A NOTE ON LOWER RADICAL CONSTRUCTIONS FOR ASSOCIATIVE RINGS

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1. Introduction. In [2], a construction for the lower radical class $R_o(\eta)$ with respect to a class η of rings was given as the union of an inductively defined ascending transfinite chain of classes of rings. It was shown there that this construction terminates, for associative rings, at ω_o, the first infinite ordinal, in the sense that if $\{\eta_\alpha: \alpha$ an ordinal$\}$ is the chain, then $R_o(\eta) = \eta_{\omega_o}$. Also, examples of classes η for which $R_o(\eta) = \eta_1, \eta_2, \eta_3$ were given.

The purpose of this note is to give an example which shows that ω_o is the best lower bound that can be obtained. We describe a class of rings η for which $R_o(\eta) = \eta_{\omega_o}$, but for which $R_o(\eta) \neq \eta_k$ for any finite ordinal k.

As a preliminary to establishing this result, we also show that, for any finite ordinal k, there are classes η for which $R_o(\eta) \geq \eta_k$. The problem of showing whether or not, for a given finite ordinal k, there is a class η for which $R_o(\eta) = \eta_k$, is still open.

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2. Notation and Preliminary Lemmas. A "ring" in this note will mean an associative ring (not necessarily possessing a unity), and an "ideal" will always mean a two sided ideal. The situation in which A is an ideal of B will be denoted $A \lhd B$.

If A and B are subrings of a ring K, and if $A \subseteq B$, the smallest ideal of B containing A will be denoted by $\langle A\rangle_B$. It is easily seen that $\langle A\rangle_B = A + BA + AB + BAB$.

For the definitions and properties of "radical properties" for associative rings we refer the reader to [1]. A class R of rings will be a radical class if and only if it is the class of $\mathcal{S}$-radical rings for some radical property $\mathcal{S}$.

Given any class η of associate rings, the lower radical class $R_o(\eta)$ is the smallest radical class containing η. Using the notation of [2], (see also [1], footnote, p. 12), $R_o(\eta) = \eta_{\omega o}$, where η_1 is the class of all homomorphic images of members in η, and η_α (α an ordinal > 1) is defined transfinitely as in [2]. Each η_α is homomorphically closed, and, if α and β are ordinals, and $\alpha \leq \beta$, then $\eta_\alpha \subseteq \eta_\beta$.

Finally, we say that a subring B of a ring K is accessible to K by a chain of length k if there is a chain

$$(1) \qquad B = A_1 \lhd A_2 \lhd A_3 \ldots \lhd A_k = K.$$

LEMMA 2.1. If B is a subring of K, if B is accessible to K by a chain of length k, and if B is in η_1, then $\langle B\rangle_{A_i}$ is in η_{i-1}, for $i = 2, 3, \ldots k$. (The A_i's are the rings in equation (1)).

Proof. The proof of the lemma, essentially an induction on i, is contained in the proof of Lemma 2 of [2].

LEMMA 2.2 Given any class η, and a finite ordinal $k > 0$, a ring K is in η_k if and only if, for any non-zero homomorphic image K' of K, there is a chain

(2) $$K' = I_k \triangleright I_{k-1} \dots \triangleright I_1 \neq 0$$

where I_1 is in η_1.

Proof. i) "Only if". This is clearly true when $k = 1$, since η_1 is homomorphically closed. Assume that it is true for all $s < k$, and let K be in η_k. If $K' \neq 0$ is a homomorphic image of K, then K' has a non-zero ideal J in η_n, for some $n < k$, and hence J is in η_{k-1}. By our inductive assumption, this gives a chain

$$K' \triangleright J = J_{k-1} \triangleright \dots \triangleright J_1 \neq 0$$

where J_1 is in η_1. This is clearly a chain of the desired form. Hence the result is true for η_k.

ii) "If" Suppose we have K' a non-zero homomorphic image of K, and a chain satisfying equation (2). Define $S = \langle I_1 \rangle_{K'}$. By Lemma 2.1, S is in η_{k-1}. Thus any non-zero homomorphic image of K has a non-zero ideal in η_{k-1} whence K is in η_k.

3. The First Example. In this section, we give examples to show that, given an integer $n > 2$, there is a class η of rings for which $R_o(\eta) \neq \eta_{n-1}$.

Let R be the field $GF(p)$ of p elements, where p is a prime, and let $F = R[x, t]$, the ring of polynomials over R in two (commuting) indeterminates. For any $n \geq 0$, let G_n be the subring of F consisting of all elements of the form

$xp(x) + \Sigma_{i=1}^{m} t^i x^n r_i(x)$, where $p(x)$ and the $r_i(x)$ are arbitrary polynomials in x, and m is an arbitrary integer ≥ 1. Thus, for example, G_o is the set of polynomials with zero constant term. Also, whenever a power of t appears in G_n, it must be multiplied by x^n. It is easily verified that

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we have

$$G_n \triangleleft G_{n-1} \triangleleft G_{n-2} \triangleleft \dots G_1 \triangleleft G_0.$$

Also, any ideal of G_0 is an ideal of F. In particular $G_1 \triangleleft F$. In fact, we have $G_1 = \langle G_n \rangle_F$. To see this, it suffices to show $G_1 \subseteq \langle G_n \rangle_F$, since $G_1 \supseteq G_n$ and $G_1 \triangleleft F$ together imply $G_1 \supseteq \langle G_n \rangle_F$. However, since any element of G_1 is of the form $xp(x) + \Sigma_{i=1}^{m} t^i x\, r_i(x)$, and since $xp(x)$ and $x\, r_i(x)$ are in G_n, we have that $xp(x) + \Sigma_{i=1}^{m} t^i x\, r_i(x) \,\varepsilon\, G_n + FG_n \subseteq \langle G_n \rangle_F$. This proves the assertion.

Let η_1 be the class of homomorphic images of G_n. Then $G_n \triangleleft G_{n-1} \triangleleft \dots \triangleleft G_1 \triangleleft F$, and $G_1 = \langle G_n \rangle_F$. By Lemma 2.1, we have $G_1 \,\varepsilon\, \eta_n$. The proof of the example will be complete if we can show $G_1 \not\varepsilon\, \eta_{n-1}$.

Suppose, to the contrary, that G_1 is in η_{n-1}. By Lemma 2.2, there must exist a chain

$$G_1 \triangleright Y_{n-2} \triangleright Y_{n-3} \triangleright \dots \triangleright Y_1 = Z \neq 0,$$

where Z is a member of η_1, that is, a homomorphic image of G_n. We show that this leads to a contradiction.

LEMMA 3.1. If φ is a non-zero homomorphism of G_n into F, then there is a unique way of extending φ to an endomorphism of F.

Proof. Recall that every element of G_n is of the form $xp(x) + \Sigma_{i=1}^{m} t^i x^n r_i(x)$. If φ is such a homomorphism, let $\varphi(x) = A$, and $\varphi(t^i x^n) = B_i$. Then $B_i B_j = \varphi(t^i x^n)\varphi(t^j x^n) = B_{i+j} A^n$.

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The ring F has no divisors of zero. Therefore, if $A = 0$, then $B_i^2 = B_{2i}A^n = 0$, and hence $B_i = 0$, for all i. Thus $A = 0$ implies that $\varphi(w) = 0$ for all w in G_n. Since we are assuming that φ is a non-zero homomorphism, we have $A \neq 0$.

An extension of φ to all of F can be found if we can find an element $Q \varepsilon F$ such that $B_i = Q^i A^n$ for all i. If we have such an element Q, setting $\psi(x) = A$ and $\psi(t) = Q$ induces an endomorphism ψ of F which is clearly an extension of φ. Furthermore, such a Q, if it exists, must be unique, since F is a unique factorization domain, and hence has no divisors of zero.

In order to find such a Q, consider the relations $B_1^2 = B_2A^n$, $B_1^3 = B_1B_2A^n = B_3A^{2n}$, ..., $B_1^k = B_kA^{(k-1)n}$. By considering the prime factors of B_1 and of A^n, we see that $[A^n]^{(k-1)}$ divides B_1^k, for all $k > 1$, implies A^n divides B_1. Suppose $B_1 = QA^n$. Then $B_1^k = B_kA^{(k-1)n}$ gives $Q^kA^{kn} = B_kA^{(k-1)n}$. Since F has no divisors of zero, we can cancel $A^{(k-1)n}$ to get $B_k = Q^kA^n$ for all integers $k \geq 1$. This completes the proof of the lemma.

We thus have $Z = Y_1 \triangleleft Y_2 \triangleleft \dots \triangleleft Y_{n-1} = G_1 \triangleleft F$, and $Z = \varphi(G_n)$ for some endomorphism f of F. We are denoting $\varphi(x)$ by A and $\varphi(t)$ by Q.

There are two possible cases which can occur - either Q and A are algebraically independent over R, or they are not.

<u>Case 1. Q and A are algebraically independent over R</u>. In this case, it follows (see [3], p.37) that the endomorphism φ of F will be one-to-one. Since $G_1 \triangleleft F$, and $A \varepsilon Z$, $Q \varepsilon F$, we have $QA \varepsilon G_1$, $QA^2 \varepsilon Y_{n-2}$, ..., $QA^{n-1} \varepsilon Y_1 = Z$.

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Since $Z = \varphi(G_n)$, there is a $g \varepsilon G_n$ such that $\varphi(g) = QA^{n-1} = \varphi(tx^{n-1})$. Since φ is one-to-one, $tx^{n-1} = g \varepsilon G_n$, a contradiction.

Case 2. Q and A are algebraically dependent over R. In this case there must be an element $B \epsilon F$ which is algebraically independent (over R) of A. For if $A \epsilon R$, then $B = x$ will do. If A is not in R, then either the degree of A in x ($\deg_x A$) is greater than or equal to 1, or $\deg_t(A) \geq 1$. Suppose that $\deg_x(A) \geq 1$. Then every non-zero $W \epsilon AF$ has $\deg_x(W) \geq 1$, and AF contains no elements which are polynomials over R in t alone. Then A and t are independent, for otherwise we would have $h_o(A)t^n + h_1(A)t^{n-1} + \ldots + h_n(A) = 0$, where each $h_i(A)$ is a polynomial in A with coefficients in R. Any common factor A^k of all the $h_i(A)$'s may be cancelled, and so we may assume that at least one $h_i(A)$ has a non-zero constant term. Collecting the terms in t alone gives $0 = q(t) + Ar(x, t)$, where $q(t)$ is a polynomial in t over R, and $r(x, t)$ is a polynomial in X and t over R. This gives $q(t) \epsilon AF$, a situation which cannot occur. Similarly, if $\deg_t(A) \geq 1$, then A and x are independent over R.

Let B and A be independent. As in Case 1, we have $BA^{n-1} \epsilon Z$. Since $\varphi(G_n) = Z$, and from the form of elements of G_n, we see that we must have $BA^{n-1} = Ap(A) + \Sigma_{i=1}^{m} QA^n r_i(A)$, where $A \neq 0$.

If $A \varepsilon R$ (i.e. if A is invertible), then B is a polynomial in Q over R. If A is not in R, since F is a UFD, it follows that the polynomial in A, $p(A)$, is divisible by A^{n-2}, and that we can write $p(A) = A^{n-2}q(A)$ where $q(A)$ is, in fact, a polynomial in A. We then obtain $B = q(A) + \Sigma_{i=1}^{m} Q^i Ar_i(A) \varepsilon R[A, Q]$. In either case $R[A, Q] \supseteq R[A, B] \supseteq R[A]$.

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However, A and Q are dependent over R, which implies R[A, Q] is algebraic over R[A]. On the other hand, A and B are independent, which implies R[A, B] is transcendental over R[A]. Again, in this case, we have a contradiction.

Thus we have $G_1 \in \eta_n$, $G_1 \notin \eta_{n-1}$.

We have actually proved slightly more; namely

LEMMA 3.2. If we have

$$G_1 \triangleright Y_{n-2} \triangleright Y_{n-3} \triangleright \ldots \triangleright Y_1 \neq 0,$$

then Y_1 cannot be a homomorphic image of G_n.

4. The Second Example. In this section we give an example, based on the previous example, of a class η and a ring K for which K is $R_o(\eta)$ radical, but K is not in η_n for each finite ordinal n.

Let $p_1, p_2, \ldots$ be an enumeration of the prime numbers, and let $G_n(p_n)$ be the example of the previous section, with $R = GF(p_n)$. We take η_1 to be the collection of all the homomorphic images of the $G_n(p_n)$ for all n, and we set $K = \bigoplus_{i=1}^{\infty} G_1(p_i)$, the (weak direct sum) ring direct sum of the $G_1(p_i)$.

Since each $G_1(p_i)$ is in the radical class $R_o(\eta)$, K is also in $R_o(\eta)$.

We claim that, for all finite n, k is not in η_n. For, since η_n is homomorphically closed, if K is in η_n, then $G_1(p_i)$ is in η_n for all i. By Lemma 2.2, this implies that we have a chain, for each i,

(1) $$G_1(p_i) \triangleright I_{n-1} \triangleright I_{n-2} \triangleright \ldots \triangleright I_1 \neq 0$$

and a homomorphism φ of $G_t(p_t)$ (for some t) onto I_1. Since every element of $G_n(p_i)$ is of characteristic p_i, we must have $i = t$.

In particular, for $i = t = n+1$, we have

$$G_1(p_{n+1}) \triangleright I_{n-1} \triangleright \dots \triangleright I_1,$$

where I_1 is a non-zero homomorphic image of $G_{n+1}(p_{n+1})$. Lemma 3.2 (with n replaced by n+1) shows this is impossible.

This completes the proof.

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