

**EXISTENCE OF OPTIMAL CONTROLS FOR SYSTEMS GOVERNED
BY SECOND ORDER LINEAR PARABOLIC PARTIAL
DELAY-DIFFERENTIAL EQUATIONS WITH FIRST BOUNDARY
CONDITIONS**

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Abstract

In this paper, we consider a class of systems governed by second order linear parabolic delay-partial differential equations with first boundary conditions. Our main results are reported in Theorems 3.1 and 3.2. As in [9, Theorems 4.1 and 4.2], the coefficients and forcing terms of the system considered in Theorem 3.1 are linear in the control variables. On the other hand, the forcing terms of the system considered in Theorem 3.2 are allowed to be nonlinear in the control variables at the expense of dropping the control variables in the cost integrand.

1. Introduction

It appears that there are only few results [5, p. 262; 9, 11] available in the literature on the existence of optimal controls for systems governed by parabolic partial delay-differential equations with controls and delayed arguments appearing in the coefficient.

In this paper, we present two existence theorems for optimal controls (Theorems 3.1 and 3.2). As in [9, Theorems 4.1 and 4.2], the coefficients and forcing terms of the system considered in Theorem 3.1 are linear in the control variables. However, Theorem 3.1 contains Theorems 4.1 and 4.2 of [9] as special cases. On the other hand, the coefficients and forcing terms of the system considered in Theorem 3.2 are allowed to be nonlinear in the control variables at the expense of dropping the control variables in the cost integrand.

It is known [10] that a class of stochastic optimal control problems can be converted into a class of optimal control problems of systems governed by parabolic partial differential equations. This reduced problem is contained as a very special case of the class of optimal control problems currently under consideration.

Let Ω be a domain in n -dimensional Euclidean space R^n , bounded by a smooth surface $\partial\Omega$ satisfying the following properties: each point of $\partial\Omega$ is locally representable by functions with the Hölder continuous second order partial derivatives.

We denote the coordinates of a point x in R^n by x_1, \dots, x_n , time by t . Let T be a fixed time instant, $T < \infty$, and h_κ ($\kappa = 0, 1, \dots, \nu$) be certain given numbers such that $0 = h_0 < h_1 < \dots < h_\nu < T$. We denote the intervals $I_0 = [-h_\nu, 0]$, $I_1 = (0, T)$, $I_2 = [-h_\nu, T]$.

Now let us consider the following second order linear partial delayed differential equations of parabolic type with a first boundary condition.

$$\left. \begin{aligned}
 \phi_t(u)(x, t) &= \sum_{i,j=1}^n a_{ij}(x, t) \cdot \phi_{x_i x_j}(u)(x, t) \\
 &+ \sum_{\kappa=0}^{\nu} \sum_{i=1}^n b_{i,\kappa}(x, t-h_\kappa, u(x, t-h_\kappa)) \cdot \phi_{x_i}(u)(x, t-h_\kappa) \\
 &+ \sum_{\kappa=0}^{\nu} c_\kappa(x, t-h_\kappa, u(x, t-h_\kappa)) \cdot \phi(u)(x, t-h_\kappa) \\
 &+ \sum_{\kappa=0}^{\nu} f_\kappa(x, t-h_\kappa, u(x, t-h_\kappa)), \quad \text{for } (x, t) \in \Omega \times I_1 \\
 \phi(u)(x, t) &= \phi_0(x, t), \quad \text{for } (x, t) \in \Omega \times I_0 \\
 \phi(u)(x, t) &= 0, \quad \text{for } (x, t) \in \partial\Omega \times I_2 \\
 u &\in D,
 \end{aligned} \right\} (1)$$

where $\phi_t = \partial\psi/\partial t$, $\psi_{x_i} = \partial\psi/\partial x_i$, $\psi_{x_i x_j} = \partial^2\psi/\partial x_i \partial x_j$ ($i, j = 1, \dots, n$), and D is the set of admissible controls to be defined later.

Let \hat{u} be a bounded measurable function from $\bar{\Omega} \times [-h_\nu, 0]$ into R^r and let U be a non-empty compact convex subset of R^r . Now let us define the set of admissible controls on $\bar{\Omega} \times [-h_\nu, T]$ by

$$\begin{aligned}
 D &= \{u: u \text{ measurable on } \bar{\Omega} \times [-h_\nu, T], u(x, t) = \hat{u}(x, t) \\
 &\text{almost everywhere on } \bar{\Omega} \times [-h_\nu, 0] \text{ and } u(x, t) \in U \\
 &\text{almost everywhere on } \bar{\Omega} \times [0, T]\}.
 \end{aligned}$$

2. Auxiliary results

With reference to system (1), it is assumed that, for each $u \in D$, the coefficients, forcing terms and data are defined and measurable on their appropriate domains. Before stating more specific assumptions, we shall introduce some useful notations.

Let $|0|$ denote the Lebesgue measure of the measurable set 0 of any finite dimensional Euclidean space. Let E be any connected subset of an s -dimensional Euclidean space R^s and denote by $C^l(E)$ the class of all l times continuously differentiable functions on E , where $1 \leq l < \infty$ is an integer. Further, let $C_0^l(E)$ be the class of functions from $C^l(E)$ with compact support on E . For any $Z \in R^s$, let $|Z| = (\sum_{i=1}^s Z_i^2)^{1/2}$. Let $z_x = [z_{x_1}, \dots, z_{x_n}]$ denote the gradient of the scalar valued function on R^n .

Let E be as before and denote by $L^\delta(E)$, $\delta \geq 1$, the Banach space consisting of all measurable functions on E that are δ th-power integrable on E . The norm on it is defined by the equalities

$$\|z\|_{\delta,E} = \left\{ \int_E |z(y)|^\delta dy \right\}^{1/\delta} \quad \text{for } 1 \leq \delta < \infty$$

and

$$\|z\|_{\infty,E} = \text{ess sup}_E |z(y)| \quad \text{for } \delta = \infty.$$

Measurability and integrability are to be understood in the sense of Lebesgue. The elements of $L^\delta(E)$ are the equivalence classes of the functions on E (functions belonging to the same equivalence class are equal almost everywhere).

Let λ be a real number such that $1 \leq \lambda < \infty$ and denote by $W_\lambda^{2,1}(E)$ the Banach space of functions from $L^\lambda(E)$ having generalized derivatives of the form $(\partial^r/\partial t^r)(\partial^s/\partial x^s)$ with any r and s satisfying the inequality $2r + s \leq 2$. The norm on it is defined by the equality

$$\|z\|_{\lambda,E}^{(2)} = \|z\|_{\lambda,E} + \left\| \frac{\partial z}{\partial t} \right\|_{\lambda,E} + \sum_{i=1}^n \left\| \frac{\partial z}{\partial x_i} \right\|_{\lambda,E} + \sum_{i,j=1}^n \left\| \frac{\partial^2 z}{\partial x_i \partial x_j} \right\|_{\lambda,E}$$

Note that if z is only a function of x defined in Ω then we denote $W_\lambda^2(\Omega)$ the Banach space of functions from $L^\lambda(\Omega)$ having generalized derivatives of the form $\partial^s/\partial x^s$ with $s = 0, 1$ and 2 . The norm on it is defined by the equality

$$\|z\|_{\lambda,\Omega}^{(2)} = \|z\|_{\lambda,\Omega} + \sum_{i=1}^n \left\| \frac{\partial z}{\partial x_i} \right\|_{\lambda,\Omega} + \sum_{i,j=1}^n \left\| \frac{\partial^2 z}{\partial x_i \partial x_j} \right\|_{\lambda,\Omega}.$$

For any non-integral positive number λ , $H^{\lambda,\lambda/2}(E)$ denotes the Banach space of functions z that are continuous on E and have derivatives of the form

$$D_i^\alpha \cdot D_x^\beta \cdot z = \frac{\partial^\alpha}{\partial t^\alpha} \cdot \left(\frac{\partial^{\beta_1 + \dots + \beta_n}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} \cdot z \right), \quad \sum_{i=1}^n \beta_i = \beta,$$

(each β_i is a non-negative integer, $2\alpha + \beta < \lambda$), and have a finite norm

$$|z|_{\bar{E}}^\lambda = \|z\|_{\bar{E}}^{(\lambda)} + \sum_{j=0}^{[\lambda]} \|z\|_{\bar{E}}^{(j)}.$$

Note that $[\lambda]$ denotes the largest integral part of λ and

$$\begin{aligned} \|z\|_{\bar{E}}^{(0)} &= |z|_{\bar{E}}^{(0)} = \max_{\bar{E}} |z(x, t)|, \\ \|z\|_{\bar{E}}^{(j)} &= \sum_{2\alpha + \beta = j} |D_t^\alpha \cdot D_x^\beta \cdot z|_{\bar{E}}^{(0)}, \\ \|z\|_{\bar{E}}^{(\lambda)} &= \|z\|_{x, \bar{E}}^{(\lambda)} + \|z\|_{t, \bar{E}}^{(\lambda/2)}, \\ \|z\|_{x, \bar{E}}^{(\lambda)} &= \sum_{2\alpha + \beta = [\lambda]} \|D_t^\alpha \cdot D_x^\beta \cdot z\|_{x, \bar{E}}^{(\lambda - (\lambda))}, \\ \|z\|_{t, \bar{E}}^{(\lambda/2)} &= \sum_{0 < \lambda - 2\alpha - \beta < 2} \|D_t^\alpha \cdot D_x^\beta \cdot z\|_{t, \bar{E}}^{(\lambda - 2\alpha - \beta)}, \\ \|z\|_{x, \bar{E}}^{(\gamma)} &= \max_{(x, t), (x', t') \in \bar{E}} \frac{|z(x, t) - z(x', t')|}{|x - x'|^\gamma}, \quad 0 < \gamma < 1, \\ \|z\|_{t, \bar{E}}^{(\gamma)} &= \max_{(x, t), (x, t') \in \bar{E}} \frac{|z(x, t) - z(x, t')|}{|t - t'|^\gamma}, \quad 0 < \gamma < 1. \end{aligned}$$

Throughout this paper, the coefficients, the forcing terms and data of the system (1) are assumed to satisfy the following assumptions which will be referred to collectively as (A).

- (i) $a_{ij}(\cdot, \cdot)$ ($i, j = 1, \dots, n$) are continuous on \bar{Q} , where $Q = \Omega \times I_1$ and \bar{Q} is the closure of Q ,
- (ii) there exist numbers $\alpha_u > \alpha_l > 0$ such that

$$\alpha_u |Z|^2 \geq \sum_{i,j=1}^n a_{ij}(x, t) \cdot Z_i \cdot Z_j \geq \alpha_l |Z|^2$$

for all $Z \in R^n$ uniformly on \bar{Q} (uniformly parabolic), where $|Z|^2 = \sum_{i=1}^n |Z_i|^2$,

(iii) $\max_{1 \leq i, j \leq n} \frac{|a_{ij}(x, t) - a_{ij}(x', t')|}{|t - t'| + |x - x'|} \leq M,$

where $t, t' \in [0, T]$; $x, x' \in \bar{\Omega}$; and M is a constant,

- (iv) $b_{i,\kappa}(\cdot, \cdot - h_\kappa, \cdot), c_\kappa(\cdot, \cdot - h_\kappa, \cdot), f_\kappa(\cdot, \cdot - h_\kappa, \cdot)$ ($i = 1, \dots, n; \kappa = 0, 1, \dots, \nu$) are bounded measurable on $\bar{Q} \times U$ and continuous on U for almost all $(x, t) \in \bar{Q}$,
- (v) $\phi_0 \in C^2(\bar{\Omega} \times [-h_\nu, 0])$ and $\phi_0(x, t) = 0$ for all $x \notin \bar{\Omega}_0$ and $t \in [-h_\nu, 0]$, where Ω_0 is a compact subset of Ω .

Note that all the results to be presented in this paper remain valid if the condition $\phi_0 \in C^2(\Omega \times [-h_\nu, 0])$ is replaced by a weaker condition $\phi_0 \in W_\lambda^{2,1}(\bar{\Omega} \times [-h_\nu, 0])$.

The reason for imposing this more restrictive assumption is merely for convenience.

With this preparation, we state the following optimal control problem P .

Given the dynamic system (1), find a control $u_0 \in D$ that minimizes the cost functional

$$J(u) = \int_Q \int G(x, t, u(x, t), \phi(u)(x, t), \phi_x(u)(x, t)) dx dt, \tag{2}$$

where

$$\begin{aligned} G(x, t, u(x, t), \phi(u)(x, t), \phi_x(u)(x, t)) \\ = G_1(x, t, u(x, t)) + G_2(x, t, \phi(u)(x, t), \phi_x(u)(x, t)). \end{aligned}$$

For brevity, the following assumptions on the cost integrand G will be referred to collectively as \mathcal{G} .

- (i) $G_1(x, t, v)$ is measurable in \bar{Q} for each $v \in R^r$, continuous in R^r for each $(x, t) \in \bar{Q}$ and convex in U for each $(x, t) \in \bar{Q}$; further, there exists a non-negative function $g_1 \in L^1(Q, R^1)$ and a constant $\omega > 0$ so that

$$|G_1(x, t, v)| \leq g_1(x, t) + \omega |v|^\sigma$$

for some constant $\sigma \geq 1$ almost everywhere on Q ; and

- (ii) there exist a real number $\gamma \geq 0$ and a non-negative measurable function $g_2 \in L^1(Q, R^1)$ so that

$$|G_2(x, t, \psi^0, \psi^1, \dots, \psi^n)| \leq g_2(x, t) + \gamma \sum_{\rho=0}^n |\psi^\rho|$$

almost everywhere on Q .

For ease in future references, the statement: “ C depends on the structure of the differential equation of system (1)” will be used to mean that C is determined by the quantities α_i, α_w, M , and the bounds of the functions $b_{i,\kappa}, c_\kappa$ ($i = 1, \dots, n; \kappa = 0, 1, \dots, n$).

Corresponding to system (1) we need the following definition.

DEFINITION 2.1. A function $\phi: \bar{\Omega} \times I_2 \rightarrow R^1$ is said to be a solution of system (1) if

- (i) $\phi(x, t) = \phi_0(x, t)$ on $\Omega \times I_0$;
- (ii) $\phi(x, t) = 0$ on $\partial\Omega \times I_2$;
- (iii) the restriction of ϕ to Q belongs to $W_{\lambda^1}^{2,1}(Q)$ ($n+2 < \lambda < \infty$); and
- (iv) ϕ satisfies the differential equation of system (1) almost everywhere in Q .

Note that, in Definition 2.1, ϕ_t and $\phi_{x_i x_j}$ are, respectively, understood to be the generalized derivative of ϕ with respect to t and the generalized derivative of ϕ_{x_i} with respect to x_j .

The existence and uniqueness of solutions of system (1) together with two *a priori* estimates are known [8, Theorem 2.3, pp. 123–124]. This result is quoted without proof in the following theorem.

THEOREM 2.1. *Under the assumptions (A), system (1) admits, for each $u \in D$, a unique solution $\phi(u)$. Further $\phi(u)$ satisfies the following estimates.*

$$\|\phi(u)\|_{\lambda, Q^{(2)}} \leq N_1; \tag{3}$$

and

$$|\phi(u)|_{\bar{Q}^{\lambda+\mu}} \leq N_2 \tag{4}$$

for all $\lambda > n + 2$ and $\mu = 1 - ((n + 2)/\lambda)$, where the positive constants N_1 and N_2 depend only on $n, \lambda, \nu, Q, \partial\Omega$, the structure of the differential equation of system (1), the bounds for the functions f_κ ($\kappa = 0, 1, \dots, \nu$), $|\phi_0|_{\bar{\Omega} \times [-h_\nu, 0]}^{(0)}$, $|\partial\phi_0/\partial x_i|_{\bar{\Omega} \times [-h_\nu, 0]}^{(0)}$ ($i = 1, \dots, n$) and $\|\phi_0(\cdot, 0)\|_{\lambda, \Omega}^{(2)}$.

In the sequel, Theorem 3.2 of [9] will be quoted without proof in the following theorem.

THEOREM 2.2. *Let $\{u_i\}_{i=1}^\infty \subset D$ and let $\{\phi(u_i)\}_{i=1}^\infty$ be the corresponding sequence of solutions of system (1). Suppose that the assumptions (A) are satisfied. Then there exists a subsequence $\{u_{i_\kappa}\}_{i_\kappa=1}^\infty \subset \{u_i\}_{i=1}^\infty$ so that*

$$b_{i,\kappa}(\cdot, \cdot - h_\kappa, u_{i_\kappa}(\cdot, \cdot - h)), c_\kappa(\cdot, \cdot - h_\kappa, u_{i_\kappa}(\cdot, \cdot - \kappa)),$$

$$f_\kappa(\cdot, \cdot - h_\kappa, u_{i_\kappa}(\cdot, \cdot - h_\kappa)) \quad (i = 1, \dots, n; \kappa = 0, 1, \dots, \nu)$$

converge, respectively, to

$$b_{i,\kappa}^*(\cdot, \cdot - h_\kappa), e_\kappa^*(\cdot, \cdot - h_\kappa), f_\kappa^*(\cdot, \cdot - h_\kappa) \quad (i = 1, \dots, n; \kappa = 0, 1, \dots, \nu)$$

in the weak * topology of $L^\infty(Q, R^1)$ as $i \rightarrow \infty$. Further,

$$\left. \begin{aligned} \phi(u_{i_\kappa}) &\rightarrow \phi^* \\ \phi_{x_i}(u_{i_\kappa}) &\rightarrow \phi_{x_i}^*, \quad i = 1, \dots, n, \end{aligned} \right\} \text{uniformly on } \bar{Q}$$

$$\left. \begin{aligned} \phi(u_{i_\kappa}) &\rightarrow \phi_i^* \\ \phi_{x_i x_j}(u_{i_\kappa}) &\rightarrow \phi_{x_i x_j}^*, \quad i, j = 1, \dots, n, \end{aligned} \right\} \text{weakly in } L^2(Q) \text{ as } i \rightarrow \infty,$$

where $\lambda > n + 2$ and ϕ^* is the unique solution of the system

$$\left. \begin{aligned} \phi_i^*(x, t) &= \sum_{i,j=1}^n a_{ij}(x, t) \phi_{x_{ij}}^*(x, t) \\ &+ \sum_{\kappa=0}^{\nu} \sum_{i=1}^n b_{i,\kappa}^*(x, t-h_\kappa) \cdot \phi_{x_i}^*(x, t-h_\kappa) \\ &+ \sum_{\kappa=0}^{\nu} c_\kappa^*(x, t-h_\kappa) \cdot \phi^*(x, t-h_\kappa) \\ &+ \sum_{\kappa=0}^{\nu} f_\kappa^*(x, t-h_\kappa), \quad (x, t) \in Q, \\ \phi^*(x, t) &= \phi_0(x, t) \quad (x, t) \in \Omega \times I_0, \\ \phi^*(x, t) &= 0, \quad (x, t) \in \partial\Omega \times I_2. \end{aligned} \right\} \quad (5)$$

For brevity, let us introduce the following notation.

$$F_\kappa(x, t, v) = [b_{1,\kappa}(x, t, v), b_{2,\kappa}(x, t, v), \dots, b_{n,\kappa}(x, t, v), c_\kappa(x, t, v), f_\kappa(x, t, v)] \quad (6)$$

($\kappa = 0, 1, \dots, \nu$);

$$\hat{F}_\kappa(x, t, v) = \begin{cases} F_\kappa(x, t, v), & (x, t) \in \Omega \times [0, T-h_\kappa], \\ F_\kappa(x, T-h_\kappa, v) & (x, t) \in \Omega \times (T-h_\kappa, T] \end{cases} \quad (7)$$

($\kappa = 0, 1, \dots, \nu$); and

$$\tilde{F}(x, t) = \{[\hat{F}_0(x, t, v), \hat{F}_1(x, t, v), \dots, \hat{F}_\nu(x, t, v)]^T : v \in U\}, \quad (8)$$

where the superscript ‘‘T’’ denotes transpose.

Clearly, \tilde{F} is a set-valued function from \tilde{Q} into $R^{(n+2)(\nu+1)}$.

In the sequel, we need

DEFINITION 2.2. *A set-valued function $\tilde{F}: \tilde{\Omega} \times [0, T] \rightarrow R^{(n+2)(\nu+1)}$ is said to be measurable if $\{(x, t) \in \tilde{\Omega} \times [0, T] : B \cap \tilde{F}(x, t) \neq \emptyset\}$ is measurable for every closed subset B of $R^{(n+2)(\nu+1)}$.*

As an immediate consequence of [3, Theorem 3, p. 281], we have the following Filippov-type lemma.

LEMMA 2.1. *Let $b_{i,\kappa}, c_\kappa, f_\kappa$ ($i = 1, \dots, n; \kappa = 0, 1, \dots, \nu$) satisfy the condition (iv) of the assumptions (A) and let z be a measurable function from \tilde{Q} into $R^{(n+2)(\nu+1)}$ so that $z(x, t) \in \tilde{F}(x, t)$ a.e. on \tilde{Q} . If \tilde{F} is a measurable set-valued function, then, there exists a measurable function u^* on \tilde{Q} such that*

$$z(x, t) = [\hat{F}_0(x, t, u^*(x, t)), \hat{F}_1(x, t, u^*(x, t)), \dots, \hat{F}_\nu(x, t, u^*(x, t))]^T$$

and $u^*(x, t) \in U$ a.e. on \tilde{Q} .

Using Theorem 2.2 and Lemma 2.1, we obtain

LEMMA 2.2. *Let the assumptions of Theorem 2.2 be satisfied and let $\tilde{F}(x, t)$ be convex for each $(x, t) \in \bar{Q}$. If, in addition, \tilde{F} is a measurable set-valued function, there exists a control $u^* \in D$ so that*

$$b_{i,\kappa}^*(x, t - h_\kappa) = b_{i,\kappa}(x, t - h_\kappa, u^*(x, t - h_\kappa)),$$

$$c_\kappa^*(x, t - h_\kappa) = c_\kappa(x, t - h_\kappa, u^*(x, t - h_\kappa)),$$

$$f_\kappa^*(x, t - h_\kappa) = f_\kappa(x, t - h_\kappa, u^*(x, t - h_\kappa))$$

($i = 1, \dots, n; \kappa = 0, 1, \dots, \nu$) almost everywhere on \bar{Q} .

PROOF. Let $\{u_i\}_{i=1}^\infty \subset \{u_j\}_{j=1}^\infty \subset D$ be as defined in Theorem 2.2. Let

$$H^i(x, t) = [\hat{F}_0(x, t, u_i(x, t)), \hat{F}_1(x, t, u_i(x, t)), \dots, \hat{F}_\nu(x, t, u_i(x, t))]^T$$

and let the set \mathcal{N} be defined by

$$\mathcal{N} = \{y: y \text{ measurable on } \bar{Q} \text{ and } y(x, t) \in \tilde{F}(x, t) \text{ for almost all } (x, t) \in \bar{Q}\}.$$

Recall that U is non-empty. Thus, D is non-empty. Now, by using appropriate assumptions of (A), we see that $[\hat{F}_0(\cdot, \cdot, u(\cdot, \cdot)), \hat{F}_1(\cdot, \cdot, u(\cdot, \cdot)), \dots, \hat{F}_\nu(\cdot, \cdot, u(\cdot, \cdot))]^T$ is measurable on G for any $u \in D$. Further, by the definition of \tilde{F} , it is clear that $[\hat{F}_0(x, t, u(x, t)), \hat{F}_1(x, t, u(x, t)), \dots, \hat{F}_\nu(x, t, u(x, t))]^T \in \tilde{F}(x, t)$ for all $(x, t) \in G$ for any $u \in D$. Thus, the set \mathcal{N} is non-empty.

By hypothesis, $\tilde{F}(x, t)$ is convex for each $(x, t) \in \bar{Q}$. Thus, it is easily observed that \mathcal{N} is convex. On the other hand, it follows from the definitions of \tilde{F} (equation 8) and \mathcal{N} and the condition (iv) of the assumptions (A) that, for each $(x, t) \in \bar{Q}$, $\tilde{F}(x, t)$ is a compact subset of $R^{(n+2)(\nu+1)}$ and there exists a constant $K > 0$ so that

$$|y(x, t)| \leq K$$

almost everywhere in \bar{Q} for all $y \in \mathcal{N}$. Let $\tilde{\mathcal{N}}$ be the class of functions defined by

$$\tilde{\mathcal{N}} = \{z: z \text{ measurable on } \bar{Q} \text{ with values in } R^{(n+2)(\nu+1)}$$

$$\text{and } |z(x, t)| \leq K \text{ for almost all } (x, t) \in \bar{Q}\}.$$

Clearly, $\mathcal{N} \subset \tilde{\mathcal{N}}$. From [2, Theorem 7.1, p. 19], the closed unit ball of $L^\infty(Q, R^{(n+2)(\nu+1)})$ is compact in the weak * topology of $L^\infty(Q, R^{(n+2)(\nu+1)})$; clearly a closed ball of any finite radius is also compact in the weak * topology of $L^\infty(Q, R^{(n+2)(\nu+1)})$. This obviously implies that $\tilde{\mathcal{N}}$ is compact in the weak * topology of $L^\infty(Q, R^{(n+2)(\nu+1)})$. Now, in view of the definitions of the \hat{F}_κ ($\kappa = 0, 1, \dots, \nu$) and H^i , we observe that $\{H^i\}_{i=1}^\infty \subset \mathcal{N} \subset \tilde{\mathcal{N}}$. On the other hand, it follows from Theorem

2.2 and the definitions of the \hat{F}_κ ($\kappa = 0, 1, \dots, \nu$) that

$$\hat{F}_\kappa(\cdot, \cdot, u_\kappa(\cdot, \cdot)) \rightarrow \hat{F}_\kappa^*(\cdot, \cdot) \quad (\kappa = 0, 1, \dots, \nu)$$

in the weak * topology of $L^\infty(Q, R^{n+2})$ as $\iota \rightarrow \infty$, where

$$\hat{F}_\kappa^*(x, t) = \begin{cases} F_\kappa^*(x, t), & (x, t) \in \Omega \times [0, T - h_\kappa], \\ F_\kappa^*(x, T - h_\kappa), & (x, t) \in \Omega \times (T - h_\kappa, T] \end{cases}$$

and

$$F_\kappa^*(x, t) = [b_{1,\kappa}^*(x, t), \dots, b_{n,\kappa}^*(x, t), c_\kappa^*(x, t), f_\kappa^*(x, t)] \quad (\kappa = 0, 1, \dots, \nu).$$

This, in turn, implies that $H^\iota(\cdot, \cdot) \rightarrow H^*(\cdot, \cdot)$ in the weak * topology of $L^\infty(Q, R^{(n+2)(\nu+1)})$, where

$$H^*(x, t) = [\hat{F}_0^*(x, t), \hat{F}_1^*(x, t), \dots, \hat{F}_\nu^*(x, t)]^T.$$

Since $\{H^\iota\}_{\iota=1}^\infty \subset \mathcal{N} \subset \tilde{\mathcal{N}}$ and $\tilde{\mathcal{N}}$ is compact in the weak * topology of $L^\infty(Q, R^{(n+2)(\nu+1)})$, it follows that $H^* \in \tilde{\mathcal{N}}$. Next, we shall show that $H^* \in \mathcal{N}$. This is equivalent to showing that $H^*(x, t) \in \tilde{F}(x, t)$ almost everywhere on \tilde{Q} . For each $\iota = 1, 2, \dots$, let

$$Q_\iota = \{(x, t) \in \tilde{Q} : u_\iota(x, t) \notin U\}.$$

Let $\tilde{Q} = \bigcup_{\iota=1}^\infty Q_\iota$. Since $u_\iota(x, t) \in U$ almost everywhere on \tilde{Q} for all integers $\iota \geq 1$, it follows that $|\tilde{Q}| = 0$. By hypothesis, \tilde{F} is a measurable set-valued function defined on \tilde{Q} . Thus, by virtue of Theorem 1 of [6, p. 857], there exists, for any $\varepsilon_1 > 0$, an open set $E_1 \subset \tilde{Q} \setminus \tilde{Q} = Q^1$ so that $|E_1| < \varepsilon_1$ and $\tilde{F}(\cdot, \cdot)$ is continuous in the Hausdorff metric topology on $Q^1 \setminus E_1$. Now, by Lusin's Theorem, there exists, for any $\varepsilon_2 > 0$, an open set $E_2 \subset Q^1$ so that $|E_2| < \varepsilon_2$ and $H^*(\cdot, \cdot)$ is continuous on $Q^1 \setminus E_2$. Let $E = E_1 \cup E_2$. Then, it is obvious that $|E| < \varepsilon_1 + \varepsilon_2 = \varepsilon_3$. Further, $\tilde{F}(\cdot, \cdot)$ is continuous in the Hausdorff metric topology on $Q^1 \setminus E = Q^2$ and $H^*(\cdot, \cdot)$ is continuous on Q^2 . Let $(x_0, t_0) \in Q^2$ be arbitrary but fixed. Then, for any $\varepsilon_4 > 0$, there exists a $\delta = \delta(\varepsilon_4)$ such that

$$\tilde{F}(x, t) \subseteq \tilde{F}^{\varepsilon_4}(x_0, t_0) \tag{9}$$

whenever $|(x, t) - (x_0, t_0)| < \delta$, where $\tilde{F}^{\varepsilon_4}(x, t)$ denotes the closed ε_4 -neighborhood of $\tilde{F}(x, t)$. We shall now show that $H^*(x_0, t_0) \in \tilde{F}(x_0, t_0)$. Since (x_0, t_0) is a continuity point of the function H^* , it follows that, for any measurable set $\Delta \subset Q^2$ containing (x_0, t_0) and contracting to the one point set $\{(x_0, t_0)\}$, we have

$$H^*(x_0, t_0) = \lim_{|\Delta| \rightarrow 0} \left\{ \frac{1}{|\Delta|} \int_\Delta \int_\Delta H^*(x, t) \, dx \, dt \right\}. \tag{10}$$

Let $M_\delta = \{(x, t) \in Q^2 : |(x, t) - (x_0, t_0)| < \delta\}$ and let $\mathcal{B} = \{B \in M_\delta : B \text{ is measurable and contains } (x_0, t_0)\}$.

Then, by virtue of the fact that $H(x_0, t_0, u_i(x_0, t_0)) \in \tilde{F}(x_0, t_0)$ for all positive integers i , it follows from (9) that, for any $B \in \mathcal{B}$,

$$H(x, t, u_i(x, t)) \in \tilde{F}^{\varepsilon_4}(x_0, t_0) \tag{11}$$

for all $(x, t) \in B$ and for all $i = 1, 2, \dots$. Now, we note that $\tilde{F}^{\varepsilon_4}(x_0, t_0)$ is convex and compact (and hence closed). Thus, it can be easily deduced from relation (11) that

$$\frac{1}{|B|} \int_B \int H(x, t, u_i(x, t)) \, dx \, dt \in \tilde{F}^{\varepsilon_4}(x_0, t_0) \tag{12}$$

for all $i = 1, 2, \dots$. Further, since $\tilde{F}^{\varepsilon_4}(x_0, t_0)$ is closed and $H(\cdot, \cdot, u_i(\cdot, \cdot))$ converges to $H^*(\cdot, \cdot)$ in the weak $*$ topology of $L^\infty(Q, R^{(n+2)(\nu+1)})$, it follows from (12) that

$$\lim_{i \rightarrow \infty} \frac{1}{|B|} \int_B \int H(x, t, u_i(x, t)) \, dx \, dt = \frac{1}{|B|} \int_B \int H^*(x, t) \, dx \, dt \in \tilde{F}^{\varepsilon_4}(x_0, t_0) \tag{13}$$

for any $B \in \mathcal{B}$. Again, by the fact that $\tilde{F}^{\varepsilon_4}(x_0, t_0)$ is closed, we deduce readily from (13) and (10) that if the set $B \in \mathcal{B}$ contracts to the one-point set $\{(x_0, t_0)\}$ then

$$H^*(x_0, t_0) = \lim_{|B| \downarrow 0} \left\{ \frac{1}{|B|} \int_B \int H^*(x, t) \, dx \, dt \right\} \in \tilde{F}^{\varepsilon_4}(x_0, t_0).$$

Since $\varepsilon_4 > 0$ is arbitrary and $\tilde{F}(x_0, t_0)$ is closed, $H^*(x_0, t_0) \in \tilde{F}(x_0, t_0)$. Further, $(x_0, t_0) \in Q^2$ is also arbitrary. Thus, $H^*(x, t) \in \tilde{F}(x, t)$ for all $(x, t) \in Q^2$. Again, since $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are arbitrary, $\varepsilon_3 > 0$ is arbitrary and hence it follows that $H^*(x, t) \in \tilde{F}(x, t)$ almost everywhere in \bar{Q} . At this stage, we can easily convince ourselves the validity of the lemma by using the definitions of the \tilde{F}_κ^* ($\kappa = 0, 1, \dots, \nu$) \mathcal{N} and Lemma 2.1.

Combining Theorem 2.2 and Lemma 2.2, we have

THEOREM 2.3. *Let $\{u_i\}_{i=1}^\infty \subset D$ and let $\{\phi(u_i)\}_{i=1}^\infty$ be the corresponding set of solutions of the system (1). Let the set-valued function \tilde{F} defined by equation (8) be measurable and let $\tilde{F}(x, t)$ be convex for each $(x, t) \in \bar{Q}$. Suppose that the assumptions (A) are satisfied. Then, there exist a subsequence $\{u_i\}_{i=1}^\infty \subset \{u_i\}_{i=1}^\infty$ and a control $u^* \in D$ so that*

$$\left. \begin{aligned} \phi(u_i) &\rightarrow \phi(u^*) \\ \phi_{x_i}(u_i) &\rightarrow \phi_{x_i}(u^*) \quad i = 1, \dots, n \end{aligned} \right\} \text{uniformly on } \bar{Q},$$

$$\left. \begin{aligned} \phi_j(u_i) &\rightarrow \phi_j(u^*) \\ \phi_{x_i x_j}(u_i) &\rightarrow \phi_{x_i x_j}(u^*), \quad i, j = 1, \dots, n \end{aligned} \right\} \text{weakly in } L^\lambda(Q)$$

as $\iota \rightarrow \infty$, where $\lambda > n + 2$ and $\phi(u^*)$ is the unique solution of system (1) with u replaced by u^* .

LEMMA 2.3. *Let the function Y satisfy the condition (i) of the assumptions \mathcal{G} . If $\{u_i\}_{i=1}^\infty \subset D$ and u_i converges in the weak * topology of $L^\infty(Q, R^r)$ to $u^* \in D$, then there exists a subsequence $\{u_i(\cdot, \cdot)\}_{i=1}^\infty \subset \{u_i(\cdot, \cdot)\}_{i=1}^\infty$ so that*

$$\lim_{\iota \rightarrow \infty} \int_Q \int Y(x, t, u_i(x, t)) \, dx \, dt \geq \int_Q \int Y(x, t, u^*(x, t)) \, dx \, dt \tag{14}$$

PROOF. By hypothesis, the sequence $\{u_i\}_{i=1}^\infty \subset D$ converges to $u^* \in D$ in the weak * topology of $L^\infty(Q, R^r)$. Since $|Q| < \infty$, the sequence also converges, in particular, weakly in $L^1(Q, R^r)$ to u^* . In view of the second part of the condition (i) of the assumptions \mathcal{G} and the fact that U is compact, we can easily show that the set defined by

$$\{Y(\cdot, \cdot, u(\cdot, \cdot)) : u \in D\}$$

satisfies the sufficient conditions stated in [1, Corollary 11, p. 294] and hence it is compact in the weak topology of $L^1(Q, R^r)$. Thus, there exists a subsequence $\{Y(\cdot, \cdot, u_i(\cdot, \cdot))\}_{i=1}^\infty \subset \{Y(\cdot, \cdot, u_i(\cdot, \cdot))\}_{i=1}^\infty$ so that it converges to $Y^*(\cdot, \cdot)$ in the weak topology of $L^1(Q, R^r)$. Obviously, $\{u_i\}_{i=1}^\infty \subset \{u_i\}_{i=1}^\infty$ also converges weakly in $L^1(Q, R^r)$ to u^* . Thus, the sequence

$$\left\{ \begin{bmatrix} u_i(\cdot, \cdot) \\ Y(\cdot, \cdot, u_i(\cdot, \cdot)) \end{bmatrix} \right\}_{i=1}^\infty \text{ converges to } \begin{bmatrix} u^*(\cdot, \cdot) \\ Y^*(\cdot, \cdot) \end{bmatrix}$$

weakly in $L^1(Q, R^{r+1})$. By the Banach–Saks Theorem [7, p. 80], we can extract a subsequence

$$\left\{ \begin{bmatrix} u_k^k(\cdot, \cdot) \\ Y(\cdot, \cdot, u_k^k(\cdot, \cdot)) \end{bmatrix} \right\}_{k=1}^\infty \subset \left\{ \begin{bmatrix} u_i(\cdot, \cdot) \\ Y(\cdot, \cdot, u_i(\cdot, \cdot)) \end{bmatrix} \right\}_{i=1}^\infty$$

so that

$$\frac{1}{\nu} \sum_{k=1}^\nu \begin{bmatrix} u_k^k(\cdot, \cdot) \\ Y(\cdot, \cdot, u_k^k(\cdot, \cdot)) \end{bmatrix} \text{ converges to } \begin{bmatrix} u^*(\cdot, \cdot) \\ Y^*(\cdot, \cdot) \end{bmatrix}$$

in the norm of $L^1(Q, R^{r+1})$. This, in turn, implies that $(1/\nu) \sum_{k=1}^\nu u_k^k(\cdot, \cdot)$ and $(1/\nu) \sum_{k=1}^\nu Y(\cdot, \cdot, u_k^k(\cdot, \cdot))$ converge, respectively, to $u^*(\cdot, \cdot)$ in the norm of $L^1(Q, R^r)$ and $Y^*(\cdot, \cdot)$ in the norm of $L^1(Q, R^1)$ as $\nu \rightarrow \infty$.

Now, by hypothesis, $Y(x, t, \cdot)$ is convex in U for each $(x, t) \in \bar{Q}$, it follows that

$$\begin{aligned} \frac{1}{\nu} \sum_{k=1}^{\nu} \int_Q \int Y(x, t, u_k^k(x, t)) dx dt &= \int_Q \int \frac{1}{\nu} \sum_{k=1}^{\nu} Y(x, t, u_k^k(x, t)) dx dt \\ &\geq \int_Q \int Y\left(x, t, \frac{1}{\nu} \sum_{k=1}^{\nu} u_k^k(x, t)\right) dx dt \end{aligned} \tag{15}$$

For ease in references, the second part of the condition (i) of the assumptions \mathcal{G} is recalled below. There exists *some* constant $\sigma \geq 1$ so that

$$|Y(x, t, v)| \leq g_1(x, t) + \omega |v|^\sigma \tag{16}$$

for almost all $(x, t) \in Q$ and for all $v \in R^r$.

Let us first consider the case when $\sigma = 1$ (the case when $\sigma > 1$ will be considered later). In this case, we obtain immediately from inequality (16) that

$$\int_Q \int |Y(x, t, u(x, t))| dx dt \leq \int_Q \int g_1(x, t) dx dt + \omega \int_Q |u(x, t)| dx dt. \tag{17}$$

Since $|Q| < \infty$ and U is compact, it is clear that Y is an operator mapping from $L^1(Q, R^r)$ into $L^1(Q, R^1)$. Thus, it follows from an obvious generalization of Theorem 2.1 of [4, p. 22] that Y is a continuous operator from $L^1(Q, R^r)$ into $L^1(Q, R^1)$. Since the sequence $\{(1/\nu) \sum_{k=1}^{\nu} u_k^k(\cdot, \cdot)\}_{\nu=1}^{\infty}$ converges to u^* in the norm of $L^1(Q, R^r)$ as $\nu \rightarrow \infty$, it follows that

$$\lim_{\nu \rightarrow \infty} \int_Q \int Y\left(x, t, \frac{1}{\nu} \sum_{k=1}^{\nu} u_k^k(x, t)\right) dx dt = \int_Q \int Y(x, t, u^*(x, t)) dx dt. \tag{18}$$

Next, we recall that the sequence $\{(1/\nu) \sum_{k=1}^{\nu} Y(\cdot, \cdot, u_k^k(\cdot, \cdot))\}_{\nu=1}^{\infty}$ converges to $Y^*(\cdot, \cdot)$ in the norm of $L^1(Q, R^1)$ as $\nu \rightarrow \infty$. Thus,

$$\lim_{\nu \rightarrow \infty} \frac{1}{\nu} \sum_{k=1}^{\nu} \int_Q \int Y(x, t, u_k^k(x, t)) dx dt = \int_Q \int Y^*(x, t) dx dt. \tag{19}$$

Further, since the sequence $\{Y(\cdot, \cdot, u_k^k(\cdot, \cdot))\}_{k=1}^{\infty}$ converges to $Y^*(\cdot, \cdot)$ weakly in $L^1(Q, R^1)$ and $|Q| < \infty$, it is obvious that

$$\lim_{i \rightarrow \infty} \int_Q \int Y(x, t, u_i^k(x, t)) dx dt = \int_Q \int Y^*(x, t) dx dt. \tag{20}$$

Thus, by combining (15), (19), (20) and (18), we obtain inequality (14) for the case when $\sigma = 1$.

To complete the proof, we shall show that the lemma remains valid when $\sigma > 1$. In fact, the proof is similar to that given for the case when $\sigma = 1$ with only the following three minor modifications.

- (i) replace [1, Corollary 11, p. 294] by [1, Corollary 4, p. 289];
- (ii) replace $L^1(Q, R^1)$ and $L^1(Q, R^r)$ by $L^\sigma(Q, R^1)$ and $L^\sigma(Q, R^r)$ respectively everywhere; and
- (iii) replace inequality (17) by the following inequality, which is obtained readily from inequality (16),

$$\int_Q \int |Y(x, t, u(x, t))| dx dt \leq \int_Q \int g_1(x, t) dx dt + \omega \int_Q \int |u(x, t)|^\sigma dx dt.$$

Thus, the proof is complete.

3. Existence theorems for optimal controls

In this section, we shall present two results on the existence of optimal controls for the problem P in Theorems 3.1 and 3.2. The main tools for Theorem 3.1 are Theorem 2.2 and Lemma 2.3. On the other hand, Theorem 3.2 is proved on a basis of Theorem 2.3, while Theorem 2.3 was obtained by using Theorem 2.2 and Lemma 2.2. Note that, as in [9, Theorems 4.1 and 4.2], the coefficients and forcing terms of system (1) considered in Theorem 3.1 are assumed linear in the control variables. However, this result contains those reported in [9, Theorems 4.1 and 4.2] as special cases. In Theorem 3.2, the coefficients and forcing terms of system (1) are allowed to be non-linear in the control variables at the expense of dropping the control variables in the cost integrand.

THEOREM 3.1. *Consider the problem P . Suppose that the assumptions (A) and \mathcal{G} are satisfied. Further, it is assumed that the coefficients and forcing terms of the system (1) are linear in the control variables. Then the corresponding problem P has a solution.*

PROOF. By the assumptions \mathcal{G} and the fact that U is compact and convex, it is easily deduced that $\text{Inf}_{u \in D} J(u) > -\infty$. Clearly, there exists a sequence $\{u_i\}_{i=1}^\infty \subset D$ so that

$$\lim_{i \rightarrow \infty} J(u_i) = \text{Inf}_{u \in D} J(u).$$

Since D is a w^* -compact subset of $L^\infty(Q, R^r)$, there exists a subsequence of $\{u_i\}_{i=1}^\infty$, which is denoted by the original sequence, so that it converges to $u^* \in D$ in the weak $*$ topology of $L^\infty(Q, R^r)$. By hypothesis, the coefficients and forcing terms of the system (1) are linear in the control variables. Thus, it can be easily shown from Theorem 2.2 that $\phi(u_i)$ and $\phi_{x_i}(u_i)$ ($i = 1, \dots, n$) converge uniformly on \bar{Q} to $\phi(u^*)$ and $\phi_{x_i}(u^*)$ ($i = 1, \dots, n$) respectively.

B

Now, let J_1 denote the first component of J . Then, by virtue of the condition (i) of \mathcal{G} , it follows from Lemma 2.3 that we can extract a subsequence $\{u_i\}_{i=1}^\infty \subset \{u_i\}_{i=1}^\infty$ so that

$$J_1(u^*) \leq \lim_{i \rightarrow \infty} J_1(u_i). \tag{21}$$

On the other hand, we can easily verify from the condition (ii) of \mathcal{G} and inequality (4) of Theorem 2.1 that the set defined by

$$H = \{G_2(\cdot, \cdot, \phi(u)(\cdot, \cdot), \phi_x(u)(\cdot, \cdot)): u \in D\}$$

satisfied the sufficient conditions stated in Corollary 11 of [1, p. 294]. Thus, it is compact in the weak topology of $L^1(Q, R^1)$. Next, by virtue of the condition (ii) of the assumptions \mathcal{G} , inequality (3) of Theorem 2.1 and the fact that $|Q| < \infty$, we can readily show that G_2 is an operator mapping from $L^\lambda(Q, R^{n+1})$ into $L^1(Q, R^1)$. Thus, it follows from an obvious generalization of [4, Theorem 2.1, p. 22] that G_2 is a continuous operator from $L^\lambda(Q, R^{n+1})$ into $L^1(Q, R^1)$. Further, since $\phi(u_i)$ and $\phi_{x_i}(u_i)$ ($i = 1, \dots, n$) converge uniformly on \bar{Q} to $\phi(u^*)$ and $\phi_{x_i}(u^*)$ ($i = 1, \dots, n$) respectively and $|Q| < \infty$, it is clear that $\phi(u_i)$ and $\phi_{x_i}(u_i)$ ($i = 1, \dots, n$) converge, respectively, to $\phi(u^*)$ and $\phi_{x_i}(u^*)$ ($i = 1, \dots, n$) in the norm of $L^\lambda(Q)$ as $i \rightarrow \infty$, where $n + 2 < \lambda < \infty$. Thus,

$$J_2(u_i) = \int_Q \int G_2(x, t, \phi(u_i)(x, t), \phi_{x_i}(u_i)(x, t)) dx dt$$

converges to

$$J_2(u^*) = \int_Q \int G_2(x, t, \phi(u^*)(x, t), \phi_{x_i}(u^*)(x, t)) dx dt$$

as $i \rightarrow \infty$.

Combining the above relation with inequality (21), we have

$$J(u^*) \leq \lim_{i \rightarrow \infty} J(u_i).$$

However,

$$\lim_{i \rightarrow \infty} J(u_i) = \inf_{u \in D} J(u).$$

Thus,

$$J(u^*) = \inf_{u \in D} J(u)$$

and the proof is complete.

REMARK 3.1. Note that Theorem 3.1 contains Theorems 4.1 and 4.2 of Reference 9 as special cases.

In the following theorem, we shall show that the problem P admits a solution without requiring the coefficients and forcing terms of the system dynamic to be

linear in the control variables if the first term of the cost integrand is the zero-valued function.

THEOREM 3.2. *Consider the problem P with $G_1 \equiv 0$. Let F_κ , \hat{F}_κ and \tilde{F} be as defined in equations (6), (7) and (8) respectively. Suppose that the assumptions (A) and the condition (ii) of \mathcal{G} are satisfied and that $\tilde{F}(x, t)$ is convex for each $(x, t) \in \bar{Q}$. Then, if \tilde{F} is a measurable set valued function, the corresponding problem P has a solution.*

PROOF. Let $\{u_i\}_{i=1}^\infty \subset D$ be a sequence so that

$$\lim_{i \rightarrow \infty} J(u_i) = \text{Inf}_{u \in D} J(u).$$

From Theorem 2.3, there exists a subsequence $\{u_{i_\nu}\}_{\nu=1}^\infty \subset \{u_i\}_{i=1}^\infty$ and a control $u^* \in D$ such that

$$\left. \begin{array}{l} \phi(u_{i_\nu}) \rightarrow \phi(u^*), \\ \phi_{x_i}(u_{i_\nu}) \rightarrow \phi_{x_i}(u^*), \quad i = 1, \dots, n, \end{array} \right\} \text{uniformly in } \bar{Q}$$

as $\nu \rightarrow \infty$. However, the objective functional J of this theorem is identical to J_2 of Theorem 3.1. Thus, it follows from precisely the same argument as that given for the corresponding part of the proof of Theorem 3.1 that $J(u_{i_\nu}) \rightarrow J(u^*)$ as $\nu \rightarrow \infty$. Since $\{u_i\}_{i=1}^\infty \subset D$ is a minimizing sequence, we conclude immediately that $J(u^*) = \text{Inf}_{u \in D} J(u)$. This completes the proof.

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