COMPOSITIO MATHEMATICA

The l-parity conjecture for abelian varieties over function fields of characteristic $p > 0$

Fabien Trihan and Seidai Yasuda

Compositio Math. **150** (2014), 507–522

doi:10.1112/S0010437X13007501

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

Compositio Math. 150 (2014) 507–522
doi:10.1112/S0010437X13007501

The ℓ-parity conjecture for abelian varieties over function fields of characteristic $p > 0$

Fabien Trihan and Seidai Yasuda

Abstract

Let A/K be an abelian variety over a function field of characteristic $p > 0$ and let ℓ be a prime number ($\ell = p$ allowed). We prove the following: the parity of the corank r_ℓ of the ℓ-discrete Selmer group of A/K coincides with the parity of the order at $s = 1$ of the Hasse–Weil L-function of A/K. We also prove the analogous parity result for pure ℓ-adic sheaves endowed with a nice pairing and in particular for the congruence Zeta function of a projective smooth variety over a finite field. Finally, we prove that the full Birch and Swinnerton-Dyer conjecture is equivalent to the Artin–Tate conjecture.

1. Introduction

Let K be a global field and let A be an abelian variety defined over K. The conjecture of Birch and Swinnerton-Dyer (BSD) asserts that the rank of the Mordell–Weil group $A(K)$ is equal to the order of vanishing of the Hasse–Weil L-function $L(A/K, s)$ as $s = 1$ (also called the analytic rank of A/K). In the number field case, at the exception of strong numerical evidences, very little is known about this conjecture. In the function field case, it was shown in [KT03] that the conjecture is equivalent to showing the finiteness of any ℓ-primary part (ℓ a prime number) of the Tate–Shafarevich group $Ш(A/K)$.

A weaker question is to know whether these two integers have at least the same parity. But even this conjecture remains unproven. The following exact sequence of cofinitely generated $\mathbb{Z}_\ell$-modules

$$0 \to A(K) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \mathrm{Sel}_{\ell^\infty}(A/K) \to Ш(A/K)[\ell^\infty] \to 0 \tag{1}$$

leads to the following conjecture.

Conjecture 1.0.1 (ℓ-parity conjecture). Let ℓ be a prime. The corank r_ℓ of the ℓ discrete Selmer group $\mathrm{Sel}_{\ell^\infty}(A/K)$ of A/K has the same parity as the order at $s = 1$ of the Hasse–Weil L-function of A/K.

In the number field case, this conjecture is now known in several cases, in particular when A is an elliptic curve and the ground field is $K = \mathbb{Q}$ by the work of the Dokchitser brothers [DD08, DD09, DD10, DD11], Kim [Kim07, Kim09], Nekovář [Nek13, Nek01, Nek09], [Nek06, Ch. 12], Česnavičius [Ces12], Coates *et al.* [CFKS10] and others.

In [TW11], the p-parity conjecture (for p equal to the characteristic of the function field) has been proved for elliptic curves. The proof of [TW11] is purely arithmetic and based on the existence of a natural p-cyclic isogeny provided in the function field case by the relative

Received 9 October 2012, accepted in final form 13 May 2013, published online 10 March 2014.
2010 Mathematics Subject Classification 11G20, 11G40 (primary).
Keywords: elliptic curves, Selmer groups, parity conjecture.

Frobenius. In particular, the proof only works for $\ell = p$ or a prime for which the elliptic curve has a l-isogeny, or in a few other cases [TW11, Theorem 2]. Moreover, the proof requires an argument due to Ulmer to reduce to the semistable case which works only so far for elliptic curves.

One of the main results of this paper is as follows.

Theorem 1.1. *Let A/K be an abelian variety over a function field in one variable over a finite field. Then the ℓ-parity conjecture holds for any prime ℓ and any abelian variety.*

We give a totally new demonstration à la Deligne–Grothendieck, using the old ($\ell \neq p$) and comparatively newer ($\ell = p$) 'Weil 2' type arguments as well as some elementary linear algebra. The case $\ell \neq p$ and $\ell = p$ are treated separately but the arguments are very symmetric. After the proof of the hard Lefschetz theorem for projective smooth varieties over finite fields and the Sato–Tate conjecture in the function field case, this is one more surprising application of the deep theorem of Deligne.

Let us mention the following easy corollaries. First, since the analytic rank is known to be greater than or equal to r_ℓ in the function field case, our main theorem immediately implies the following corollary.

Corollary 1.0.2. *If A/K is an abelian variety with analytic rank 0 or 1, then r_ℓ is equal to the analytic rank of A.*

We are also able to deduce from our main theorem and from the short exact sequence (1) the following new piece of information on the Tate–Shafarevich group.

Corollary 1.0.3. *The corank of the ℓ-primary part of the Tate–Shafarevich group of A/K has the same parity for any prime ℓ.*

Incidentally, we also give a new (p-adic) functional equation of the Hasse–Weil L-function of A/K without Euler factors at the places of bad reduction (see Remark 4.2.1).

In the last section, we extend our main theorem to any smooth ℓ-adic sheaf F_ℓ if $\ell \neq p$ and any overconvergent F-isocrystal F_p, both pure of weight -1 and endowed with a skew-symmetric pairing $F_\ell \times F_\ell \to \mathbb{Q}_\ell(1)$ (see Theorem 5.1) and to compatible families of such objects (see Theorem 5.2). When $\ell \neq p$ we generalize our technique to higher-dimensional varieties (Theorem 5.3). Finally, we consider the case of the congruence zeta function of varieties over finite fields. In the special case of surfaces, we have the following theorem.

Theorem 1.2 (Theorem 5.5). *The Artin–Tate conjecture is equivalent to the BSD conjecture in the function field case.*

This last result is merely a remark after the precursory work of Tate, the results of [KT03, LLR05] and a final observation of Ulmer.

We finish by proving the analogue of Theorem 1.1 for projective smooth varieties over finite fields (see Corollary 5.6.2).

2. Setting and first reductions

Notation. Let A/K be an abelian variety over K a function field in one variable over the field $\mathbb{F}_q$, where q is some power of a prime p. We fix $\overline{K}$ a separable closure of K and let $C/\mathbb{F}_q$ be the proper smooth connected curve with function field K. Let $f : \mathcal{A} \to C$ be the Néron model of A/K. We denote by U the open subset of C where we have removed the places of bad reduction.

https://doi.org/10.1112/S0010437X13007501 Published online by Cambridge University Press

We recall the following facts of [KT03]. For convenience we will follow the same notation as much as possible. The only facts we need to know are summed up as follows.

By [KT03, 3.3.5], we have a short exact sequence for any l

$$0 \to \operatorname{coker}_{0,\ell} \to H^1_{\mathrm{ar},V}\{\ell\} \to \ker_{1,\ell} \to 0,$$

where $\ker_{i,\ell}/\operatorname{coker}_{i,\ell}$ is the kernel/cokernel of some map

$$\varphi_\ell - \mathbf{1} : H^i_{1,\mathbb{Q}_\ell/\mathbb{Z}_\ell} \to H^i_{2,\mathbb{Q}_\ell/\mathbb{Z}_\ell}$$

and the modules $H^*_{k,\mathbb{Q}_\ell/\mathbb{Z}_\ell}$ and $H^1_{\mathrm{ar},V}\{\ell\}$ are cofinitely generated torsion $\mathbb{Z}_\ell$-modules (see [KT03, ch. 3 and 6] for the definition of these modules).

The crucial point is that $\operatorname{corank}(H^1_{\mathrm{ar},V}\{\ell\}) = \operatorname{corank}(\mathrm{Sel}_{\ell^\infty}(A/K)) = r_\ell$ by the short exact sequence [KT03, 2.5.2]. Moreover, $\operatorname{coker}_{0,\ell}$ is a finite group for any ℓ so that $r_\ell = \operatorname{corank}(\ker_{1,\ell})$.

On the other hand, we can relate this corank with the rank of the $\mathbb{Q}_\ell$-coefficient cohomology theory as follows: recall that the abelian variety A/K induces an ℓ-adic smooth sheaf $V_\ell(\mathcal{A})$ for the ℓ-adic étale cohomology and an F-overconvergent isocrystal $R^1 f_* O^\dagger_{\mathcal{A}/U}$ for the rigid cohomology of Berthelot. We denote by $D^\dagger(A)$ the dual F-overconvergent isocrystal of $R^1 f_* O^\dagger_{\mathcal{A}/U}$. In the category ab/fab (of abelian groups modulo finite abelian groups), we have a short exact sequence

$$0 \to L_{i,\ell} \to H^i_{\mathbb{Q}_\ell} \to H^i_{k,\mathbb{Q}_\ell/\mathbb{Z}_\ell} \to 0,$$

$k = 1, 2$, where $L_{i,\ell}$ is a $\mathbb{Z}_\ell$-lattice of the finite dimensional $\mathbb{Q}_\ell$-vector space

$$H^i_{\mathbb{Q}_\ell} := H^i_{\text{ét},c}(\overline{U}, V_\ell(\mathcal{A})), \quad (\ell \neq p)$$

and

$$H^i_{\mathbb{Q}_p} := H^i_{\mathrm{rig},c}(U, D^\dagger(A)), \quad (\ell = p).$$

Note that for any ℓ, $H^i_{\mathbb{Q}_\ell}$ is endowed with a Frobenius operator also denoted φ_ℓ for simplicity. For $\ell \neq p$, φ_ℓ is induced by the geometric Frobenius, while for $\ell = p$, the operator is q^{-1} times the Frobenius operator induced by the Frobenius of $D^\dagger(A)$. The operator $\mathbf{1} - \varphi_\ell : H^i_{1,\mathbb{Q}_\ell/\mathbb{Z}_\ell} \to H^i_{2,\mathbb{Q}_\ell/\mathbb{Z}_\ell}$ is compatible with $\mathrm{id} - \varphi_\ell$ acting on $H^i_{\mathbb{Q}_\ell}$ so that

$$r_\ell = \dim_{\mathbb{Q}_\ell}(\ker(\mathrm{id} - \varphi_\ell, H^1_{\mathbb{Q}_\ell})).$$

The space $\ker(\mathrm{id} - \varphi_\ell, H^1_{\mathbb{Q}_\ell})$ is denoted $I_{2,\ell}$ in [KT03]. We also denote by $I_{3,\ell}$ (as in [KT03]) the part of $H^1_{\mathbb{Q}_\ell}$ where φ_ℓ acts unipotently. We have

$$I_{2,\ell} \subset I_{3,\ell} \subset H^1_{\mathbb{Q}_\ell},$$

with equality between $I_{2,\ell}$ and $I_{3,\ell}$ if and only if the semisimplicity conjecture holds for $(H^1_{\mathbb{Q}_\ell}, \varphi_\ell)$.

Finally note that the Hasse–Weil L-function of A/K without Euler factors at the places of bad reduction is obtained by

$$L(U, A, s+1) = \prod_{i=0}^{2} \det(1 - q^{-s}\varphi_\ell, H^i_{\mathbb{Q}_\ell})^{(-1)^{i+1}}$$

and the left-hand side of the equation is independent of the prime ℓ. Moreover, the dimension of $I_{3,\ell}$ is equal to the analytic rank r of the abelian variety, that is, the order of the zero at $s = 1$ of the Hasse–Weil L-function so that it is also independent of ℓ (see [KT03, 3.5.3]).

Therefore our main theorem, Theorem 1.1, can be reformulated as follows.

https://doi.org/10.1112/S0010437X13007501 Published online by Cambridge University Press

Theorem 2.1. *For any prime ℓ,*

$$\dim_{\mathbb{Q}_\ell} I_{2,\ell} \equiv \dim_{\mathbb{Q}_\ell} I_{3,\ell} \quad \text{modulo } 2.$$

3. Proof of Theorem 2.1 when $\ell \neq p$

3.1 We first treat the case $\ell \neq p$. Define $\mathcal{H}^i_{\mathbb{Q}_\ell}$ as $H^i_{\text{ét}}(\overline{C}, j_* V_\ell(\mathcal{A}))$, where $j : U \hookrightarrow C$ denotes the canonical inclusion. Similarly to the previous section, we also denote by $\mathcal{I}_{3,\ell}$ the part of $\mathcal{H}^1_{\mathbb{Q}_\ell}$ where φ_ℓ acts unipotently and set $\mathcal{I}_{2,\ell} := \ker(\mathrm{id} - \varphi_\ell, \mathcal{H}^1_{\mathbb{Q}_\ell})$.

3.2 Recall [Sch82] that the Hasse–Weil L-function of A/K is defined as

$$L(C, A, s) := \prod_x \det(1 - \varphi_x q^{(1-s)\deg(x)}, V_\ell(A)^{I_x})^{-1},$$

where x runs over all closed points of C, $\deg(x) = [k(x) : \mathbb{F}_q]$, φ_x is the (geometric) Frobenius at x given by sending y to $y^{q^{-\deg(x)}}$, $V_\ell(A) = (\lim_{\longleftarrow n} A(\overline{K})[\ell^n]) \otimes \mathbb{Q}_\ell$ and I_x is the inertia group at x, where we have fixed a place $\overline{x} \in \overline{K}$ above x.

Lemma 3.2.1. *We have*

$$L(C, A, s) = \prod_{i=0}^{2} \det(1 - q^{1-s}\varphi_\ell, \mathcal{H}^i_{\mathbb{Q}_\ell})^{(-1)^{i+1}}.$$

Proof. To see that, put $F = j_* V_\ell(\mathcal{A})$. By the Grothendieck–Lefschetz trace formula the right-hand side is equal to the product

$$\prod_x \det(1 - q^{(1-s)\deg(x)}\varphi_x, F_{\overline{x}})^{-1},$$

where $F_{\overline{x}}$ denotes the stalk of F at a geometric point $\overline{x}$ of C over x, so it is enough to prove that for any closed point x in C,

$$F_{\overline{x}} \simeq V_\ell(A)^{I_x}.$$

Let L be the I_x-fixed part of $\overline{K}$ and let $\overline{x}$ denote the unique place of L below the fixed place of $\overline{K}$. Let $\mathcal{O}_{L,\overline{x}} \subset L$ denote the valuation ring with respect to the valuation corresponding to $\overline{x}$. Then the ring $\mathcal{O}_{L,\overline{x}}$ is a strict henselization of the local ring $\mathcal{O}_{C,x}$ at x. We may and will assume that the geometric point $\overline{x}$ above x is equal to the closed point of $\operatorname{Spec} \mathcal{O}_{L,\overline{x}}$. We write L as a union $L = \bigcup_j L_j$ of finite subextensions of L/K. By the definition of the stalk $F_{\overline{x}}$, we have

$$F_{\overline{x}} = \left(\varprojlim_n \varinjlim_j H^0(\operatorname{Spec} L_j, A[\ell^n]) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \left(\varprojlim_n A(L)[\ell^n] \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

It follows from the definition of L that $A(L)[\ell^n]$ is equal to the I_x-fixed part of the $\mathrm{Gal}(\overline{K}/K)$-module $A(\overline{K})[\ell^n]$. Hence the claim follows. □

Lemma 3.2.2. *We have for $k = 2, 3$ and for any prime $\ell \neq p$*

$$\mathcal{I}_{k,\ell} \simeq I_{k,\ell}.$$

Proof. Let $i : C \backslash U \to C$ denote the closed immersion of the complement of U. We abbreviate $V_\ell(\mathcal{A})$ by V_ℓ. We have an exact sequence

$$0 \to j_! V_\ell \to j_* V_\ell \to i_* i^* j_* V_\ell \to 0. \tag{2}$$

https://doi.org/10.1112/S0010437X13007501 Published online by Cambridge University Press

Since the sheaf $F = i_*i^*j_*V_\ell$ is supported on $C\backslash U$, we have $H^i(\overline{C}, F) = 0$ for $i \geqslant 1$. Hence the map $H^1_c(\overline{U}, V_\ell) \to H^1(\overline{C}, j_*V_\ell)$ is surjective and its kernel is a quotient of $H^0(\overline{C}, F)$. Note that $\det(1-\varphi_\ell q^{1-s}, H^0(\overline{C}, F))^{-1}$ is equal to the product of the local L-factors of $L(A, s)$ at the places not belonging to U. By [Del80, 1.8.1], the eigenvalues of φ_ℓ on $H^0(\overline{C}, F)$ have complex absolute values q^r with $r \leqslant -1/2$. It follows that $H^0(\overline{C}, F)$ does not have a non-trivial subquotient on which φ_ℓ acts unipotently. This shows that $I_{3,\ell}$ is isomorphic to $\mathcal{I}_{3,\ell}$. Since this isomorphism is compatible with the Frobenius operator, we deduce immediately that $\mathcal{I}_{2,\ell} \simeq I_{2,\ell}$. □

Lemma 3.2.3. *For any prime $\ell \neq p$, we have a perfect pairing compatible with the Frobenius action*

$$\langle\,.\,,.\rangle_\ell : \mathcal{H}^1_{\mathbb{Q}_\ell} \times \mathcal{H}^1_{\mathbb{Q}_\ell,t} \to \mathbb{Q}_\ell,$$

where $\mathcal{H}^1_{\mathbb{Q}_\ell,t}$ is the analogue of $\mathcal{H}^1_{\mathbb{Q}_\ell}$ associated to A^t, the dual abelian variety, instead of A.

Proof. The perfectness follows from the duality and the fact that the shift $j_*V_\ell(\mathcal{A})[1]$ is the intermediate extension (in the sense of [KW01]) of the perverse sheaf $V_\ell(\mathcal{A})[1]$ on U with respect to the inclusion j. More precisely, the perfectness can be proved by combining [KW01, Corollary III.5.3, p. 149] (which says that the intermediate extension commutes with taking duals) and the statement (in which the authors give an explicit description of the intermediate extension) in [KW01, p. 153, Example]. □

3.3 Fix a polarization of A inducing an isogeny $\lambda : A \to A^t$. Then, λ induces an isomorphism $V_\ell(\mathcal{A}) \simeq V_\ell(\mathcal{A}^t)$ and therefore an isomorphism

$$\lambda : \mathcal{H}^1_{\mathbb{Q}_\ell} \cong \mathcal{H}^1_{\mathbb{Q}_\ell,t}.$$

Lemma 3.3.1. *The pairing $\langle\,.\,,.\rangle_\ell$ induces by composition with the isomorphism λ a non-degenerate symmetric pairing compatible with the Frobenius action*

$$(\,.\,,.)_\ell : \mathcal{H}^1_{\mathbb{Q}_\ell} \times \mathcal{H}^1_{\mathbb{Q}_\ell} \to \mathbb{Q}_\ell,$$

for any prime $\ell \neq p$.

Proof. The perfectness of $(\,.\,,.)_\ell$ follows immediately from the perfectness of $\langle\,.\,,.\rangle_\ell$ since $\lambda : \mathcal{H}^1_{\mathbb{Q}_\ell} \cong \mathcal{H}^1_{\mathbb{Q}_\ell,t}$ is an isomorphism. To prove that $(\,.\,,.)_\ell$ is also symmetric, it suffices to show that the homomorphism $j_*V_\ell(\mathcal{A})[1] \otimes^L j_*V_\ell(\mathcal{A})[1] \to \mathbb{Q}_\ell(1)[2]$ induced by the isomorphism $D(j_*V_\ell(\mathcal{A})[1]) \cong j_*V_\ell(\mathcal{A}^t)[1] \cong j_*V_\ell(\mathcal{A})[1]$ is symmetric, where we denote by D the dualizing functor [KW01, II, Definition 7.2]. Let us consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{End}(j_*V_\ell(\mathcal{A})) & \xrightarrow{\ \cong\ } & \mathrm{End}(V_\ell(\mathcal{A})) \\
\cong\downarrow & & \cong\downarrow \\
\mathrm{Hom}(j_*V_\ell(\mathcal{A}) \otimes^L j_*V_\ell(\mathcal{A}), \mathbb{Q}_\ell(1)) & \longrightarrow & \mathrm{Hom}(V_\ell(\mathcal{A}) \otimes^L V_\ell(\mathcal{A}), \mathbb{Q}_\ell(1))
\end{array}$$

where the horizontal arrows are given by the restrictions to $U \subset C$. Thus we only need to show that the homomorphism $V_\ell(\mathcal{A}) \otimes^L V_\ell(\mathcal{A}) \cong V_\ell(\mathcal{A}) \otimes^L V_\ell(\mathcal{A}^t) \to \mathbb{Q}_\ell(1)$ is anti-symmetric. This last statement is proved in [Mil84, 16.2]. □

3.4 The previous pairing induces by restriction a perfect pairing symmetric and compatible with the Frobenius action

$$(\,.\,,.)_\ell : \mathcal{I}_{3,\ell} \times \mathcal{I}_{3,\ell} \to \mathbb{Q}_\ell.$$

We can see φ_ℓ as an unipotent element of the orthogonal group $O((\,.\,,.)_\ell)$.

https://doi.org/10.1112/S0010437X13007501 Published online by Cambridge University Press

Lemma 3.4.1. *We have*

$$\dim_{\mathbb{Q}_\ell}(\mathcal{I}_{2,\ell}) \equiv \dim_{\mathbb{Q}_\ell}(\mathcal{I}_{3,\ell}) \quad \text{modulo } 2.$$

Proof. It is known (cf. [Nek07, Lemma 2.2.2]) that for any quadratic space (V, q) over a field L of characteristic not equal to 2 and for any element u in the orthogonal group $O(V, q)$, we have $\det(-u) = (-1)^{\dim_L \ker(1-u)}$. Applying this to $V = \mathcal{I}_{3,\ell}$ and $u = \varphi_\ell$, we obtain the claim. □

4. Proof of Theorem 2.1 when $\ell = p$

4.1 Clearly, for $\ell = p$, Lemma 3.4.1 will hold without change so that we are reduced to proving analogues of Lemmas 3.2.2, 3.2.3 and 3.3.1. We can lift C and U to smooth proper formal schemes $\pi : \mathcal{C} \to \mathrm{Spf}(W(\mathbb{F}_q))$ and an open formal subscheme $\mathcal{U} \subset \mathcal{C}$. We denote $F := \mathrm{Frac}(W(\mathbb{F}_q))$ and let $\mathcal{C}_F^{\mathrm{an}}$ be the rigid analytic space associated to $\mathcal{C}$ and denote by $\mathrm{sp} : \mathcal{C}_F^{\mathrm{an}} \to \mathcal{C}$ the specialization map. We also fix ι, an embedding of F in $\mathbb{C}$.

4.2 We recall some facts concerning the rigid cohomology with coefficients over a smooth curve. The constructions are due to Crew [Cre98]; see also [Ked06, 2.6]. We denote [Cre98, 7.1.1 and 7.2.1]

$$A_U^\dagger := \varinjlim_V \Gamma(V, O_V)$$

and for $x \in C \backslash U$,

$$A_U(x) := \varinjlim_V \Gamma(V \cap]x[, O_{\mathcal{C}_F^{\mathrm{an}}}),$$

where V is running through a cofinal set of strict neighborhood of $]U[:= \mathrm{sp}^{-1}\mathcal{U}$. We have a short split exact sequence

$$0 \to A_U^\dagger \to A_U^{\mathrm{loc}} := \bigoplus_{x \in C\backslash U} A_U(x) \to A_U^{\mathrm{qu}} \to 0 \tag{3}$$

where the algebra on the right is defined by this short exact sequence. Note that since $D^\dagger(A)$ is overconvergent over the affine curve U, it can be seen as a finite locally free $A_U^\dagger$-module endowed with a connection

$$\nabla : D^\dagger(A) \to D^\dagger(A) \otimes_{A_U^\dagger} \Omega^1_{A_U^\dagger}$$

horizontal with respect to the Frobenius operator.

For $R = A_U^\dagger$, $R' = A_U^{\mathrm{loc}}$ or A_U^{qu} and (M, ∇) a R-module with connection, we denote by $DR_{R'}(M) := [M \otimes_R R' \to M \otimes_R \Omega^1_R \otimes_R R']$, the complex concentrated in degree 0 and 1.

The short exact sequence (3) induces a short exact sequence of de Rham complexes (see [Cre98, 9.5.2])

$$0 \to DR_{A_U^\dagger}(D^\dagger(A)) \to DR_{A_U^{\mathrm{loc}}}(D^\dagger(A)) \to DR_{A_U^{\mathrm{qu}}}(D^\dagger(A)) \to 0. \tag{4}$$

We therefore deduce a distinguished triangle

$$DR_{A_U^{\mathrm{qu}}}(D^\dagger(A))[-1] \to DR_{A_U^\dagger}(D^\dagger(A)) \to DR_{A_U^{\mathrm{loc}}}(D^\dagger(A)) \to DR_{A_U^{\mathrm{qu}}}(D^\dagger(A)). \tag{5}$$

We have by [Cre98, 8.1.1 and 8.1.3]

$$H^i(DR_{A_U^\dagger}(D^\dagger(A))) \simeq H^i_{\mathrm{rig}}(U, D^\dagger(A)),$$
$$H^i(DR_{A_U^{\mathrm{qu}}}(D^\dagger(A))[-1]) \simeq H^i_{\mathrm{rig},c}(U, D^\dagger(A))$$

https://doi.org/10.1112/S0010437X13007501 Published online by Cambridge University Press

inducing therefore a long exact sequence

$$\cdots \to H^i_{\mathrm{rig},c}(U, D^\dagger(A)) \to H^i_{\mathrm{rig}}(U, D^\dagger(A)) \to H^i(DR_{A_U^{\mathrm{loc}}}(D(A)^\dagger)) \to \cdots . \tag{6}$$

Remark 4.2.1. We can obtain the following p-adic functional equation for the Hasse–Weil L-function of A/K without Euler factors outside U:

$$\prod_{i=0}^{2} \det(-q^s \varphi_p^{-1})^{(-1)^i} L(U, A^t, 1-s) = L(U, A, s+1) \cdot \prod_{i=0}^{1} \det(1 - q^{-s}\varphi_p, H^i(DR_{A_U^{\mathrm{loc}}}(D^\dagger(A))))^{(-1)^{i+1}}.$$

In order to obtain this formula, recall that

$$L(U, A, s+1) = \prod_{i=0}^{2} P_{c,i}(A, t)^{(-1)^{i+1}}, \tag{7}$$

where $t = q^{-s}$ and $P_{c,i}(A, t) = \det(1 - t\varphi_p; H^i_{\mathrm{rig},c}(U, D^\dagger(A)))$. Hence we have

$$L(U, A^t, 1-s) = \prod_{i=0}^{2} P_{c,i}(A^t, t^{-1})^{(-1)^{i+1}}. \tag{8}$$

We define the action of φ_p on $H^i_{\mathrm{rig}}(U, D^\dagger(A))$ in such a way that the pairing

$$H^i_{\mathrm{rig}}(U, D^\dagger(A)) \times H^{2-i}_{\mathrm{rig},c}(U, D^\dagger(A^t)) \to F$$

is φ_p-invariant. It then follows that the natural homomorphism of the long exact sequence (6) from $H^i_{\mathrm{rig},c}$ to H^i_{rig} is compatible with the action of φ_p. Since the operator $1 - t^{-1}\varphi_p$ on $H^{2-i}_{\mathrm{rig},c}$ is adjoint to the operator $1 - t^{-1}\varphi_p^{-1}$ on H^i_{rig} with respect to this pairing, we have

$$\begin{aligned} P_{c,2-i}(A^t, t^{-1}) &= \det(1 - t^{-1}\varphi_p^{-1}; H^i_{\mathrm{rig}}(U, D^\dagger(A))) \\ &= \det(-t^{-1}\varphi_p^{-1}; H^i_{\mathrm{rig}}(U, D^\dagger(A))) \det(1 - t\varphi_p; H^i_{\mathrm{rig}}(U, D^\dagger(A))) \\ &= \det(-t^{-1}\varphi_p^{-1}; H^i_{\mathrm{rig}}(U, D^\dagger(A))) \cdot P_i(A, t), \end{aligned}$$

with $P_i(A, t) = \det(1 - t\varphi_p; H^i_{\mathrm{rig}}(U, D^\dagger(A)))$. On the other hand, taking the alternating product of the characteristic polynomial of φ_p on each of de Rham complexes of the distinguished triangle (5) and using the Poincaré duality as well as [EL93, Main Theorem, II], we deduce

$$\prod_{i=0}^{2} P_i(A, t)^{(-1)^{i+1}} = \prod_{i=0}^{2} P_{i,c}(A, t)^{(-1)^{i+1}} \cdot \prod_{i=0}^{1} \det(1 - t\varphi_p, H^i(DR_{A_U^{\mathrm{loc}}}(D^\dagger(A))))^{(-1)^{i+1}},$$

which we can rewrite, using the formulas (7) and (8), as

$$\prod_{i=0}^{2} \det(-q^s \varphi_p^{-1})^{(-1)^i} L(U, A^t, 1-s) = L(U, A, s+1) \cdot \prod_{i=0}^{1} \det(1 - q^{-s}\varphi_p, H^i(DR_{A_U^{\mathrm{loc}}}(D^\dagger(A))))^{(-1)^{i+1}},$$

which is nothing other than the desired formula. This fact will not be used in our proof.

https://doi.org/10.1112/S0010437X13007501 Published online by Cambridge University Press

Setting $\mathcal{H}^1_{\mathbb{Q}_p} := \mathrm{Im}(H^1(DR_{A^{\mathrm{qu}}_U}(D^\dagger(A))[-1])) \to H^1(DR_{A^\dagger_U}(D^\dagger(A)))$, we deduce a surjective map $H^1_{\mathbb{Q}_p} \to \mathcal{H}^1_{\mathbb{Q}_p}$, whose kernel is a quotient of $H^0(DR_{A^{\mathrm{loc}}_U}(D^\dagger(A)))$.

Lemma 4.2.2. *The dual F-overconvergent isocrystal of $D^\dagger(A)$ is $D^\dagger(A^t)$.*

Proof. Observe that the assertion holds at the level of Dieudonné crystals by [BBM82, 5.1]. In particular it also holds at the level of the associated convergent F-isocrystals, via the natural functor

$$\{F-\text{crystals over } U\} \to \{F-\text{convergent isocrystal over } U\}.$$

Finally, using the fully faithfulness of the functor

$$\mathrm{res} : \{F-\text{overconvergent isocrystals over } U\} \to \{F-\text{convergent isocrystal over } U\}$$

(see [Ked04]), we deduce that the assertion holds also at the level of overconvergent F-isocrystals. □

Recall [Ked06] that an overconvergent F-isocrystal $E^\dagger$ is ι-pure of weight w, if for any $x \in U$ and for any eigenvalue α of φ_p acting on the fiber at x of $E^\dagger$, we have $|\iota(\alpha)| = q^{w/2}$. Recall also that a finite dimensional F-vector space V endowed with a linear operator f is called ι-mixed of weight less than w if for all eigenvalues α of f, we have $|\iota(\alpha)| = q^{(w+i)/2}$ for some integer $i = i(\alpha) \leqslant 0$.

Lemma 4.2.3. *We have $(D^\dagger(A), \varphi_p)$ is ι-pure of weight -1.*

Proof. By the previous lemma, we know that $D^\dagger(A)$ is equal to $R^1 f^t_* O^\dagger_{\mathcal{A}^t/U}$ so the fiber at any x in U is $H^1_{\mathrm{crys}}(\mathcal{A}^t_x/W(k(x)))[1/p]$ which is pure of weight 1 [KM74] since the fiber $\mathcal{A}^t_x$ of $\mathcal{A}^t$ at x is an abelian variety over $k(x)$. Since φ_p is q^{-1} times the Frobenius operator, we conclude that the absolute value of the eigenvalues is $q^{-1+1/2} = q^{-1/2}$ and the assertion is proved. □

Lemma 4.2.4. *We have for $k = 2, 3$*

$$\mathcal{I}_{k,p} \simeq I_{k,p}.$$

Proof. We can reason as in the case $\ell \neq p$. We are therefore reduced to showing that 1 is not an eigenvalue of $(H^0(DR_{A^{\mathrm{loc}}_U}(D^\dagger(A))), \varphi_p)$. By the previous lemma and [Ked06, 6.4.4], we see that $(H^0(DR_{A^{\mathrm{loc}}_U}(D^\dagger(A))), \varphi_p)$ is ι-mixed of weight $\leqslant -1$, so the assertion is clear. □

Lemma 4.2.5. *We have a perfect pairing compatible with the Frobenius action*

$$\langle .\, , . \rangle_p : \mathcal{H}^1_{\mathbb{Q}_p} \times \mathcal{H}^1_{\mathbb{Q}_p,t} \to \mathbb{Q}_p,$$

where $\mathcal{H}^1_{\mathbb{Q}_p,t}$ is the analogue of $\mathcal{H}^1_{\mathbb{Q}_p}$ associated to A^t, the dual abelian variety, instead of A.

Proof. Note that $D^\dagger(A)$ is quasi-unipotent [MT04] and therefore 'strict' in the sense of Crew [Cre98, 10.2]. Thus the assertion follows from [Cre98, 9.5] and Lemma 4.2.2. □

4.3 As before, the isogeny $\lambda : A \to A^t$ induces an isomorphism

$$\lambda : \mathcal{H}^1_{\mathbb{Q}_p} \cong \mathcal{H}^1_{\mathbb{Q}_p,t}.$$

Lemma 4.3.1. *The pairing $\langle .\, , . \rangle_p$ induces by composition with the isomorphism λ a non-degenerate symmetric pairing compatible with the Frobenius action*

$$(.\, , .)_p : \mathcal{H}^1_{\mathbb{Q}_p} \times \mathcal{H}^1_{\mathbb{Q}_p} \to \mathbb{Q}_p.$$

Proof. The perfectness of $(.\,,.)_p$ follows immediately from the perfectness of $\langle .\,,.\rangle_p$. To prove that $(.\,,.)_p$ is also symmetric, it suffices to look at the level of Dieudonné crystals or even at the level of p-divisible groups. By [Oda69, Proposition 1.12], the polarization λ induces a skew-symmetric map of p-divisible groups $\lambda : \mathcal{A}|_U[p^\infty] \to \mathcal{A}^t|_U[p^\infty]$, in the sense that the Cartier dual λ^t of this map coincides with $-\lambda$. Applying the covariant Dieudonné functor we deduce a map of Dieudonné crystals $D(\lambda) : D(\mathcal{A}|_U) \to D(\mathcal{A}^t|_U)$. By [BBM82, 5.1], we have $D(\lambda)^\vee = D(\lambda^t) = -D(\lambda)$, where we write $.^\vee$ for the dual F-crystal. Now the map λ also induces a map $D^\dagger(\lambda) : D^\dagger(A) \to D^\dagger(A^t)$ such that $\mathrm{res}(D^\dagger(\lambda)) = D(\lambda)$. By fully faithfulness of res we deduce that the map $D^\dagger(\lambda)$ is such that $D^\dagger(\lambda)^\vee = -D^\dagger(\lambda)$. The rest of the proof is formal. □

4.4 Again, the previous pairing induces by restriction a perfect pairing symmetric and compatible with the Frobenius action

$$(.\,,.)_p : \mathcal{I}_{3,p} \times \mathcal{I}_{3,p} \to \mathbb{Q}_p.$$

Finally, the Theorem 2.1 in the case $\ell = p$ is deduced as in the case $\ell \neq p$ from the p-adic analogue of Lemma 3.4.1.

5. Extensions of the method

5.1 Let ℓ be a prime, F_ℓ a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on U when $\ell \neq p$ and an overconvergent F-isocrystal on U when $\ell = p$. We denote $H^i_{\mathbb{Q}_\ell}(F_\ell) = H^i_{\text{ét},c}(\overline{U}, F_\ell)$ when $\ell \neq p$ and $H^i_{\mathbb{Q}_p}(F_p) := H^i_{\mathrm{rig},c}(U, F_p)$.

Similarly to the previous section, we also use $I_{3,\ell,i}(F_\ell)$ for the part of $H^i_{\mathbb{Q}_\ell}(F_\ell)$ where φ_ℓ acts unipotently and set $I_{2,\ell,i}(F_\ell) = \ker(\mathrm{id} - \varphi_\ell, H^i_{\mathbb{Q}_\ell}(F_\ell))$. Let $r_{\mathrm{an}}(F_\ell)$ denote the analytic rank of F_ℓ, defined as the order of the zero of

$$L(U, F_\ell, s) := \prod_{x \in U} \det(1 - \mathrm{Frob}_x\, q^{-s \cdot \deg(x)}; F_{\ell,x})^{-1}$$

(where $F_{\ell,x}$ is the fiber at a fixed geometric point above x when $\ell \neq p$ and the usual fiber at x when $\ell = p$) at $s = 0$.

5.2 We define for any ℓ the Selmer complex $\mathrm{Sel}(F_\ell)$ associated to F_ℓ by the following distinguished triangles:

$$\mathrm{Sel}(F_\ell) \to \mathbb{R}\Gamma_{\text{ét},c}(\overline{U}, F_\ell) \stackrel{1-\varphi_\ell}{\to} \mathbb{R}\Gamma_{\text{ét},c}(\overline{U}, F_\ell) \to \mathrm{Sel}(F_\ell)[1] \tag{9}$$

and

$$\mathrm{Sel}(F_p) \to \mathbb{R}\Gamma_{\mathrm{rig},c}(U, F_p) \stackrel{1-\varphi_p}{\to} \mathbb{R}\Gamma_{\mathrm{rig},c}(U, F_p) \to \mathrm{Sel}(F_p)[1] \tag{10}$$

and set $r(F_\ell) := \sum_i (-1)^i i.\dim_{\mathbb{Q}_\ell} H^i(\mathrm{Sel}(F_\ell))$.

Remark 5.2.1. With the present definition, the Selmer complex is only defined up to isomorphisms. It is however possible to define this complex up to a canonical isomorphism. When $\ell = p$ the construction of such a complex is easy, using de Rham complexes which compute $\mathbb{R}\Gamma_{\mathrm{rig},c}(U, F_p)$. When $\ell \neq p$, this can also be done by using a Godement resolution of $j_! F_{\ell,n}$ for each n. Here $(F_{\ell,n})_n$ is a system of smooth $\mathcal{O}_E/\ell^n$-sheaves on U which represents F_ℓ, where $\mathcal{O}_E$ is the ring of integers of a finite extension $E \subset \overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

By definition, we have

$$r_{\mathrm{an}}(F_\ell) = \sum_i (-1)^{i+1} \dim_{\mathbb{Q}_\ell}(I_{3,\ell,i}(F_\ell))$$

and conjecturally (if we assume the semisimplicity of the Frobenius acting on $H^i_{\mathbb{Q}_\ell}(F_\ell)$)

$$r_{\mathrm{an}}(F_\ell) = \sum_i (-1)^{i+1} \dim_{\mathbb{Q}_\ell}(I_{2,\ell,i}(F_\ell)) = r(F_\ell),$$

where the second equality is deduced from the short exact sequences

$$0 \to \mathrm{coker}(1-\varphi_{\ell,i-1}) \to H^i(\mathrm{Sel}(F_\ell)) \to \ker(1-\varphi_{\ell,i}) \to 0,$$

where $\varphi_{\ell,i}$ denotes φ_ℓ acting on $I_{3,\ell,i}(F_\ell)$.

5.3 We assume that the sheaf F_ℓ is pure of weight -1. In this case, $1-\varphi_\ell$ is an isomorphism on $H^i_{\mathbb{Q}_\ell}(F_\ell)$ for $i \neq 1$. We say that $\mathcal{H}^1_{\mathbb{Q}_\ell}(F_\ell) = H^1_{\text{ét}}(\overline{C}, j_* F_\ell)$ when $\ell \neq p$ and $\mathcal{H}^1_{\mathbb{Q}_p}(F_p) = \mathrm{Im}(H^1_{\mathrm{rig},c}(U, F_p) \to H^1_{\mathrm{rig}}(U, F_p))$. For $k = 2, 3$, we write

$$I_{k,\ell}(F_\ell) = I_{k,\ell,1}(F_\ell).$$

We set $\mathcal{I}_{2,\ell}(F_\ell) = \ker(\mathrm{id} - \varphi_\ell, \mathcal{H}^1_{\mathbb{Q}_\ell})$ and write $\mathcal{I}_{3,\ell}(F_\ell)$ for the part of $\mathcal{H}^1_{\mathbb{Q}_\ell}(F_\ell)$ where φ_ℓ acts unipotently.

Lemma 5.3.1. *Assume that F_ℓ is pure of weight -1 (ℓ any prime). Then we have for $k = 2, 3$*

$$\mathcal{I}_{k,\ell}(F_\ell) \simeq I_{k,\ell}(F_\ell).$$

Proof. The proof is completely analogous to the proof of Lemmas 3.2.2 and 4.2.4. □

Lemma 5.3.2. *Let F_ℓ be as above, pure of weight -1.*

(i) *For any prime $\ell \neq p$, we have* $\mathrm{Sel}(F_\ell) = \mathbb{R}\Gamma_{\text{ét},c}(U, F_\ell)$.

(ii) *For any prime ℓ, we have* $\dim_{\mathbb{Q}_\ell}(\mathcal{I}_{2,\ell}(F_\ell)) = r(F_\ell)$.

Proof. We deduce from the Hochschild–Serre spectral sequence for ℓ-adic cohomology the long exact sequence

$$\cdots \to H^i_{\text{ét},c}(U, F_\ell) \to H^i_{\text{ét},c}(\overline{U}, F_\ell) \stackrel{1-\varphi_\ell}{\to} H^i_{\text{ét},c}(\overline{U}, F_\ell) \to \cdots$$

and so the first claim is clear. The second results from Lemma 5.3.1 and the fact that $1-\varphi_\ell$ is an isomorphism on $H^i_{\mathbb{Q}_\ell}(F_\ell)$ for $i \neq 1$. □

We deduce, as in the previous two sections, the following theorem.

Theorem 5.1. *Assume that F_ℓ is pure of weight -1 and equipped with a skew-symmetric non-degenerate pairing $F_\ell \times F_\ell \to \overline{\mathbb{Q}}_\ell(1)$. Then*

$$r_{\mathrm{an}}(F_\ell) \cong r(F_\ell) \quad \text{modulo } 2.$$

Remark 5.3.3. (i) If we are interested in the behavior of the Hasse–Weil L-function of F_ℓ at $s = r$, then by replacing F_ℓ with the twist $F_\ell(r)$, defined as the twist of F_ℓ by the unramified character which sends the geometric Frobenius to q^{-r} when $\ell \neq p$ and as $(F_p, \varphi_p(r))$ when $\ell = p$, we may assume that $r = 0$. Then it follows from the weight argument that the Hasse–Weil L-function does not have a zero at $s = 0$ unless F is of weight -1 (with respect to a fixed isomorphism between $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$).

(ii) More generally, one can ask when $r_{\mathrm{an}}(F_\ell)$ will be equal to $r(F_\ell)$. This should hold for $\ell \neq p$, when F_ℓ is a semi-simple object in the category of smooth $\overline{\mathbb{Q}}_\ell$-sheaves on U. It follows from the global Langlands conjecture (Lafforgue's theorem) for GL_n over K [Laf02] that any irreducible smooth $\overline{\mathbb{Q}}_\ell$-sheaf is an unramified twist of a pure $\overline{\mathbb{Q}}_\ell$-sheaf. Hence in order to verify the expectation for all F_ℓ as above, it suffices to verify it for all F_ℓ that are irreducible and pure.

The weight argument shows that, to verify the expectation for a fixed F_ℓ, it suffices to prove that φ_ℓ acts semi-simply on $H^1(\overline{C}, j_*F_\ell)$. Note also that $H^1(\overline{C}, j_*F_\ell)$ behaves better than $H^1(\overline{U}, F_\ell)$ and $H^1_c(\overline{U}, F_\ell)$ in the following senses: first, if F_ℓ is pure of weight w then $H^1(\overline{C}, j_*F_\ell)$ is pure of weight $w+1$. Moreover, the group $H^1(\overline{C}, j_*F_\ell)$ does not depend on the choice of U, that is, the group for F_ℓ is isomorphic to the group for the restriction of F_ℓ to any non-empty open subscheme of U.

We can also consider families of ℓ-adic sheaves, where the prime ℓ varies. Following Serre, we make the following definition.

DEFINITION 5.3.4. Let E be a finite extension of $\mathbb{Q}$ and let S be a set of pairs (ℓ, ι) of a prime number ℓ and an embedding $\iota : E \to \overline{\mathbb{Q}}_\ell$. A strict compatible (E, S)-family of ℓ-adic sheaves (ℓ any prime) corresponds to the data $(F_{\ell,\iota})_{(\ell,\iota)\in S}$ of smooth $\overline{\mathbb{Q}}_\ell$-sheaves $F_{\ell,\iota}$ on U ($\ell \neq p$) and of an overconvergent F-isocrystal $(F_{p,\iota}, \varphi_p)$ over $U/\overline{\mathbb{Q}}_p$ such that for any $x \in U$, $\det(1 - \mathrm{Frob}_x\, q^{-s\cdot\deg(x)}; F_{\ell,\iota,x}) = \iota(P_x)$ (where $F_{\ell,\iota,x}$ is defined as above) for some $P_x \in E[q^{-s}]$ which is independent of $(\ell, \iota) \in S$.

Remark 5.3.5. For a finite extension E of $\mathbb{Q}$ let S^p_E denote the set of pairs (ℓ, ι) of a prime number $\ell \neq p$ and an embedding $\iota : E \to \overline{\mathbb{Q}}_\ell$. Let $(\ell_0, \iota_0) \in S^p_E$ and let F_{ℓ_0,ι_0} be an irreducible smooth $\overline{\mathbb{Q}}_{\ell_0}$-sheaf on U whose determinant character takes values in $\iota_0(E)^\times$. Then it follows from Lafforgue's theorem that there exist a finite extension E' of E and an embedding $\iota'_0 : E' \to \overline{\mathbb{Q}}_\ell$ extending ι_0 such that for any $x \in U$, $\det(1 - \mathrm{Frob}_x\, q^{-s\cdot\deg(x)}; F_{\ell_0,\iota_0,x}) \in \iota(E')[q^{-s}]$, and moreover that F_{ℓ_0,ι_0} is uniquely extended to a strict compatible $(E', S^p_{E'})$-family of ℓ-adic sheaves $(F'_{\ell,\iota})$ with $F'_{\ell_0,\iota'_0} = F_{\ell_0,\iota_0}$.

THEOREM 5.2. *Let $F = (F_\ell)_\ell$ be a strict compatible system of ℓ-adic sheaves pure of weight -1. Then $r_{\mathrm{an}}(F_\ell)$ is independent of ℓ and we denote it $r_{\mathrm{an}}(F)$. Assume moreover that all ℓ-adic sheaves are endowed with a skew-symmetric non-degenerate pairing $F_\ell \times F_\ell \to \overline{\mathbb{Q}}_\ell(1)$. Then*

$$r_{\mathrm{an}}(F) \cong r(F_\ell) \quad \text{modulo } 2.$$

Proof. By hypothesis, the L-functions of F_ℓ for varying primes ℓ have the same local factors so that $L(U, F_\ell, s) = L(U, F_{\ell'}, s)$ for any distinct primes ℓ and ℓ'. The first assertion follows. The second assertion results from Theorem 5.1. □

5.4 Obviously Theorem 5.1 is a generalisation of our main theorem. Here is another example where this theorem applies. We assume $\ell \neq p$ (but we have no doubt that we can construct a similar example in the case $\ell = p$). Let $f : X \to U$ be a smooth projective morphism of pure relative dimension d. A fixed embedding $X \to \mathbb{P}^n_U$ gives a Lefschetz operator $N : R^i f_*\overline{\mathbb{Q}}_\ell \to R^{i+2} f_*\overline{\mathbb{Q}}_\ell(1)$ for each i. Let m be an odd integer with $1 \leqslant m \leqslant d$. We let F_ℓ denote the primitive part of $R^m f_*\overline{\mathbb{Q}}_\ell((m+1)/2)$, that is, F_ℓ is the kernel of the $d-m+1$-fold iteration

$$N^{d-m+1} : R^m f_*\overline{\mathbb{Q}}_\ell((m+1)/2) \to R^{2d-m+2} f_*\overline{\mathbb{Q}}_\ell(d-(m-3)/2)$$

of the Lefschetz operators. Then F_ℓ is a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on U which is pure of weight -1. It follows from the hard Lefschetz theorem that the cup product

$$- \cup - : R^m f_*\overline{\mathbb{Q}}_\ell((m+1)/2) \times R^{2d-m} f_*\overline{\mathbb{Q}}_\ell(d-(m-1)/2) \to R^{2d} f_*\overline{\mathbb{Q}}_\ell(d+1) \cong \overline{\mathbb{Q}}_\ell(1)$$

induces a skew-symmetric non-degenerate pairing $F_\ell \times F_\ell \to \overline{\mathbb{Q}}_\ell(1)$ which sends a local section (x, y) of $F_\ell \times F_\ell$ to $x \cup N^{d-m} y$.

5.5 Our argument can be also proved useful for higher-dimensional varieties, but in this case, and since we have to deal with more φ_ℓ-$\mathbb{Q}_\ell$-vector spaces, we need to work with the pure φ_ℓ-$\mathbb{Q}_\ell$-vector spaces $\mathcal{H}^i_{\mathbb{Q}_\ell}$ rather than with the mixed φ_ℓ-$\mathbb{Q}_\ell$-vector spaces $H^i_{\mathbb{Q}_\ell}$. In the rest of this section we will restrict to the case $\ell \neq p$. The same reasoning applies to the case $l = p$ by using the intermediate extensions as defined in the recent work of [AC13].

Let X be a proper and smooth variety of dimension d over a finite field $\mathbb{F}_q$ and let U be an open dense subscheme of X. Let F be a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on U pure of odd (respectively even) weight w. Let G denote the intermediate extension of F to X. Then G is a pure perverse sheaf on X. We define the completed L-function $L(X, F, s)$ to be the Euler product

$$\prod_x \det(1 - q_x^{-s} \operatorname{Frob}_x, G_{\overline{x}})^{-1}$$

where x runs over the closed points of X. For $i \geqslant 0$ we put $\mathcal{H}^i = H^i_{\text{ét}}(\overline{X}, G)$. By the Grothendieck Lefschetz trace formula we have $L(X, F, s) = \prod_{i=0}^{2d} \det(1 - \varphi_\ell q^{-s}; \mathcal{H}^i)^{(-1)^{i+1}}$. Here φ_ℓ denotes the geometric Frobenius. Since each $\mathcal{H}^i$ is pure of weight $i + w$ [KW01, I, Corollary 7.3], the function may admit a zero (respectively a pole) at the integer $s = n := (2k+1+w)/2$, for $k = 0, \ldots, [d/2]-1$ (respectively $s = n = (2k + w)/2$ for $k = 0, \ldots, d$). The order $r_{\text{an}}(F, n)$ of the zero (respectively pole) of $L(X, F, s)$ at $s = n$ is equal to the multiplicity of the eigenvalue 1 for the action of φ_ℓ on $\mathcal{H}^{2n-w}(n)$.

We set

$$\operatorname{Sel}(F, n) := H^{2n-w}_{\text{ét}}(X, G(n))$$

and $r(F, n) := \dim_{\mathbb{Q}_\ell}(\operatorname{Sel}(F, n))$.

Theorem 5.3. *Let F be as above. Assume that d and w are odd (respectively even) and that F is endowed with a skew-symmetric (respectively symmetric) non-degenerate pairing $(\,,) : F \otimes F \to \overline{\mathbb{Q}}_\ell(-w)$. We have*

$$r_{\text{an}}(F, (d + w)/2) \cong_r (F, (d + w)/2) \quad \text{modulo } 2.$$

Proof. Let $\mathcal{I}_{2,\ell}$ denote the kernel of $1 - \varphi_\ell$ on $\mathcal{H}^d((d + w)/2)$ and let $\mathcal{I}_{3,\ell}$ denote the part of $\mathcal{H}^d((d + w)/2)$ on which φ_ℓ acts unipotently. Then we have $\mathcal{I}_{2,\ell} \cong \operatorname{Sel}(F)$ and $\dim_{\overline{\mathbb{Q}}_\ell} \mathcal{I}_{3,\ell}$ is equal to the order of pole or zero of $L(X, F, s)$ at $s = (d + w)/2$. Thus it suffices to prove that $\dim \mathcal{I}_{2,\ell}$ and $\dim \mathcal{I}_{3,\ell}$ have the same parity. Since the intermediate extension commutes with taking duals, the pairing $F_\ell \times F_\ell \to \overline{\mathbb{Q}}_\ell(-w)$ extends to a non-degenerate pairing $G \otimes^{\mathbb{L}} G \to \overline{\mathbb{Q}}_\ell(-w)$ which is symmetric (respectively skew-symmetric) if d is even (respectively odd). The cup product $\mathcal{H}^d((d + w)/2) \otimes \mathcal{H}^d((d + w)/2) \to H^{2d}_{\text{ét}}(\overline{X}, G \otimes^{\mathbb{L}} G(d + w))$ combined with this pairing induces a symmetric pairing $\mathcal{H}^d((d + w)/2) \times \mathcal{H}^d((d + w)/2) \to H^{2d}_{\text{ét}}(\overline{X}, \overline{\mathbb{Q}}_\ell(d)) \cong \overline{\mathbb{Q}}_\ell$ which is invariant under the action of φ_ℓ. Therefore the argument in § 3 shows that $\dim \mathcal{I}_{2,\ell}$ and $\dim \mathcal{I}_{3,\ell}$ have the same parity. □

5.6 We turn now our attention to the congruence Zeta function of a projective smooth variety V over a finite field $\mathbb{F}_q$ of pure dimension d. Let ℓ be a prime distinct from p. By the Grothendieck–Lefschetz trace formula, the Zeta function of V is a rational function in the variable q^{-s}. More precisely, we have

$$Z(V, s) = \prod_{i=0}^{2d} P_i(q^{-s})^{(-1)^{i+1}},$$

with $P_i(t) = \det(1 - t\varphi_\ell, H^i_{\text{ét}}(\bar{V}, \mathbb{Q}_\ell))$.

The Riemann hypothesis asserts that all the zeroes are on the lines $\mathrm{Re}(s) = 1/2, 3/2, \ldots, (2d-1)/2$ and the poles are on the lines $\mathrm{Re}(s) = 0, 1, 2, \ldots, d$. Let $r(V, n) = r(n)$ denote the order of the pole of $Z(V, s)$ at $s = n$, for $n = 0, \ldots, d$.

The Artin–Tate conjecture [Tat94] predicts the following result.

Conjecture 5.6.1 ($T(n)$). We have:

(i) $r(n) = \dim_{\mathbb{Q}_\ell} H^{2n}_{\text{ét}}(V, \mathbb{Q}_\ell(n))$;

(ii) $r(n)$ is equal to the rank of the numerical equivalence classes of cycles of codimension n.

This conjecture is related to some arithmetic invariants of the variety in a very similar way to the case of abelian varieties. Consider the short exact sequence

$$0 \to \mathbb{Z}_\ell \otimes \mathrm{Pic}(V) \to H^2_{\text{ét}}(V, \mathbb{Z}_\ell(1)) \to \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathrm{Br}(V)) \to 0. \tag{11}$$

Theorem 5.4 [Tat94, 4.3]. *Let V be a projective smooth variety over $\mathbb{F}_q$. The following assertions are equivalent:*

(1) *$T(1)$ holds;*

(2) *for all primes l (respectively for one prime l), the ℓ-primary part of $\mathrm{Br}(V)$ is finite;*

(3) *for all primes l (respectively for one prime l), the map $\mathbb{Z}_\ell \otimes \mathrm{Pic}(V) \to H^2_{\text{ét}}(V, \mathbb{Z}_\ell(1))$ of the short exact sequence (11) is an isomorphism;*

(4) *$r(1) = \mathrm{rank}(\mathrm{Pic}(V))$.*

In fact it is possible to deduce the full BSD conjecture from the Artin–Tate conjecture.

Theorem 5.5. *The conjecture $T(1)$ for projective smooth surfaces over finite fields is equivalent to the BSD conjecture for abelian varieties over function fields in one variable over finite fields of characteristic $p > 0$.*

Proof. Clearly, if the BSD conjecture is true for abelian varieties, it is true for Jacobians and therefore by [Gro68] the Brauer group is finite for any smooth projective surface over $\mathbb{F}_q$. Hence the conjecture $T(1)$ holds for smooth projective surfaces. Conversely, let $C/\mathbb{F}_q$ be a projective smooth connected curve with function field K. Assume that the conjecture $T(1)$ for surfaces over finite fields is known. Let $X/\mathbb{F}_q$ be a smooth proper and geometrically connected surface endowed with a proper flat map $f : X \to C$ such that the generic fiber X_K/K is a proper smooth geometrically connected curve of genus g. As is observed by Saper (see [LLR05, Theorem 2]), the main result of [KT03] implies that the BSD conjecture is true for the Jacobian of X_K/K. Now take any abelian variety over a function field that is a Jacobian of a curve X_K/K. We can find a proper flat model X/C ([Liu06, 10.1, Remark 1.8]). Thanks to [Liu06, 10.1 Remark 1.9], we can moreover assume that X is regular. Since $\mathbb{F}_q$ is perfect, it follows that X is smooth over $\mathbb{F}_q$. Now, by the classical theorem of Zariski, which says that any proper regular surface over an algebraically closed field is projective (see [Har77, II, Remark 4.10.2]), we conclude that a proper regular surface over a perfect field is projective [EGAII, Corollaire (6.6.5)]. Therefore our model $X/\mathbb{F}_q$ is a projective smooth surface. Assuming the Tate conjecture for surfaces, we deduce the BSD conjecture for Jacobians. The last input is due to Ulmer [Ulm12]. Since his argument is short we reproduce it integrally. Observe that given an abelian variety A over K, there is another abelian variety A_0 over K and a Jacobian J over K with an isogeny $J \to A \times A_0$. If BSD holds for Jacobians, then it also holds for $A \oplus A_0$. But since we have an inequality rank $\leqslant$ ord for abelian varieties over function fields, equality for the direct sum implies equality for the factors. Thus BSD (which is equivalent to the rank conjecture in the function field case) holds for A as well. $\square$

In the direction of these conjectures, our method leads to the following parity result.

Corollary 5.6.2. *Suppose that V is projective and smooth of pure dimension d over a finite field of characteristic p and let $\ell \neq p$. Then for each integer i with $0 \leqslant i \leqslant d$, we have* $r(i) \cong \dim_{\mathbb{Q}_\ell} H^{2i}_{\text{ét}}(V, \mathbb{Q}_\ell(i))$ modulo 2.

Proof. By Poincaré duality we may assume that $0 \leqslant i \leqslant d/2$. Fix an embedding of V to a projective space and for each integer j with $0 \leqslant j \leqslant d$ let P^j denote the primitive part of $H^j_{\text{ét}}(\bar{V}, \mathbb{Q}_\ell)$. By the hard Lefschetz theorem, we have a direct sum decomposition $H^{2i}_{\text{ét}}(\bar{V}, \mathbb{Q}_\ell(i)) \cong P^{2i}(i) \oplus P^{2i-2}(i-1) \oplus \cdots \oplus P^0$. Since we have a non-degenerate symmetric pairing $P^{2j}(j) \times P^{2j}(j) \to \mathbb{Q}_\ell$ on each j, the claim follows from Theorem 5.3. □

Remark 5.6.3. (i) Using the log de Rham–Witt cohomology of Illusie (see for example [Mil84]) instead of the group $H^{2i}_{\text{ét}}(V, \mathbb{Q}_\ell(i))$, we also get a p-parity statement. Note that for $i = 1$, the crystalline analogue of $H^2_{\text{ét}}(V, \mathbb{Q}_\ell(1))$ is just $H^2_{fl}(V, \mathbb{Q}_p(1)) := \lim_{\longleftarrow n} H^2_{fl}(X, \mu_{p^n}) \otimes \mathbb{Q}_p$ (see [Mil84, p. 309]).

(ii) This corollary covers in particular the case of a surface and should be considered in this case as equivalent to our Theorem 1.1 for Jacobians. To see that, consider a projective smooth surface $X/\mathbb{F}_q$ equipped with a flat morphism $f : X \to C$ whose geometric fiber is a smooth projective geometrically connected curve X_K/K and denote by J_X/K its Jacobian. Let l be a prime number and denote $H^2(X, \mathbb{Q}_\ell(1)) = H^2_{\text{ét}}(V, \mathbb{Q}_\ell(1))$ if $l \neq p$ and $H^2_{fl}(V, \mathbb{Q}_p(1))$ otherwise. Set:

(a) $r(X) = \mathrm{rank}_{\mathbb{Z}} NS(X)$;
(b) $r(J_X) = \mathrm{rank}_{\mathbb{Z}} J_X(K)$;
(c) $r_\ell(X) := \dim_{\mathbb{Q}_\ell} H^2(X, \mathbb{Q}_\ell(1))$;
(d) $r_\ell(J_X) := \mathrm{corank}_{\mathbb{Z}_\ell} \mathrm{Sel}_{\ell^\infty}(J_X/K)$;
(e) $r_{\text{an}}(X)$ the order of the pole at $s = 1$ of the Zeta function of X;
(f) $r_{\text{an}}(J_X)$ the order of the zero at $s = 1$ of the Hasse–Weil L-function.

Then it can be shown (see [Ulm12, 6.2.4]) that

$$r(X) - r(J_X) = r_\ell(X) - r_\ell(J_X) = r_{\text{an}}(X) - r_{\text{an}}(J_X) = 2 + \sum_{v \in C} (f_v - 1),$$

where f_v is the number of irreducible components of X_v and so the equivalence of the two results is now clear.

References

AC13 T. Abe and D. Caro, *Theory of weights in p-adic cohomology*, Preprint (2013), arXiv:1303.0622.

BBM82 P. Berthelot, L. Breen and W. Messing, *Théorie de Dieudonné Cristalline II*, Lecture Notes in Mathematics, vol. 930 (Springer, 1982).

CFKS10 J. Coates, T. Fukaya, K. Kato and R. Sujatha, *Root numbers, Selmer groups, and non-commutative Iwasawa theory*, J. Algebraic Geom. **19** (2010), 19–97.

Ces12 K. Česnavičius, *The p-parity conjecture for elliptic curves with a p-isogeny*. Preprint (2012), arXiv:1207.0431v2.

Cre98 R. Crew, *Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve*, Ann. Sci. Éc. Norm. Supér. (4) **31** (1998), 717–763.

Del80 P. Deligne, *La conjecture de Weil : II*, Publ. Math. Inst. Hautes Études Sci. **52** (1980), 137–252.

DD08 T. Dokchitser and V. Dokchitser, *Parity of ranks for elliptic curves with a cyclic isogeny*, J. Number Theory **128** (2008), 662–679.

DD09 T. Dokchitser and V. Dokchitser, *Regulator constants and the parity conjecture*, Inv. Math. **178** (2009), 23–71.

DD10 T. Dokchitser and V. Dokchitser, *On the Birch–Swinnerton-Dyer quotients modulo squares*, Ann. of Math. (2) **172** (2010), 567–596.

DD11 T. Dokchitser and V. Dokchitser, *Root numbers and parity of ranks of elliptic curves*, J. Reine Angew. Math. **2011** (658) (2011), 39–64.

EGAII A. Grothendieck and J. Dieudonné, *Eléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : II. Etude globale élémentaire de quelques classes de morphismes*, Publ. Math. Inst. Hautes Études Sci. **8** (1961), 5–222.

EL93 J.-Y. Etesse and B. Le Stum, *Fonctions L associés aux F-isocristaux surconvergents I. Interprétation cohomologique*, Math. Ann. **296** (1993), 557–576.

Gro68 A. Grothendieck, *Le groupe de Brauer III*, in *Dix expos'es sur la cohomologie des schémas* (North Holland, 1968).

Har77 R. Hartshorne, *Algebraic geometry* (Springer, New York, 1977), corrected 6th printing, 1993.

KT03 K. Kato and F. Trihan, *On the conjecture of Birch and Swinnerton-Dyer in characteristic* $p > 0$, Invent. Math. **153** (2003), 537–592.

KM74 N. Katz and W. Messing, *Some consequences for the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77.

Ked04 K. Kedlaya, *Full faithfulness for overconvergent F-isocrystals*, Geometric Aspects of Dwork Theory, vol. II (de Gruyter, Berlin, 2004), 819–835.

Ked06 K. Kedlaya, *Fourier transform and p-adic Weil II*, Compositio Math. **142** (2006), 1426–1450.

KW01 R. Kiehl and R. Weissauer, *Weil conjectures, perverse sheaves and l-adic Fourier transform*, Ergebnisse de Mathematik und ihere Grenzgebiete vol. 42 (Springer, Berlin, 2001).

Kim07 B. D. Kim, *The parity conjecture for elliptic curves at supersingular reduction primes*, Compositio Math. **143** (2007), 47–72.

Kim09 B. D. Kim, *The symmetric structure of the plus/minus Selmer groups of elliptic curves over totally real fields and the parity conjecture*, J. Number Theory **129** (2009), 1149–1160.

Laf02 L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. **147** (2002), 1–242.

Liu06 Q. Liu, *Algebraic geometry and arithmetic curves*, Transl. by Reinie Ern. (English) Oxford Graduate Texts in Mathematics, vol. 6 (Oxford University Press, Oxford, 2006).

LLR05 Q. Liu, D. Lorenzini and M. Raynaud, *On the Brauer group of a surface*, Invent. Math. **159** (2005), 673–676.

MT04 S. Matsuda and F. Trihan, *Image directe supérieure et unipotence*, J. Reine Angew. Math. **2004** (2004), 47–54.

Mil84 J. Milne, *Abelian varieties*, in *Proc. conf. on arithmetic geometry, Storrs, 1984* (Springer, New York, 1984), 103–150.

Nek01 J. Nekovář, *On the parity of ranks of Selmer groups II*, C. R. Acad. Sci. Paris, Ser. I **332** (2001), 99–104.

Nek06 J. Nekovář, *Selmer complexes*, Astérisque **310** (2006).

Nek07 J. Nekovář, *On the parity of ranks of Selmer groups III*, Doc. Math. **12** (2007), 243–274; Erratum: Doc. Math. **14** (2009), 191–194.

Nek09 J. Nekovář, *On the parity of ranks of Selmer groups IV*, Compositio Math. **145** (2009), 1351–1359; with an appendix by J.-P. Wintenberger.

Nek13 J. Nekovář, *Some consequences of a formula of Mazur and Rubin for arithmetic local constants*, Algebra Number Theory **7** (2013), 1101–1120.

Oda69 T. Oda, *The first de Rham cohomology group and Dieudonné modules*, Ann. Sci. Éc. Norm. Supér. (4) (1969), 63–135.

Sch82 P. Schneider, *Zur Vermutung von Birch und Swinnerton-Dyer über globalen Funktionskörpern.*, Math. Ann. **260** (1982), 495–510.

Tat94 J. Tate, *Conjectures on algebraic cycles in ℓ-adic cohomology*, in *Motives. Proceedings of the Summer Research Conference on Motives, University of Washington, Seattle, WA, USA, July 20–August 2, 1991*, Proceedings of Symposia in Pure Mathematics, vol. 55, eds U. Jannsen *et al.*, (American Mathematical Society, Providence, RI, 1994), 71–83; Pt. 1.

TW11 F. Trihan and C. Wuthrich, *Parity conjectures for elliptic curves over global fields of positive characteristic*, Compositio Math. **147** (2011), 1105–1128.

Ulm12 D. Ulmer, *Curves and Jacobians over function fields*, Preprint (2012), http://people.math.gatech.edu/~ulmer/research/preprints/C.pdf.

Fabien Trihan f.trihan@exeter.ac.uk
School of Engineering, Computing and Mathematics, University of Exeter, EX4 4QF, UK

Seidai Yasuda s-yasuda@math.sci.osaka-u.ac.jp
Department of Mathematics, Graduate School of Science, Osaka University,
Toyonaka 560-0043, Japan

https://doi.org/10.1112/S0010437X13007501 Published online by Cambridge University Press