $n$ of one model of set theory can often be viewed as nonstandard or even uncountable from another model. Indeed, every set-theoretic argument can take place in a model, which from the inside appears to be totally fine, but actually, the model is seen to be ill-founded from the external perspective of another, better model. Under the universe view, this problem terminates in the absolute set-theoretic background universe, which provides an accompanying absolute standard of well-foundedness. But the multiverse view allows for many different set-theoretic backgrounds, with varying concepts of the well-founded, and there seems to be no reason to support an absolute notion of well-foundedness. Thus, we seem to be led to the following principle:

**Well-foundedness Mirage.** Every universe $V$ is ill-founded from the perspective of another, better universe.

This principle appears to be abhorrent to many set theorists—and so I hope to have succeeded in my goal to be provocative—but I have a serious purpose in mind with it. The point is that we know very well that the concept of well-foundedness depends on the set-theoretic background, and so how can we accept philosophical talk focussing on well-founded models, using this term as though it had an absolute meaning independent of any set-theoretic background? And once we give up on an absolute set-theoretic background, then any grounding we may have had for an absolute background concept of well-foundedness falls away. On the multiverse view, therefore, for any universe there may be other universes with an improved concept of well-foundedness, and others with still better concepts, and we may apply this idea even to the natural numbers of the models.

Consider next the reverse embedding axiom. When we are living in a universe $V$ and have an embedding $j : V \to M$, such as an ultrapower embedding, then we know very well how to iterate this map forwards

$$V \to M \to M_2 \to M_3 \to \cdots$$

The later models $M_n$ have their iterated version of the embedding $j_n : M_n \to M_{n+1}$, and do not realize that these maps have already been iterated many times. On the multiverse view, every embedding is like this, having already been iterated an enormous number of times.

**Reverse Embedding Axiom.** For every universe $V$ and every embedding $j : V \to M$ in $V$, there is a universe $W$ and embedding $h$

$$W \xrightarrow{h} V \xrightarrow{j} M$$

such that $j$ is the iterate of $h$.

We saw earlier how every countable transitive model of set theory can be continued to a model of $V = L$. On the multiverse view, we expect every universe to be similarly absorbed into $L$, so that every universe is a countable transitive model inside a much larger $L$. In this sense, $L$ is a rewarder of the patient, for given sufficient ordinals your favorite set and indeed, your whole universe, will become both countable and constructible.

**Absorption into $L$.** Every universe $V$ is a countable transitive model in another universe $W$ satisfying $V = L$.

I have described a number of multiverse axioms, sketching out a vision of the universe existence principles we expect to find in the multiverse. A natural question, of course, is whether this vision is coherent. One way to mathematize this coherence question is to consider the multiverse axioms in their toy model interpretation: can one have a set collection of models of ZFC that satisfy all of the closure properties expressed by the multiverse axioms above? And for this toy model version of the question, the answer is yes. Victoria Gitman and I proved in Gitman & Hamkins (2010) that the collection of countable

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computably saturated models of ZFC satisfies all of the multiverse axioms above. With acknowledgement for the limitations of the toy model perspective as a guide to the full, true, higher-order multiverse, we take this as evidence that the multiverse vision expressed by the multiverse axioms above is coherent.

In conclusion, let me remark that this multiverse vision, in contrast to the universe view with which we began this article, fosters an attitude that what set theory is about is the exploration of the extensive range of set-theoretic possibilities. And this is an attitude that will lead any young researcher to many interesting set-theoretic questions and, I hope, fruitful answers.

**§10. Appendix: multiverse-inspired mathematics.** The mathematician's measure of a philosophical position may be the value of the mathematics to which it leads. Thus, the philosophical debate here may be a proxy for: where should set theory go? Which mathematical questions should it consider? The multiverse view in set theory leads one to consider how the various models of set theory interact, or how a particular world sits in the multiverse. This would include such mainstream topics as absoluteness, independence, forcing axioms, indestructibility, and even large cardinal axioms, which concern the relation of the universe $V$ to certain inner models $M$, such as an ultrapower. A few recent research efforts, however, have exhibited the multiverse perspective more explicitly, and I would like briefly to describe two such projects in which I have been involved, namely, the modal logic of forcing and set-theoretic geology. This appendix is adapted from Hamkins & Löewe (2008); Fuchs *et al.* (submitted) and the surveys Hamkins (2009, in press).

The modal logic of forcing (see Hamkins & Löewe, 2008, for a full account) is concerned with the most general principles in set theory relating forcing and truth. Although we did not begin with the intention to introduce modal logic into set theory, these principles are most naturally expressed in modal logic, for the collection of forcing extensions forms a robust Kripke model of possible worlds. Each such universe serves as an entire mathematical world and every ground model has a degree of access, via names and the forcing relation, to the objects and truths of its forcing extensions. Thus, this topic engages explicitly with a multiverse perspective. The natural forcing modalities (introduced in Hamkins, 2003b) assert that a sentence $\varphi$ is *possible* or *forceable*, written $\Diamond\varphi$, when it holds in a forcing extension and *necessary*, written $\Box\varphi$, when it holds in all forcing extensions. Thus, $\varphi$ is forceably necessary, if it is forceable that $\varphi$ is necessary; in other words, if there is a forcing extension in which $\varphi$ holds and continues to hold in all further extensions.

The initial inquiry centered on the question: Could the universe be completed with respect to what is forceably necessary? That is, could there be a model over which any possibly necessary assertion is already necessary? The *Maximality Principle* is the scheme expressing exactly that:

$$\Diamond\Box\varphi \implies \Box\varphi$$

This principle asserts that the universe has been completed with respect to forcing in the sense that everything permanent achievable by forcing has already been achieved. It turns out to be equiconsistent with ZFC.

THEOREM 10.1 (Hamkins, 2003b, also Stavi & Väänänen, 2003, independently). *The Maximality Principle is relatively consistent with* ZFC.

Another way to state the theorem is that if ZFC is consistent, then there is a model of ZFC in which the implication $\Diamond\Box\varphi \implies \Box\varphi$ is a valid principle of forcing, in the sense that it holds under the forcing interpretation for all sentences $\varphi$ in the language of set theory. This led to a deep question:

**Question 2.** *What are the generally correct principles of forcing? Which modal principles are valid in all models of set theory? Which can be valid?*

This question and its solution are deeply engaged with the multiverse perspective. Löwe and I were inspired by Solovay's (1976) modal analysis of provability, in which he discovered the corresponding valid principles of provability, and we aimed to do for forceability what Solovay did for provability.

To get started, the following principles are trivially valid principles of forcing:

$$\begin{array}{rl}
K & \Box(\varphi \implies \psi) \implies (\Box\varphi \implies \Box\psi) \\
Dual & \Box\neg\varphi \iff \neg\Diamond\varphi \\
S & \Box\varphi \implies \varphi \\
4 & \Box\varphi \implies \Box\Box\varphi \\
.2 & \Diamond\Box\varphi \implies \Box\Diamond\varphi
\end{array}$$

For example, K asserts that if an implication holds in every forcing extension and also its hypothesis, then so does its conclusion. Axioms Dual, S and 4 are similarly easy. Axiom .2 asserts that every possibly necessary statement is necessarily possible, and this is true because if $\mathbb{P}$ forces $\varphi$ to be necessary, and $\mathbb{Q}$ is some other forcing notion, then forcing with $\mathbb{P}$ over $V^{\mathbb{Q}}$ is equivalent by product forcing to forcing with $\mathbb{Q}$ over $V^{\mathbb{P}}$, and so $\varphi$ is possible over $V^{\mathbb{Q}}$, as desired. Since the assertions above axiomatize the modal theory known as S4.2 (and because the ZFC provable validities are closed under necessitation), it follows that any S4.2 modal assertion is a valid principle of forcing.

Going beyond S4.2, the following is a sample of modal assertions not derivable in S4.2—any modal logic text will contain hundreds more—each of which is important in some variant modal context. For example, the Löb axiom figures in Solovay's solution to provability logic, and expresses the provability content of Löb's theorem. Does the reader think that any of these express valid principles of forcing?

$$\begin{array}{rl}
5 & \Diamond\Box\varphi \implies \varphi \\
M & \Box\Diamond\varphi \implies \Diamond\Box\varphi \\
W5 & \Diamond\Box\varphi \implies (\varphi \implies \Box\varphi) \\
.3 & \Diamond\varphi \wedge \Diamond\psi \implies (\Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\psi \wedge \Diamond\varphi)) \\
Dm & \Box(\Box(\varphi \implies \Box\varphi) \implies \varphi) \implies (\Diamond\Box\varphi \implies \varphi) \\
Grz & \Box(\Box(\varphi \implies \Box\varphi) \implies \varphi) \implies \varphi \\
\text{Löb} & \Box(\Box\varphi \implies \varphi) \implies \Box\varphi \\
H & \varphi \implies \Box(\Diamond\varphi \implies \varphi)
\end{array}$$

It is a fun forcing exercise to show that actually, none of these assertions is valid for the forcing interpretation in all models of ZFC, although some are valid in some but not all models and others are never valid. The resulting modal theories can be arranged by strength into the following diagram, which originally focused our attention on S4.2 as a possible answer to question §10, since this theory is valid but all stronger theories in the diagram fail in some models of set theory.

Löwe and I were very pleased to prove that this answer was indeed correct:

THEOREM 10.2 (Hamkins & Löewe, 2008). *If* ZFC *is consistent, then the* ZFC*-provably valid principles of forcing are exactly S4.2.*

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We observed above that S4.2 is valid; the difficult part of the theorem was to show that nothing else is valid. If $S4.2 \nvdash \varphi$, then we must provide a model of set theory and set-theoretic assertions $\psi_i$ such that $\varphi(\psi_0, \dots, \psi_n)$ fails in that model. But what set-theoretic content should these assertions express? It seems completely mysterious at first. I would like to mention a few of the ideas from the proof. Two attractively simple concepts turn out to be key. Specifically, a statement $\varphi$ of set theory is said to be a *switch* if both $\varphi$ and $\neg\varphi$ are necessarily possible. That is, a switch is a statement whose truth value can always be turned on or off by further forcing. In contrast, a statement is a *button* if it is (necessarily) possibly necessary. These are the statements that can be forced true in such a way that they remain true in all further forcing extensions. The idea here is that once you push a button, you cannot unpush it. The Maximality Principle, for example, asserts that every button has already been pushed. Although buttons and switches may appear at first to be very special kinds of statements, nevertheless every statement in set theory is either a button, a switch, or the negation of a button. (After all, if you can't always switch $\varphi$ on and off, then it will either get stuck on or stuck off, and product forcing shows these possibilities to be mutually exclusive.) A family of buttons and switches is *independent*, if the buttons are not yet pushed and (necessarily) each of the buttons and switches can be controlled without affecting the others. Under $V = L$, it turns out that there is an infinite independent family of buttons and switches.

The proof of Theorem 10.2 rests in part on a detailed understanding of the modal logic S4.2 and its complete sets of Kripke frames. A (propositional) *Kripke model* is a collection of propositional *worlds* (essentially a truth table row, assigning propositional variables to true and false), with an underlying accessibility relation called the *frame*. A statement is possible or necessary at a world, accordingly as it is true in some or all accessible worlds, respectively. Every Kripke model built on a frame that is a directed partial preorder will satisfy the S4.2 axioms of modal logic, and in fact the finite directed partial preorders are *complete* for S4.2 in the sense that the statements true in all Kripke models built on such frames are exactly the statements provable from S4.2. An improved version of this, proved in Hamkins & Löewe (2008), is that the finite prelattices, and even the finite pre-Boolean algebras, are complete for S4.2. The following lemma, a central technical claim at the core of Hamkins & Löewe (2008), shows that any model of set theory with an independent family of buttons and switches is able to simulate any given Kripke model built on a finite prelattice frame.

LEMMA 10.3. *If $W \models$ ZFC has sufficient independent buttons and switches, then for any Kripke model $M$ on a finite prelattice frame, any $w \in M$, there is a translation of the propositional variables $p_i \mapsto \psi_i$ to set-theoretic assertions $\psi_i$, such that for any modal assertion $\varphi(p_1, \dots, p_n)$:*

$$(M, w) \models \varphi(p_1, \dots, p_n) \quad \Longleftrightarrow \quad W \models \varphi(\psi_1, \dots, \psi_n).$$

*Each $\psi_i$ is a Boolean combination of the buttons and switches.*

Consequently, if $S4.2 \nvdash \varphi$, then since we proved that there is a Kripke model $M$ built on a finite prelattice frame in which $\varphi$ fails, it follows that in any model of set theory $W$ having independent buttons and switches, which we proved exist, the corresponding assertion $\varphi(\psi_1, \dots, \psi_n)$ fails. This exactly shows that $\varphi$ is not a provably valid principle of forcing, as desired to prove Theorem 10.2. The proof is effective in the sense that if $S4.2 \nvdash \varphi$, then we are able explicitly to provide a model $W \models$ ZFC and the particular set-theoretic substitution instance $\varphi(\psi_1, \dots, \psi_n)$ which fails in $W$. To answer our earlier query about what the content of the sentences $\psi_i$ might be, the answer is that they are certain complex Boolean combinations of the independent buttons and switches relating to the Jankov–Fine formula describing the underlying frame of a propositional Kripke model in which $\varphi$ fails. The buttons and switches can be combined so as to fail under the forcing interpretation in just exactly the way that the modal assertion $\varphi$ fails in a finite Kripke model.

Although Theorem 10.2 tells us what are the ZFC-provably valid principles of forcing, it does not tell us that all models of ZFC exhibit only those validities. Indeed, we know that this isn't the case, because we know there are models of the Maximality Principle, for which the modal theory S5 is valid, and this is strictly stronger than S4.2. So different models of set theory can exhibit different valid principles of forcing. The proof of Theorem 10.2 adapts to show, however, that the valid principles of forcing over any fixed model of set theory lies between $S4.2$ and $S5$. This result is optimal because both of these endpoints occur: the theory $S4.2$ is realized in any model of $V = L$, and $S5$ is realized in any model of the Maximality Principle. This result also shows that the Maximality Principle is the strongest possible axiom of this type, expressible in this modal language as valid principles of forcing. A number of questions remain open, such as: Is there a model of ZFC whose valid principles of forcing form a theory other than S4.2 or S5? If $\varphi$ is valid in $W$, is it valid in all extensions of $W$? Can a model of ZFC have an unpushed button, but not two independent buttons?

Let me turn now to the topic of set-theoretic geology, another topic engaged with the multiverse perspective (see Fuchs *et al.*, submitted, for a full introductory account). Although forcing is ordinarily viewed as a method of constructing *outer* as opposed to *inner* models of set theory—one begins with a ground model and builds a forcing extension of it—a simple switch in perspective allows us to use forcing to describe inner models as well. The idea is simply to consider things from the perspective of the forcing extension rather than the ground model and to look downward from the universe $V$ for how it may have arisen by forcing. Given the set-theoretic universe $V$, we search for the possible ground models $W \subseteq V$ such that there is a $W$-generic filter $G \subseteq \mathbb{P} \in W$ such that $V = W[G]$. Such a perspective quickly leads one to look for deeper and deeper grounds, burrowing down to what we call bedrock models and deeper still, to what we call the mantle and the outer core, for the topic of set-theoretic geology.

Haim Gaifman has pointed out (with humor) that the geologic terminology takes an implicit position on the question of whether mathematics is a natural process or a human activity, for otherwise we might call it set-theoretic *archeology*, aiming to burrow down and uncover the remains of ancient set-theoretic civilizations.

Set-theoretic geology rests fundamentally on the following theorem, a surprisingly recent result considering the fundamental nature of the question it answers. Laver's proof of this theorem makes use of my work on the approximation and covering properties (Hamkins, 2003a).

THEOREM 10.4 (Laver, 2007, independently Woodin, 2004). *Every model of set theory $V \models$ ZFC is a definable class in all of its set forcing extensions $V[G]$, using parameters in $V$.*

After learning of this theorem, Jonas Reitz and I introduced the following hypothesis:

DEFINITION 10.5 (Hamkins, 2005; Reitz, 2006, 2007). *The ground axiom GA is the assertion that the universe is not obtained by nontrivial set forcing over any inner model.*

Because of the quantification over ground models, this assertion appears at first to be second order, but in fact the ground axiom is expressible by a first-order statement in the language of set theory (see Reitz, 2006, 2007). And although the ground axiom holds in many canonical models of set theory, such as $L$, $L[0^\sharp]$, $L[\mu]$ and many instances of $K$, none of the regularity features of these models are consequences of GA, for Reitz proved that every ZFC model has an extension, preserving any desired $V_\theta$ (and mild in the sense that every new set is generic for set forcing), which is a model of ZFC plus the ground axiom. Reitz's method obtained GA by forcing very strong versions of $V = \mathrm{HOD}$, but in a three-generation collaboration between Reitz, Woodin, and myself (Hamkins *et al.*, 2008), we proved that every model of set theory has an extension which is a model of GA plus $V \neq \mathrm{HOD}$.

A *ground* of the universe $V$ is a transitive class $W \models \mathrm{ZFC}$ over which $V$ was obtained by forcing, so that $V = W[G]$ for some $W$-generic $G \subseteq \mathbb{P} \in W$. A ground $W$ is a *bedrock* of $V$ if it is a minimal ground, that is, if there is no deeper ground inside $W$. Equivalently, $W$ is a bedrock of $V$ if it is a ground of $V$ and satisfies the ground axiom. It is an open question whether a model can have more than one bedrock model. However, Reitz (2006) constructs a *bottomless* model of ZFC, one having no bedrock at all. Such a model has many grounds, over which it is obtained by forcing, but all of these grounds are themselves obtained by forcing over still deeper grounds, and so on without end. The principal new set-theoretic concept in geology is the following:

DEFINITION 10.6. *The mantle is the intersection of all grounds.*

The analysis at this point engages with an interesting philosophical position, the view I call *ancient paradise*. This position holds that there is a highly regular core underlying the universe of set theory, an inner model obscured over the eons by the accumulating layers of debris heaped up by innumerable forcing constructions since the beginning of time. If we could sweep the accumulated material away, on this view, then we should find an ancient paradise. The mantle, of course, wipes away an entire strata of forcing, and so if one subscribes to the ancient paradise view, then it seems one should expect the mantle to exhibit some of these highly regular features. Although I find the position highly appealing, our initial main theorem appears to provide strong evidence against it.

THEOREM 10.7 (Fuchs *et al.*, submitted). *Every model of* ZFC *is the mantle of another model of* ZFC.

The theorem shows that by sweeping away the accumulated sands of forcing, what we find is not a highly regular ancient core, but rather: an arbitrary model of set theory. In particular, since every model of set theory is the mantle of another model, one can prove nothing special about the mantle, and certainly, it need be no ancient paradise.

The fact that the mantle M is first-order definable follows from the following general fact, which reduces second-order quantification over grounds to first order quantification over indices.

THEOREM 10.8. *There is a parameterized family of classes $W_r$ such that*

*(1) Every $W_r$ is a ground of $V$ and $r \in W_r$.*

*(2) Every ground of $V$ is $W_r$ for some $r$.*

*(3) The relation "$x \in W_r$" is first order.*

For example, the ground axiom is first order expressible as $\forall r\, W_r = V$; the model $W_r$ is a bedrock if and only if $\forall s\, (W_s \subseteq W_r \implies W_s = W_r)$; and the mantle is defined by

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$M = \{ x \mid \forall r\, (x \in W_r) \}$, and these are all first-order assertions. When looking downward at the various grounds, it is very natural to inquire whether one can fruitfully intersect them. For example, the question of whether there can be distinct bedrock models in the universe is of course related to the question of whether there is a ground in their intersection. Let us define that the grounds are *downward directed* if for every $r$ and $s$ there is $t$ such that $W_t \subseteq W_r \cap W_s$. In this case, we say that the downward directed grounds hypothesis (DDG) holds. The strong DDG asserts that the grounds are downward set directed, that is, that for every set $A$ there is $t$ with $W_t \subseteq \bigcap_{r \in A} W_r$. These hypotheses concern fundamental questions about the structure of ground models of the universe—fundamental properties of the multiverse—and these seem to be foundational questions about the nature and power of forcing. In every model for which we can determine the answer, the strong DDG holds, although it is not known to be universal. In any case, when the grounds are directed, the mantle is well behaved.

THEOREM 10.9.

*(1) If the grounds are downward directed, then the mantle is constant across the grounds and a model of ZF.*
*(2) If the grounds are downward set-directed, then the mantle is a model of ZFC.*

It is natural to consider how the mantle is affected by forcing, and since every ground of $V$ is a ground of any forcing extension of $V$, it follows that the mantle of any forcing extension is contained in the mantle of the ground model. In the limit of this process, we arrive at:

DEFINITION 10.10. *The* generic mantle *of a model of set theory $V$ is the intersection of all ground models of all set forcing extensions of $V$.*

The generic mantle has proved in several ways to be a more robust version of the mantle. For example, for any model of ZFC, the generic mantle gM is always a model at least of ZF, without need for any directedness hypothesis. If the DDG holds in all forcing extensions, then in fact the mantle and the generic mantle are the same. If the strong DDG also holds, then the generic mantle is a model of ZFC.

Set-theoretic geology is naturally carried out in a context that includes all the forcing extensions of the universe, all the grounds of these extensions, all forcing extensions of these resulting grounds, and so on. The *generic multiverse* of a model of set theory, introduced by Woodin (2004), is the smallest family of models of set theory containing that model and closed under both forcing extensions and grounds. There are numerous philosophical motivations to study to the generic multiverse. Indeed, Woodin introduced it specifically in order to criticize a certain multiverse view of truth, namely, truth as true in every model of the generic multiverse. Although I do not hold such a view of truth, nevertheless I want to investigate the fundamental features of the generic multiverse, a task I place at the foundation of any deep understanding of forcing. Surely the generic multiverse is the most natural and illuminating background context for the project of set-theoretic geology.

If the DDG holds in all forcing extensions, then the grounds are dense below the generic multiverse, and so not only does $\mathrm{M} = \mathrm{gMantle}$, but the generic multiverse is exhausted by the ground extensions $W_r[G]$, the forcing extensions of the grounds. In this case, any two models in the generic multiverse have a common ground, and are therefore at most two steps apart on a generic multiverse path. There is a toy model example, however, showing that the generic grounds—the ground models of the forcing extensions—may not necessarily exhaust the generic multiverse.

The generic mantle gM is a parameter-free uniformly definable class model of ZF, containing all ordinals and invariant by forcing. Because it is invariant by forcing, the generic mantle gM is constant across the generic multiverse. In fact, it follows that the generic mantle gM is precisely the intersection of the generic multiverse. Therefore, it is the largest forcing-invariant class. Surely this makes it a highly canonical class in set theory; it should be the focus of intense study. Whenever anyone constructs a new model of set theory, I should like to know what is the mantle and generic mantle of this model.

The *generic* HOD, introduced by Fuchs in an attempt to identify a very large canonical forcing-invariant class, is the intersection of all HODs of all forcing extensions $V[G]$.

$$\mathrm{gHOD} = \bigcap_G \mathrm{HOD}^{V[G]}$$

This class is constant across the generic multiverse. Since the HODs of all the forcing extensions are downward set-directed, it follows that gHOD is locally realized and $\mathrm{gHOD} \mathrel{=\!\!\models} \mathrm{ZFC}$. The following diagram shows the basic inclusion relations.

The main results of Fuchs *et al*. (submitted) attempt to separate and control these classes.

THEOREM 10.11 (Fuchs *et al*., submitted).

1. *Every model of set theory $V$ has an extension $\overline{V}$ with*
$$V = M^{\overline{V}} = gM^{\overline{V}} = \mathrm{gHOD}^{\overline{V}} = \mathrm{HOD}^{\overline{V}}$$
2. *Every model of set theory $V$ has an extension $W$ with*
$$V = M^{W} = \mathrm{gM}^{\mathrm{W}} = \mathrm{gHOD}^{\mathrm{W}} \qquad \textit{but} \quad \mathrm{HOD}^{\mathrm{W}} = \mathrm{W}$$
3. *Every model of set theory $V$ has an extension $U$ with*
$$V = \mathrm{HOD}^{\mathrm{U}} = \mathrm{gHOD}^{\mathrm{U}} \qquad \textit{but} \quad \mathrm{M}^{\mathrm{U}} = \mathrm{U}$$
4. *Lastly, every $V$ has an extension $Y$ with*
$$Y = \mathrm{HOD}^{\mathrm{Y}} = \mathrm{gHOD}^{\mathrm{Y}} = \mathrm{M}^{\mathrm{Y}} = \mathrm{gM}^{\mathrm{Y}}$$

In particular, as mentioned earlier, every model of ZFC is the mantle and generic mantle of another model of ZFC. It follows that we cannot expect to prove any regularity features about the mantle or the generic mantle, beyond what can be proved about an arbitrary model of ZFC. It also follows that the mantle of $V$ is not necessarily a ground model of $V$, even when it is a model of ZFC. One can therefore iteratively take the mantle of the mantle and so on, and we have proved that this process can strictly continue. Indeed, by iteratively computing the mantle of the mantle and so on, what we call the *inner* mantles, we might eventually arrive at a model of ZFC plus the ground axiom, where the process would naturally terminate. If this should occur, then we call this termination model the *outer core* of the original model. Generalizing the theorem above, Fuchs, Reitz, and I have conjectured that every model of ZFC is the $\alpha^{\text{th}}$ inner mantle of another model of ZFC, for arbitrary ordinals $\alpha$ (or even $\alpha = \mathrm{ORD}$ or beyond), and we have proved this for $\alpha < \omega^2$. These inner mantles speak to a revived version of the ancient paradise position, which replies to Theorems 10.7 and 10.11 by saying that if the mantle turns out itself to be a forcing extension, then one hasn't really stripped away all the forcing yet, and one should proceed with the inner mantles, stripping away additional strata of forcing residue. If the conjecture is correct, however, then any fixed number of iterations of this will still arrive essentially at an arbitrary model of set theory, rather than an ancient set-theoretic paradise.

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