ON THE BASIS AND CHROMATIC NUMBER OF A GRAPH

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Introduction. The basis theorem for directed graphs is, in effect, a result on weakly ordered sets, and, in §1, a proof is given, based on Zorn's lemma, that generalizes, and perhaps clarifies the exposition in (**1**, Chapter 2). In §2, a graph G^* is defined, on an arbitrary collection Q of non-void subsets of a set X (which includes all its one-element subsets), in such a way that the partitions of X into Q-sets correspond to the kernels of G^*. Applied to the collection Q of non-null internally stable subsets of a graph G without loops, this identifies the chromatic number of G with the least cardinal number of any kernel of G^*.

1. The basis theorem. A non-null set X is *weakly ordered* by a relation $(\leqslant)$ in case

(R) $x \leqslant x$, $x \in X$, and

(T) $x \leqslant y$ and $y \leqslant z \Rightarrow x \leqslant z$;

partly ordered if also

(A) $x \leqslant y$ and $y \leqslant x \Rightarrow x = y$.

In a weakly ordered set, the relation $x \equiv y$, defined as $x \leqslant y$ and $y \leqslant x$, is an equivalence relation, which induces a partition of X into a set $P = P(X)$ of classes $[x] = \{y; y \equiv x\}$. The order relation $[x] \leqslant [y]$ defined by $x \leqslant y$ is well defined on classes, and partly orders the set P (**2**, Chapter I, Theorem 3).

A subset B of a weakly ordered set X will be called a *basis* in case

$(B1)$ $b \leqslant b_1$; $b, b_1 \in B$ implies $b = b_1$, and

$(B2)$ for every $x \in X$, there is a $b \in B$ such that $x \leqslant b$.

THEOREM 1. *If a weakly ordered set X has a basis B, then B is a representation system for the set M of maximal classes m $(m \leqslant p \Rightarrow m = p)$ of $P = P(X)$. Hence all bases have the same cardinal number. Moreover, the set M has the covering property*

(C): for every $p \in P$ there is an $m \geqslant p$, $m \in M$.

Conversely, if M satisfies (C), then every representation system of M is a basis for X.

Proof. If B is a basis, we show that $b \to [b]$ is one–one on B to all of M; hence B is a representation system for M. For, $[b] \leqslant [x]$ implies $b \leqslant x$, while

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$x \leqslant b_1 \in B$ by $(B2)$. Thus $b \leqslant b_1$ by (T), $b = b_1$ by $(B1)$, and $b \equiv x$, $[x] = [b] \in M$. Since $[b] = [b_1]$ implies $b = b_1$ by $(B1)$, $b \to [b]$ is one-one. If $[m] \in M$, then $m \leqslant b \in B$ by $(B2)$ so $[m] \leqslant [b]$, $[m] = [b]$, and $b \to [m]$. Moreover, for $[x] \in P$, we have $x \leqslant b \in B$ by $(B2)$, so that $[x] \leqslant [b]$ where $[b] \in M$, and (C) holds.

Conversely, if M satisfies (C), and B is a representation system for M, then B is a basis. Under the conditions of (B1), we have $[b] \leqslant [b_1] \in M$, $[b] = [b_1]$, and $b = b_1$ since B is a representation system. By (C), we have $[x] \leqslant [m] = [b]$, $x \leqslant b \in B$, and $(B2)$ holds.

COROLLARY 1. *A weakly ordered set X has a basis if and only if the set M of maximal classes of $P(X)$ has the covering property (C).*

We now invoke the axiom of choice in the following well-known form (**2**, p. 42):

LEMMA 1 (Zorn). *If every chain C of a partly ordered set P has an upper bound u $(\geqslant C)$ in P, then P contains a maximal element m, and indeed, the set M of all such has the covering property (C).*

Remark. A chain is a non-null partly ordered set with $x \leqslant y$ or $y \leqslant x$ for all x, y. The second statement of the lemma follows at once from the first applied to the partly ordered subset of all $p_1 \geqslant p$; cf. Theorem 1.

Thus we have

COROLLARY 2 (Basis Theorem). *A weakly ordered set X has a basis if every chain in $P(X)$ has an upper bound.*

That the conditions of Lemma 1 and Corollary 2 are by no means necessary is shown by

Example 1. The set $X = \{\pm 2^i, \pm 2^i p_i; i = 1, 2, \ldots\}$, where p_i is the ith odd prime, is weakly ordered by the relation $x \leqslant y$ meaning x divides y, and $P(X)$ consists of the classes $\{\pm 2^i\}$, $\{\pm 2^i p_i\}$. Although the chain of elements $\{\pm 2^i\}$ has no upper bound in $P(X)$, the set M of maximal classes $\{\pm 2^i p_i\}$ has property (C), and X has a continuum of bases.

One can prove a partial *converse* of Lemma 1; cf. (**2**, Chapter 2, Theorem 2).

LEMMA 2. *If the set M of a partly ordered set P has property (C) and is finite, then every chain of P has an upper bound.*

Proof. To each c of a chain C, assign a maximal element $m(c) \geqslant c$. Then at least one of the finite set of values m^* of $m(c)$ is an upper bound of C. Otherwise, for each m^* there is a C-element $c(m^*)$ not $\leqslant m^*$. The finite set of such chain elements may be written $c_1 \leqslant \ldots \leqslant c_k$. Then $c_k \leqslant m(c_k) = m_0^*$, and $c(m_0^*) = c_i$ is not $\leqslant m_0^*$. But $c_i \leqslant c_k \leqslant m_0^*$, a contradiction.

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COROLLARY 3. *If a weakly ordered set X has a finite basis, then every chain C in $P(X)$ has an upper bound.*

In view of Corollary 1, and Example 1, it may be of interest to note the following "generalization" of Zorn's lemma:

LEMMA 3. *The set M of a partly ordered set P has property (C) if and only if (C'): every $p \in P$ is contained in a maximal chain (there is at least one) that has an upper bound.*

Proof. Zorn's lemma is trivially equivalent to that of Hausdorff: every chain in a partly ordered set is contained in a maximal chain (**2**, p. 42). Thus, for every $p \in P$, $\{p\}$ is a chain, and there is always a maximal chain $C \supset \{p\}$.

If (C') is true, let $p \in C \leqslant u$, C maximal. Then $u \in C$, since C is a maximal chain, and $u \in M$ for the same reason.

Conversely, if M satisfies (C), and $p \leqslant m \in M$, the chain $\{p, m\}$ is contained in a maximal chain C (Hausdorff).

Since $m \in M$, $m \geqslant C$, we have

COROLLARY 4. *A weakly ordered set has a basis if and only if its partly ordered set $P(X)$ has property (C').*

The sufficient condition required by Zorn's lemma cannot be restricted to countable chains, as shown by the following example.

Example 2. The set X of all countable subsets x of real numbers is already *partly* ordered under the relation of set-inclusion $x \subset y$. Thus the elements of the associated set $P = P(X)$ are classes of the form $\{x\}$, consisting of one element each. Every countable chain $C \subset P$ has an upper bound $\{u\} \in P$ where the countable set u is the union of all sets x involved in C. Nevertheless, P contains *no* maximal element (hence not *every* chain has an upper bound), and X has *no* basis.

Note that a "countable chain" means a chain of (at most) countably many elements c_n, and need not be of the sequential form $c_1 \leqslant c_2 \leqslant \ldots$; for example, concentric circles of rational radii. Indeed we shall cite below (cf. **3**, p. 5).

LEMMA 4. *A countable chain $C = \{c_n\}$ either contains a maximal element, or it contains a sequential subchain $c_1' < c_2' < \ldots$ which is co-final with C; i.e., for every $c_n \in C$, there is a $c_j' \geqslant c_n$.*

Proof. Suppose C contains *no* maximal element. Define $c_1' = c_1$. Assume that c_j' is already defined for a particular $j \geqslant 1$, so that $c_j \leqslant c_j'$. Then there exists a $c_n > c_j'$ and we need only define $c_{j+1}' = \max(c_n, c_{j+1})$.

Now let $G = (X, F)$ be a *directed graph*, namely a non-null set X with a function $F(x) \subset X$ on elements of X to definite subsets (null-set $\emptyset$ allowed) of X. One defines

$$F(A) = \cup \{F(a); a \in A \subset X\},$$

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and, inductively,

$$F^{n+1}(x) = F(F^n(x)), \qquad n \geqslant 1.$$

The set $R(x) = \{x\} \cup F(x) \cup F^2(x) \cup \ldots$ thus consists of x and all its "progeny." The relation $x \leqslant y$ meaning $y \in R(x)$ is a weak order on X, and a *basis B of G* is by definition a basis for the weakly ordered set $(X, \leqslant)$. The preceding results therefore apply in a way sufficiently obvious. Moreover, *every* weakly ordered set $(X, \leqslant)$ is that of a (transitive) graph $G = (X, F)$, namely that defined by $F(x) = \{y; x \leqslant y\}$, and the examples given are relevant for graph theory. In particular, Example 2 indicates that the basis theorem as stated in (**1**, Chapter 2, Theorem 1) is either false, or is intended to apply only to graphs of some restricted type. We give next a result of this sort which appears to be correct.

Let $G = (X, F)$ be called *inductive* (**1**, p. 13) if, for every sequence $\{x_i; i \geqslant 1\}$ with $x_{i+1} \in F(x_i)$, there exists a $z \geqslant x_i$, $i \geqslant 1$, $z \in X$.

COROLLARY 5. *If $G = (X, F)$ is inductive, and if the partly ordered set $P(X)$ associated with its weak order contains only countable chains, then G has a basis. In particular, this is true if X or even $P(X)$ is (at most) countable.*

Proof. By Corollary 2, it suffices to prove that every chain $C \subset P$ has an upper bound. Now C either already *contains* an upper bound, or, by Lemma 4, it contains a co-final subchain $c_1 \leqslant c_2 \leqslant \ldots$, where $c_j = [y_j]$, and hence $y_j \leqslant y_{j+1}$, i.e., $y_{j+1} \in R(y_j)$. This implies that the sequence $\{y_j\}$ has a refinement $\{x_i\}$, where $x_{i+1} \in F(x_i)$, $i \geqslant 1$. Since G is inductive, we have a $z \geqslant x_i$, $i \geqslant 1$; hence $z \geqslant y_j$, and $[z] \geqslant [y_j] = c_j$, $j \geqslant 1$. Since $\{c_j\}$ is co-final with C, it follows that $[z] \geqslant C$.

2. The chromatic number. If $G = (X, F)$ is a graph, a subset $Y \subset X$ is called *internally stable* if its elements are unrelated, i.e. $Y \cap F(Y) = \emptyset$; *externally stable* if for every $x \notin Y$, $F(x) \cap Y \neq \emptyset$; and a *kernel* if it has both properties (**1**, pp. 35, 40, 45). For example, the bases of a weakly ordered set coincide with the kernels of the associated graph with

$$F'(x) = \{y; x \leqslant y \text{ and } x \neq y\}.$$

A more interesting example arises in set theory. Suppose that Q is any collection of non-null subsets q of a set X, including all its one-element subsets $\{x\}$. Let $G^* = (Q, F^*)$ denote the graph on Q defined by

$$F^*(q) = \{q_1; q_1 \neq q, q_1 \cap q \neq \emptyset\}.$$

Concerning G^* we have

THEOREM 2. *A subset $Q_1 \subset Q$ is internally stable if and only if its distinct elements q_1 are disjoint, and externally stable if and only if its elements cover X. Hence the partitions of a set X into Q-sets are defined by the kernels of the graph G^*.*

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Proof. If Q_1 is internally stable and $q_1 \neq q$ are in Q_1, they are unrelated by F^*, and hence disjoint. The converse is equally obvious. If Q_1 is externally stable and $x \in X$, then $\{x\} \in Q$; either $\{x\} \in Q_1$ and x is covered, or $\{x\} \notin Q_1$ and, since Q_1 is externally stable, there exists a $q_1 \in F^*\{x\} \cap Q_1$, hence $q_1 \cap \{x\} \neq \emptyset$, and $x \in q_1$.

Conversely, suppose X is covered by the q_1 of Q_1. If $q \notin Q_1$, let $x \in q$. Then $x \in q_1$ for some $q_1 \in Q_1$, $q_1 \cap q \neq \emptyset$, and certainly $q_1 \neq q$. Hence $q_1 \in F^*(q)$ and Q_1 is externally stable.

If $G = (X, F)$ is a graph, a partition of X is said to be *chromatic* when every two adjacent vertices x, y ($x \neq y$, and $y \in F(x)$ or $x \in F(y)$) fall in different classes (**3**, Chapter 14). We may call the least cardinal number (**2**, p. 44) of classes in any such partition the *chromatic number* $\gamma(G)$. For a graph without loops ($x \notin F(x)$, all x) it is self-evident that a partition is chromatic if and only if its classes are internally stable subsets of G. Thus we have

COROLLARY 6. *The chromatic partitions of a graph $G = (X, F)$ without loops are defined by the kernels of the graph $G^* = (Q, F^*)$, where Q is the collection of all non-null internally stable subsets of G, and F^* is defined as above. Thus the chromatic number $\gamma(G)$ is the least cardinal number of any kernel of G^*.*

The set S of all internally stable subsets of a graph $G = (X, F)$ is partly ordered under set inclusion. Since the union of a chain of internally stable sets is internally stable, the set M of maximal elements of S has the covering property (C) by Lemma 1. Since an externally stable set obviously cannot be properly contained in an internally stable set, every kernel of G is necessarily an element of M. If G has no loops, and is symmetric ($y \in F(x) \Rightarrow x \in F(y)$), then conversely every element of M is externally stable and hence a kernel (**1**, p. 46). For such a graph we conclude that the kernels of G are simply its maximal internally stable subsets, every internally stable subset being contained in a kernel.

Since the graph G^* is symmetric without loops, we have the following Corollary, under the provisions of Theorem 2 and Corollary 6:

COROLLARY 7. *The partitions of X into Q-sets are defined by the maximal internally stable subsets of G^*. In particular, the chromatic number of a graph G without loops is the least cardinal of any maximal internally stable subset of G^*.*

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