STANDARD REPRESENTATIONS OF SIMPLE LIE ALGEBRAS

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Summary. Let L be any simple finite-dimensional Lie algebra (defined over the field K of complex numbers). Cartan's theory of weights is used to define sets of (algebraic) representations of L that can be characterized in terms of left ideals of the universal enveloping algebra of L. These representations, called standard, generalize irreducible representations that possess a dominant weight. The newly obtained representations are all infinite-dimensional. Their study is initiated here by obtaining a partial solution to the problem of characterizing them by means of sequences of elements in K.

Notation. K denotes the field of complex numbers. Whenever the word *algebra* is used, it will mean *algebra over* K. The words *basis*, *dimension*, and *dual*, when applied to an algebra, will refer to the underlying linear space of the algebra. An associative algebra will always be assumed to have a multiplicative unit. The multiplication in a Lie algebra will be denoted by square brackets and that in an associative algebra by juxtaposition (with the usual conventions).

1. Introduction. With any Lie algebra L an associative algebra $U(L)$, called the universal enveloping algebra of L, can be associated (**4**). It is defined as the algebra $T(L)/I$, where $T(L)$ is the tensor algebra of the space L and I is the two-sided ideal of $T(L)$ generated by all elements of the form $xy - yx - [x, y]$ for $x, y \in L \subseteq T(L)$. The space L may be considered as a subspace of $U(L)$. Any representation of L induces a representation of $U(L)$ and any representation of $U(L)$ subduces a representation of L. This correspondence is one-to-one and preserves irreducibility. Complete information about the irreducible representations of a Lie algebra L may therefore be obtained by studying the irreducible representations of the associative algebra $U(L)$.

The irreducible representations of any associative algebra A may be analysed in terms of the inherent structure of A (**7**, Chapter 2, §2). For that purpose the left regular representation $\rho_{A/J}$ of A modulo J, for any left ideal J of A, is defined as the mapping $a \to \rho_{A/J}(a)$ on A, where $\rho_{A/J}(a)$ is the operator on the quotient space A/J given by $\rho_{A/J}(a) \,.\, (b + J) = (ab) + J$, for each $b \in A$. The representation $\rho_{A/J}$ is irreducible if and only if J is a maximal left ideal of A. Now let (ρ, V) be any irreducible representation of A. Let V_0 be any one-

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dimensional subspace of V and let J_0 be the annihilator ideal of V_0, that is, $J_0 = \{x \in A \mid \rho(x)\mathbf{v} = 0$ for any non-zero $\mathbf{v} \in V_0\}$. Then J_0 is a maximal left ideal of A and ρ is equivalent to the left regular representation of A modulo J_0.

Since, in general, different one-dimensional subspaces of a representation space V have different annihilator ideals, the above correspondence between the set of inequivalent irreducible representations of A and the set of maximal left ideals of A is not one-to-one. However, if for a set $\mathbf{S}$ of irreducible representations of A, there can be specified a mapping ψ that associates with each $(\rho, V) \in \mathbf{S}$ a unique one-dimensional subspace V_0 of V, then the correspondence $(\rho, V) \rightarrow J_0$ between $\mathbf{S}$ and the set $\mathbf{J}(\mathbf{S}, \psi)$ of corresponding maximal left ideals of A will be one-to-one. The characterization problem for $\mathbf{S}$ will then be equivalent to that for $\mathbf{J}(\mathbf{S}, \psi)$. The following problems thus arise:

(a) to define pairs of the form $(\mathbf{S}, \psi)$, and

(b) to characterize the elements of the corresponding sets $\mathbf{J}(\mathbf{S}, \psi)$.

When A has the form $U(L)$, where L is a simple Lie algebra satisfying $1 < \dim L < \infty$, Cartan's theory of weights may be used to define suitable pairs $(\mathbf{S}, \psi)$.

Chevalley (**2**) and Harish-Chandra (**3**) introduced and characterized the set $\mathbf{S}(\emptyset)$ (our notation) of all those irreducible representations of $U(L)$ that possess a dominant weight. (For a very clear and concise formulation of this work, see (**8**, Exposé 17).) With the set $\mathbf{S}(\emptyset)$ we may associate the mapping $\psi(\emptyset)$ that maps each $(\rho, V) \in \mathbf{S}(\emptyset)$ to the (one-dimensional) subspace of V that belongs to the dominant weight of ρ. In this paper the concept of a dominant weight is generalized (Definition (3.1)), in a natural way, to give rise to more general pairs $(\mathbf{S}(\Delta), \psi(\Delta))$, where the index Δ ranges through all subsets of a given set of simple roots of L. The sets $\mathbf{S}(\Delta)$, for $\Delta \neq \emptyset$, thus consist of irreducible infinite-dimensional representations that do not possess dominant weights. (They may, for instance, be seen to include the "continuous classes" $C_q^{\,0}$ and $C_q^{\,\frac{1}{2}}$ of irreducible representations of (a real form of) A_1 obtained by Bargmann (**1**).) The author calls the elements of $\mathbf{S}(\Delta)$ Δ-*standard* representations of $U(L)$. The problem of characterizing these representations is solved only partially. A formula is established (§4) by means of which, for each given L and Δ, the elements of a certain subset $\mathbf{S}'(\Delta)$ (conjectured to be not proper) of $\mathbf{S}(\Delta)$ may be characterized. In §§5 and 6 this characterization process is carried through explicitly for all cases covered by: (i) L arbitrary, $|\Delta| = 1$, and (ii) $L = A_2$, $|\Delta| = 2$. For the cases (i) it is shown that $\mathbf{S}'(\Delta) = \mathbf{S}(\Delta)$, so that when Δ consists of a single element, the problem of characterizing the Δ-standard representations of $U(L)$ is completely solved. Two conjectures, borne out by these examples, are stated in the final §7.

The methods of proof and exposition used in this paper are based upon those in (**8**, Exposé 17). As to the extent of our generalization we note that for irreducible representations that possess a dominant weight, condition (b) of Theorem (4.4) is trivially satisfied while condition (c) of the same theorem is satisfied vacuously.

The author takes this opportunity to acknowledge that the paper **(2)** by Chevalley has been the main stimulus for the study presented here. He is greatly indebted to his Ph.D. supervisor, Professor A. J. Coleman (Queen's University), for his interest and suggestions.

2. Preliminaries. Let L be a simple Lie algebra satisfying $1 < m = \dim L < \infty$ and let D be a fixed Cartan subalgebra of L. Let $n = \dim D$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots of L (defined with respect to D) and let P be the corresponding set of positive roots of L. The set S forms a basis of D^*, the space dual to D.

The space L has a basis of the form (É. Cartan)

$$B = \{e_i | \ i = 1, \ldots, m\} = \{d_\alpha, q_\beta, p_\beta | \ \alpha \in S, \beta \in P\},$$

where the d_α ($\alpha \in S$) form a basis of D, where an element p_β (or q_β), for each $\beta \in P$, belongs (as an eigenvector of the restriction to D of the adjoint representation of L) to the positive root β (or, respectively, the negative root $-\beta$), and where they satisfy certain well-known commutation relations **(8**, p. 17–01**)**. We may assume (as we do in Lemma (2.7) in particular) that these elements are so normed (with respect to the Cartan–Killing form) that $[p_\alpha, q_\alpha] = d_\alpha$, $[d_\alpha, p_\alpha] = 2p_\alpha$, and $[d_\alpha, q_\alpha] = -2q_\alpha$ for all $\alpha \in S$. When $\alpha = \alpha_i$ ($i = 1, \ldots, s$), we shall write d_i, p_i, q_i for $d_\alpha, p_\alpha, q_\alpha$, respectively.

Let $e_1 < \ldots < e_m$ denote any linear order on B. Then the Birkhoff–Witt theorem **(8**, p. 1–07, théorème 1′**)** states that a basis of $U(L)$ is given by the set of all elements u of the form

$$u = e_1^{u(1)} \ldots e_m^{u(m)} \tag{2.1}$$

where the exponents $u(j)$ are integers $\geqslant 0$.

The following lemma is useful for exhibiting different bases of $U(L)$. It is a direct consequence of **(8**, p. 1–06, théorème 1**)**, which is a basis independent form of the Birkhoff–Witt theorem.

(2.2) LEMMA. *Any set of elements which can be derived (bijectively) from the set of all elements of the form* (2.1) *by allowing the factors* $e_i (i = 1, \ldots, m)$ *to commute appears as a basis of* $U(L)$.

Let F denote any basis of $U(L)$ as determined by (2.2).

For any $u \in F$ and any $\beta \in P$, let $n(\beta)$ (or $m(\beta)$) be the number of times that the factor p_β (or, respectively, q_β) is contained in u. Then the *mass* $\mu\{u\}$ of u is defined to be the element $\sum_{\beta \in P} (n(\beta) - m(\beta))\beta$ of D^*. If an element z in $U(L)$ is such that its linear decomposition with respect to the basis F consists of summands of equal mass, then the mass $\mu\{z\}$ of z is defined to be the mass of any one of its summands. The masses of elements of $U(L)$ occur as the points of the integral lattice through the origin in D^*, that is, they are the elements $\sum_{i=1}^{n} m_i \alpha_i$ where the m_i are integers. For any ξ in D^* of this form the set of elements of $U(L)$ that have mass ξ forms an (infinite-dimensional) subspace M_ξ

of $U(L)$. We may state: (i) $U(L)$ is the direct sum of the mass spaces M_ξ, and (ii) $z \in M_\xi, z' \in M_\eta \Rightarrow zz' \in M_{\xi+\eta}$. (i) clearly holds while (ii) follows from the fact that the linear decomposition of zz' with respect to the basis F proceeds by application of the commutation relations of L and these are mass-preserving. (i) and (ii) state that $U(L)$ is a graded algebra, graded by the mass.

From (ii) it follows that the mass of a product is independent of the order of its factors, which implies that the existence and the value of the mass of an element of $U(L)$ do not depend upon the particular basis F chosen for $U(L)$.

An element z in $U(L)$ is called a *cycle* in $U(L)$ if $\mu\{z\} = 0$. The space M_0 forms a subalgebra of $U(L)$ which will be called the *cycle algebra* of L.

By induction on the lengths of words we verify that

$$[d, x] = \xi(d)x \tag{2.3}$$

for any $x \in M_\xi$ and any $d \in D$ (where $[d, x]$ denotes $dx - xd$). From (2.3) it is seen that M_0 is the centralizer algebra in $U(L)$ of the universal enveloping algebra $U(D)$ of D.

The significance of the (mass-)graded structure of $U(L)$ as far as the theory of (weight-possessing) irreducible representations of $U(L)$ is concerned is due to equation (2.4) below. If an irreducible representation (ρ, V) of $U(L)$ has at least one weight, then V is the direct sum of weight spaces V_μ. For any fixed weight λ of ρ and any fixed non-zero vector $\mathbf{w}$ in V_λ, the irreducibility of ρ implies that every vector $\mathbf{v}$ in V has the form $\rho(u)\mathbf{w}$ with $u \in U(L)$. Using (2.3) it is seen that this condition implies that

$$V_\mu = \rho(M_{\mu-\lambda})\mathbf{w} \tag{2.4}$$

for any weight μ of ρ. In particular, (2.4) implies that the weights of ρ occur as a subset of the integral lattice through λ in D^*, that is, each weight of ρ has the form $\lambda + \sum_{i=1}^n m_i\, \alpha_i$, where the m_i are integers.

We now discuss the concept of primitivity as defined for elements of a basis F of $U(L)$. For any $u \in F$, let $u(j)$ be the number of times that the factor e_j is contained in $u (j = 1, \ldots, m)$. With any $u \in F$ we associate its (ordered) *m-tuple of exponents*: $[u] = (u(j)) = (u(1), \ldots, u(m))$. This defines a bijective mapping from F to the set $I(m)$ of all m-tuples of integers $\geqslant 0$. In $I(m)$ we introduce the following partial order: $\mathbf{m} = (m(j)) \geqslant \mathbf{n} = (n(j))$ if and only if $m(j) \geqslant n(j)$ for each $j = 1, \ldots, m$, while $\mathbf{m} > \mathbf{n}$ if and only if $\mathbf{m} \geqslant \mathbf{n}$ and $\mathbf{m} \neq \mathbf{n}$.

(2.5) DEFINITION. *For any $u, u' \in F$, u is said to* contain (*or* properly contain) *u' if $[u] \geqslant [u']$* (or $[u] > [u']$). *An element $u \neq 1$ of F is called a* primitive element of *F if it does not properly contain a cycle $\neq 1$.*

The proof of the following lemma has been communicated to the author by G. E. Burger:

(2.6) LEMMA. *For any mass space M_ξ the number of primitive elements of F belonging to M_ξ is finite.*

Proof. We first note that for an element u in F of mass ξ the condition of primitivity is equivalent to the condition that u does not properly contain any element $u' \neq 1$ satisfying $\mu\{u'\} = \xi$. Now let W be any subset of $B = \{e_1, \ldots, e_m\}$. Let $E(W)$ be the set of all those primitive elements z of F that belong to M_ξ and that have the property that each element of W occurs with a non-zero exponent in z while all other elements of B occur with zero exponents. If $E(W) \neq \emptyset$, let $x \in E(W)$. Let a_1 be the highest exponent occurring in the m-tuple $[x]$ of exponents of x. We now construct, recursively, a sequence of positive integers $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_r (r = |W|)$ such that any element of $E(W)$ has at least t non-zero exponents $\leqslant a_t$ for each $t = 1, 2, \ldots, r$. The first step has just been described (that a_1 has the desired property follows from the condition of primitivity on the elements of $E(W)$). For the kth step we assume that a_{k-1} has been found. We then form $W^{(k-1)}$, the set of all subsets of W having $k - 1$ elements. The linear order on B induces a linear order on each of these subsets so that each element of $W^{(k-1)}$ may be regarded as a $(k - 1)$-tuple of elements of W. Define an *allowable* $(k - 1)$*-tuple of positive integers* to be a $(k - 1)$-tuple $(h_1, \ldots, h_{k-1})$ of integers with $0 < h_i \leqslant a_{k-1}$ for each i. To each element $(x_{i(1)}, \ldots, x_{i(k-1)})$ in $W^{(k-1)}$ we now assign consecutively all allowable $(k - 1)$-tuples of positive integers. For each such assignment, keeping $(x_{i(1)}, \ldots, x_{i(k-1)})$ momentarily fixed, consider the set of all elements of $E(W)$ in which $x_{i(j)}$ occurs with exponent h_j, for each $j = 1, \ldots, k - 1$. If this set is non-empty, let x be an element of it. (If it is empty, ignore it.) Write down the highest exponent appearing in $[x]$. Repeat this process for all allowable $(k - 1)$-tuples of positive integers, and for all elements of $W^{(k-1)}$. Let a_k be the greatest of all the positive integers that have been written down. We prove now that every element of $E(W)$ has at least k non-zero exponents $\leqslant a_k$. By the induction hypothesis every element of $E(W)$ has at least $k - 1$ non-zero exponents $\leqslant a_{k-1}$. Now, $a_k \geqslant a_{k-1}$. Therefore, if the above assertion is false, there must exist a $y \in E(W)$ such that $[y]$ has exactly $k - 1$ non-zero entries $\leqslant a_k$. This means that y has exponents $m_1, \ldots, m_{k-1}$ (occurring, respectively, with $y_{i(1)}, \ldots, y_{i(k-1)} \in W$, say) with $0 < m_i \leqslant a_{k-1} (i = 1, \ldots, k - 1)$ and all other non-zero entries of $[y] > a_k$. By the method of construction of a_k there exists a y' in $E(W)$ in which the elements $y_{i(1)}, \ldots, y_{i(k-1)}$ appear with the same exponents $m_1, \ldots, m_{k-1}$ but in which all other exponents are $\leqslant a_k$. Then $[y] > [y']$. But then $y \notin E(W)$ (by the condition of primitivity). This contradiction proves the assertion. It follows that a_r is an upper bound for the exponents appearing in any $[z]$, $z \in E(W)$. Therefore $E(W)$ is finite ($|E(W)| \leqslant a_r^r$). Since this is true for any $W \subseteq B$, the lemma is proved.

Finally, for later reference, we state:

(2.7) LEMMA. *For any simple root α of L, the following relations hold in $U(L)$:*

$$q_\alpha{}^t p_\alpha{}^t = \prod_{i=1}^{t} (q_\alpha p_\alpha - (t-i)(t-i+1+d_\alpha))$$

and

$$p_\alpha{}^t q_\alpha{}^t = \prod_{i=1}^{t} (q_\alpha p_\alpha - (t+1-i)(t-i-d_\alpha)),$$

for any integer $t > 0$.

Proof. Let q, p, d denote q_α, p_α, d_α, respectively. The relations follow by induction, writing $q^{t+1}p^{t+1} = q(q^t p)p^t$, $p^{t+1}q^{t+1} = p(p^t q)q^t$, and using the identities, also proved by induction:

$$q^t p = -t(t-1)q^{t-1} - tdq^{t-1} + pq^t$$

and

$$p^t q = -t(t-1)p^{t-1} + tdp^{t-1} + qp^t.$$

3. Standard representations of L. Let Δ denote any fixed subset of the set S of simple roots of L. Let $|\Delta| = s$. The simple roots of L will be considered to be numbered in such a way that $\Delta = \{\alpha_1, \ldots, \alpha_s\}$. Let Δ^+ denote the set of positive roots of L spanned by Δ and let Δ^0 be the complement in P of Δ^+.

(3.1) DEFINITION. *An irreducible representation ρ of $U(L)$ (or, equivalently, of L) is called* Δ-standard *if it possesses a weight λ such that:*

(i) *for each $\alpha \in \Delta$, the α-ladder through λ is doubly infinite, that is, $\lambda + m\alpha$ occurs as a weight of ρ for every integer m,*

(ii) *the $\beta \in \Delta^0$ are roots that annihilate λ, that is, ρ contains no weight of the form $\lambda + \beta$ $(\beta \in \Delta^0)$,*

(iii) *the λ weight space of ρ has dimension equal to one,*

(iv) *if $s \geqslant 1$, then for $j = 1, \ldots, s$, the coefficients $k(j)$ of*

$$\lambda = \sum_{i=1}^{n} k(i)\alpha_i \qquad (k(i) \in K)$$

satisfy $0 \leqslant$ *(real part of $k(j)$)* < 1. *A weight λ of ρ satisfying* (i) *to* (iv) *is called a* characteristic weight of ρ. *The order s of Δ is called the* order of ρ. *A representation of $U(L)$ is called* standard *if it is Δ-standard, for some $\Delta \subseteq S$.*

The set of all the Δ-standard representations of $U(L)$ will be denoted by $\mathbf{S}(\Delta)$.

For $s = 0$, Δ equals the empty set $\emptyset$ and $\Delta^0 = P$. The set $\mathbf{S}(\emptyset)$ consists of all those irreducible representations of $U(L)$ that possess a *dominant weight* (**8**, Exposé 17). ((3.1) (iii) then follows as a property.) The characterization problem for $\mathbf{S}(\emptyset)$ has been completely solved (see **8**, Exposé 17): each $\rho \in \mathbf{S}(\emptyset)$ has a unique dominant weight λ and the corresponding mapping $\rho \to \lambda$ from $\mathbf{S}(\emptyset)$ to D^* is bijective.

It follows from (3.1) (i) that if $\Delta \neq \emptyset$, then each Δ-standard representation of $U(L)$ is infinite-dimensional.

Remark. We briefly comment on condition (iv) of (3.1). If Conjectures (7.1) and (7.2) are true, then (for $s \geqslant 1$) there will exist an infinity of weights λ satisfying (3.1) (i) to (iii), while exactly one of them will satisfy (3.1) (iv). It is therefore anticipated that condition (iv) of (3.1) will not act as a restriction on the representations to be considered, but simply as a prescription for associating a unique weight with each representation.

In analysing the Δ-standard representations of $U(L)$ the structure of $U(L)$ is to be studied in terms of the favoured role accorded in Definition (3.1) to the subset Δ of S.

From (2.1) it follows that a basis of $U(L)$ is given by the set of all elements of the form:

$$(3.2)\qquad \prod_{\gamma} q_\gamma^{m(\gamma)} \left[\prod_{i=1}^{s} d_i^{m_i} \prod_{\beta} q_\beta^{m(\beta)} \prod_{\beta} p_\beta^{n(\beta)} \right] \prod_{i=s+1}^{n} d_i^{m_i} \prod_{\gamma} p_\gamma^{n(\gamma)}$$

where the ranges of β, γ are Δ^+, Δ^0, respectively, where the exponents are integers $\geqslant 0$ and where each of the product signs respects a fixed order on its index set.

$L'(\Delta)$, or simply L', will denote the subalgebra of L generated (as a Lie algebra) by the set $\{q_\alpha, p_\alpha | \alpha \in \Delta\}$. A basis for L' is given by the set $\{d_\alpha, q_\beta, p_\beta | \alpha \in \Delta, \beta \in \Delta^+\}$ (the proof being similar to the proof that U_+ is an algebra, in (**8**, p. 17–01)). A basis for $U(L')$ is thus given by the set of all elements of the form as given within square brackets in (3.2). The *cycle algebra* $C(L')$ *of* L' is the subalgebra of $U(L')$ whose underlying linear space is the space $M_0 \cap U(L')$. A basis for $C(L')$ is provided by the set of all those basis elements of $U(L')$ that have mass equal to zero. If an element of $U(L')$ (or $C(L')$) is a primitive element of the basis (3.2), it is called a *primitive element* (or, respectively, a *primitive cycle*) of L'.

By Lemma (2.6) the number of primitive cycles of L' is finite. Let π denote the set of all the primitive cycles of L' other than the elements $d_i (i = 1, \ldots, s)$. For any element λ of D^* and any mapping $\mathbf{k}$ from π to K, $J(\lambda, \mathbf{k})$ will denote the left ideal of $U(L)$ generated by the elements

$$(3.3)\qquad \begin{array}{rl} d_i - \lambda(d_i) \,.\, 1 & (i = 1, \ldots, n), \\ p_\gamma & (\gamma \in \Delta^0), \\ c - \mathbf{k}(c) \,.\, 1 & (c \in \pi). \end{array}$$

(3.4) LEMMA. *Any Δ-standard representation ρ of $U(L)$ is equivalent to a left regular representation $\rho_{U/J}$ of $U(L)$ modulo J, where J is a maximal left ideal of $U(L)$ containing a left ideal of the form $J(\lambda, \mathbf{k})$.*

Proof. Let λ be a characteristic weight of ρ and let J be the annihilator ideal of the one-dimensional λ weight space of ρ. Then $\rho \simeq \rho_{U/J}$ (§1). Since ρ, possessing at least one weight, is not the zero representation, it follows that $J \neq U(L)$. Since ρ is irreducible, J is maximal. By Definition (3.1) (ii) and (iii), and the form of the generators (3.3) of $J(\lambda, \mathbf{k})$, it follows (using (2.4)) that J contains an ideal of the form $J(\lambda, \mathbf{k})$. This proves the lemma.

By (3.4) we have that to each Δ-standard representation ρ of $U(L)$ there corresponds a couple $(\lambda, \mathbf{k})$, where λ is an element of D^* and where $\mathbf{k}$ is a mapping from π to K. The correspondence is defined by:

$$f\colon \rho \to J \to J(\lambda, \mathbf{k}) \to (\lambda, \mathbf{k}) \tag{3.5}$$

where λ is a characteristic weight of ρ, and J, containing $J(\lambda, \mathbf{k})$, is the annihilator ideal of the λ weight space of ρ. We shall show (Theorem (3.12)) that f is an injective mapping, so that the Δ-standard representations of $U(L)$ will be characterized by the couples $(\lambda, \mathbf{k})$ associated with them.

(3.6) THEOREM. *For $\lambda \in D^*$ and a mapping $\mathbf{k}$ from π to K let $J(\lambda, \mathbf{k}) \neq U(L)$. Let J be any proper left ideal of $U(L)$ containing $J(\lambda, \mathbf{k})$. Let ρ^* denote the left regular representation of $U(L)$ modulo J. Then:*

(a) *each weight of ρ^* has the form:*

$$\lambda + \sum_{j=1}^{s} m_j \alpha_j - \sum_{i=s+1}^{n} m'_i \alpha_i,$$

where the m'_i are integers $\geqslant 0$ and where the m_j are any integers,

(b) *the weight spaces of ρ^* are finite-dimensional, and*

(c) *the λ weight space of ρ^* has dimension equal to one.*

Proof. It follows from the form of the first two sets of generators (3.3) of $J(\lambda, \mathbf{k})$ that any element u of the form (3.2) reduces modulo $J(\lambda, \mathbf{k})$, and thus modulo J, to a K-multiple of an element

$$v = \prod\nolimits_{\gamma \in \Delta^0} q_\gamma^{m(\gamma)} x(u) \tag{3.7}$$

with $x(u) \in U(L')$, so that U/J is spanned by classes of the form $v + J$. Modulo J we have

$$\begin{aligned} dv &= vd + (\mu\{v\})(d)v && \text{for each } d \in D \text{ (by (2.3))} \\ &\equiv (\lambda + \mu\{v\})(d)v && \text{(since } d \equiv \lambda(d)\,.\,1 \bmod J(\lambda, \mathbf{k})). \end{aligned}$$

Since, in the expression for $\mu\{v\}$, the coefficients of the $\alpha_i (i = s + 1, \ldots, n)$ are all negative or zero, (a) is proved. By supposition, $J \neq U(L)$, that is, $1 \not\equiv 0 \bmod J$. Also, $1 + J \in (U/J)_\lambda$ (by the form of the first set of the generators (3.3) for $J(\lambda, \mathbf{k})$). The element 1 generates U/J (as left $U(L)$-module modulo J) and by (2.4) we have that for any weight $\lambda + \xi$ of ρ^*, the $(\lambda + \xi)$ weight space of ρ^* is spanned by the classes of the form $v + J$ where $\mu\{v\} = \xi$. From (3.7) we may state that

$$\xi = \mu\{v\} = -\sum\nolimits_{\gamma \in \Delta^0} m(\gamma)\gamma + \mu\{x(u)\}. \tag{3.8}$$

Comparing coefficients of the $\alpha_i (i = s + 1, \ldots, n)$ in (3.8), all the $m(\gamma)$ appear in the equations while no coefficients from $\mu\{x(u)\}$ appear (since $x(u) \in U(L')$). Since the $m(\gamma)$ are positive integers, it follows that for fixed ξ there are only a finite number of solutions $(m(\gamma))$ for the equations of the comparison. Thus, by (3.8), $\mu\{x(u)\}$ can assume only a finite number of values. It remains to be shown that for any fixed mass ζ, the solutions x of the equation $\zeta = \mu\{x\}$, where x is any basis element of $U(L')$, determine only a finite number of classes $x + J$. By allowing factors to commute, each element x of the form as given within square brackets in (3.2) and of mass ζ may be brought into the form

$$x = p(x) \prod_{c \in \pi} c^{m(c)} \prod_{i=1}^{s} d_i^{m_i}, \tag{3.9}$$

where $p(x)$ is a primitive element of L' having mass equal to ζ, where the exponents are integers $\geqslant 0$, and where the first product sign respects a fixed order on π. (For $\zeta = 0$, $p(x)$ is defined to be the constant 1.) By an application of Lemma (2.2), as restricted to the algebra $U(L')$, we find that *a basis for $M_\zeta \cap U(L')$ is provided by a set of elements of the form* (3.9). Since the primitive cycles of L' all reduce, modulo J, to multiples of 1, an application of Lemma (2.6) yields (b). For $\xi = 0$, the coefficients $m(\gamma)$ in (3.8) are all 0, so that $\mu\{x(u)\} = 0$. But for $\zeta = 0$ in (3.9), $p(x) = 1$. This proves (c) and the theorem.

For any Δ-standard representation ρ of L and for any characteristic weight λ of ρ, we have by Lemma (3.4) (and its proof) that Theorem (3.6) (a) and (b) hold for ρ and λ. Using this we may state:

(3.10) COROLLARY. *Each $\rho \in \mathbf{S}(\Delta)$ has exactly one characteristic weight and each weight space of ρ is finite-dimensional.*

Proof. We still need to prove the first statement. This follows from the fact that condition (iv) of Definition (3.1) uniquely determines λ within the weight set of ρ as given by Theorem (3.6) (a).

The mapping $\psi(\Delta)$, defined on $\mathbf{S}(\Delta)$ (see §1), may now be considered to be given by $\psi(\Delta)$: $\rho \rightarrow$ (the one-dimensional space belonging to the characteristic weight λ of ρ).

(3.11) COROLLARY. *If $\Delta \neq \Delta'$, then the sets $\mathbf{S}(\Delta), \mathbf{S}(\Delta')$ are disjoint.*

Proof. Let $\rho \in \mathbf{S}(\Delta)$, $\rho' \in \mathbf{S}(\Delta')$. By Theorem (3.6) (a) and condition (i) of Definition (3.1) the weight sets of ρ and ρ' are distinct, so that ρ and ρ' cannot be equivalent representations.

(3.12) THEOREM. *The Δ-standard representations ρ of $U(L)$ are characterized by the couples* $(\lambda, \mathbf{k})$ *that, by* (3.5), *are associated with them.*

Proof. Consider the correspondence f given by (3.5). From Corollary (3.10) and Definition (3.1) (iii) it follows that the first composant of f is a mapping. Since $\rho \simeq \rho_{U/J}$ (§1), this mapping is injective. That the last two composants of f are injective mappings is a consequence of the following two assertions. These assertions are mild extensions of results in (**8**, Exposé **17**), and the proofs are omitted: (1) *Two proper ideals of the form $J(\lambda, \mathbf{k})$ are equal if and only if the corresponding couples $(\lambda, \mathbf{k})$ are equal.* (2) *Any (proper) maximal left ideal of $U(L)$ containing an ideal of the form $J(\lambda, \mathbf{k})$ contains exactly one such ideal.*

Conversely, if $J(\lambda, \mathbf{k}) \neq U(L)$, *then there exists exactly one maximal left ideal* J *of* $U(L)$ *containing it.* (The one-dimensionality of the λ weight space of $\rho_{U/J(\lambda,\mathbf{k})}$ (Theorem (3.6) (c)) is used in the proof of the second part of this assertion). This proves the theorem.

Let Δ be any subset of S. Then $L'(\Delta)$ and the set π of primitive cycles of $L'(\Delta)$ are defined. Let $l = |\pi|$ and let the elements of π be linearly ordered, say $\pi = \{c(1), \ldots, c(l)\}$. Then by Theorem (3.12) and Corollary (3.11) we may state:

(3.13) COROLLARY. *The standard representations of* L *are characterized by the* $(2 + l)$*-tuples* $(\Delta, \lambda, \mathbf{k}(c(1)), \ldots, \mathbf{k}(c(l)))$ *associated with them.*

4. The characterization problem. For any given L and Δ, the problem remains to characterize (in terms of λ and $\mathbf{k}$ explicitly) those couples $(\lambda, \mathbf{k})$ that determine (by Theorem (3.12)) the Δ-standard representations of L.

Given any couple $(\lambda, \mathbf{k})$, with $\lambda \in D^*$ and with $\mathbf{k}$ a mapping from π to K, choosing J to be the maximal left ideal of $U(L)$ containing $J(\lambda, \mathbf{k})$, it follows from Theorem (3.6) that to $(\lambda, \mathbf{k})$ there corresponds an irreducible representation of L, denoted by $\rho(\lambda, \mathbf{k})$, which (if it is not the zero representation) satisfies (ii) and (iii) of Definition (3.1). The couple $(\lambda, \mathbf{k})$ therefore determines a Δ-standard representation of L if and only if the following three conditions hold:

(A) the non-triviality condition: $J(\lambda, \mathbf{k}) \neq U(L)$;

(B) $\rho(\lambda, \mathbf{k})$ satisfies (i) of Definition (3.1); and

(C) λ satisfies (iv) of Definition (3.1).

We need to express (A) and (B) explicitly in terms of λ and $\mathbf{k}$, that is, as (algebraic) relations in K holding between the image elements of the mappings λ and $\mathbf{k}$.

We first consider *condition* (A):

(4.1) DEFINITION. *A* c-basis *of* $U(L)$ *is a basis of* $U(L)$ *consisting of elements of the form:*

$$u = v(u) \prod_{c\in\pi} c^{m(c)} \prod_{i=1}^{n} d_i^{m_i} \prod_{\gamma\in\Delta^0} p_\gamma^{n(\gamma)} \tag{4.2}$$

where the exponents are integers $\geqslant 0$, *where each product sign respects a fixed order on its index set, and where the elements* $v(u)$ *are primitive elements of* L *of the form* (3.7).

By an application of Lemma (2.2) it follows that c-bases of $U(L)$ exist.

Using (3.9) it is seen that if c', c'' are any two primitive cycles of L', then the linear decomposition of $c'c''$ with respect to any given c-basis of $U(L)$ has the form

(4.3) $\qquad c'c'' =$ a K-linear sum of elements of the form

$$\prod_{c\in\pi} c^{m(c)} \prod_{i=1}^{s} d_i^{m_i}.$$

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(4.4) THEOREM. *For any couple* $(\lambda, \mathbf{k})$ *the following three conditions are equivalent:*

(a) $J(\lambda, \mathbf{k}) \neq U(L)$.

(b) *The mapping defined by*

$$d_i \to \lambda(d_i) \quad (i = 1, \ldots, s), \qquad c \to \mathbf{k}(c) \quad (c \in \pi),$$

can be extended to a (one-dimensional) representation of $C(L')$.

(c) *Given any c-basis of* $U(L)$, *for each pair* (c', c'') *of primitive cycles of* L', *the corresponding equation* (4.3) *holds with each* $c \in \pi$ *replaced by* $\mathbf{k}(c)$ *and each* $d_i (i = 1, \ldots, s)$ *replaced by* $\lambda(d_i)$.

Proof. (a) $\Rightarrow$ (b): By letting x in (3.9) be any basis cycle in $U(L')$ (so that $p(x) = 1$), it follows that $C(L')$ *is generated, as an algebra, by the set* $\pi \cup \{d_i | i = 1, \ldots, s\}$ *of all primitive cycles of* L'. Let $\xi((d_i), (c)) = 0$ be any algebraic relation holding in $C(L')$ between the primitive cycles of L'. Modulo $J(\lambda, \mathbf{k})$ the relation takes the form $\xi((\lambda(d_i)), (\mathbf{k}(c))) \cdot 1^* = 0^*$, where x^* denotes the class of x modulo $J(\lambda, \mathbf{k})$. Since $1 \notin J(\lambda, \mathbf{k})$, we have that $1^* \neq 0^*$, so that $\xi((\lambda(d_i)), (\mathbf{k}(c))) = 0$ holds in K. This implies (b).

(b) $\Rightarrow$ (c): Trivial.

(c) $\Rightarrow$ (a): Set $g(c) = c - \mathbf{k}(c)$ for each $c \in \pi$, and $g(d_i) = d_i - \lambda(d_i)$ for each $i = 1, \ldots, n$. Using the binomial theorem it can be seen that a basis for $U(L)$ is given by all those elements that are obtainable from the elements (4.2) of a c-basis by replacing each c by $g(c)$ and each d_i by $g(d_i)$. Such a basis will be called a g-*basis* of $U(L)$. Now suppose $1 \in J(\lambda, \mathbf{k})$, that is, 1 has the form:

$$1 = \sum_{c \in \pi} u(c) g(c) + \sum_{i=1}^{n} u_i g(d_i) + \sum_{\gamma \in \Delta^0} u(\gamma) p_\gamma \tag{4.5}$$

where the elements $u(c)$, u_i, $u(\gamma)$ are in $U(L)$ and considered to be expressed in terms of that g-basis of $U(L)$ which is determined by the couple $(\lambda, \mathbf{k})$ and the given c-basis of $U(L)$. It will be shown, using (c), that the decomposition of the right-hand side of (4.5), with respect to the chosen g-basis of $U(L)$, contains no non-zero K-multiple of 1 as summand, so that (4.5) contradicts the uniqueness property of decomposition with respect to a basis.

By the form of the elements of a g-basis it will suffice to prove that any product of two elements, selected from the set

$$\{g(c), g(d_i), p_\gamma | c \in \pi; i = 1, \ldots, n; \gamma \in \Delta^0\},$$

contains, in its decomposition with respect to the g-basis, no non-zero constant term. We proceed to do so.

(i) $p_\gamma p_{\gamma'} = [p_\gamma, p_{\gamma'}] + p_{\gamma'} p_\gamma$. Now in L, $[p_\gamma, p_{\gamma'}]$ is either zero or a K-multiple of $p_{\gamma+\gamma'}$. Also, $\gamma, \gamma' \in \Delta^0 \Rightarrow \gamma + \gamma' \in \Delta^0$. Thus no g-basis summand of $p_\gamma p_\gamma{}'$ is a non-zero multiple of 1.

(ii) Let b be any element of the set $\pi \cup \{d_i | i = 1, \ldots, n\}$. For any $\delta \in \Delta^0$, the mass of $p_\delta b$ is defined and equal to δ. For any element u of the c-basis of

$U(L)$, the equation $\mu\{u\} = \delta$ has as solutions u only elements for which at least one of the exponents $n(\gamma)$ in (4.2) is different from zero (by the definitions of mass, Δ^+ and Δ^0). By the mass-graded structure of $U(L)$ it follows that each c-basis summand of $p_\delta\, b$ has mass δ and thus contains a factor of the form p_γ, $\gamma \in \Delta^0$. It now easily follows that each g-basis summand of $p_\delta\, g(b)$ contains a factor of the form p_γ, $\gamma \in \Delta^0$.

(iii) Let b, b' be any two cycles of the form b as in (ii). If any one (or both) of b and b' is of the form $d_i (i = 1, \ldots, n)$, then $bb' = b'b$ (by (2.3)), so that $g(b)g(b') = g(b')g(b)$. There remains the case where both of b and b' are in π. It is at this point that condition (c) is used. Letting $b = c$, $b' = c'$, we have:

$$g(c)g(c') = cc' - \mathbf{k}(c')c - \mathbf{k}(c)c' + \mathbf{k}(c)\mathbf{k}(c').$$

Using (4.3), then substituting $g(c) + \mathbf{k}(c)$ for each c and $g(d_i) + \lambda(d_i)$ for each d_i, it follows from the binomial theorem and condition (c) that the constant term of the g-basis decomposition of the right-hand side of this equation is zero.

All cases are covered by the products in (i), (ii), and (iii), so that (c) $\Rightarrow$ (a). This completes the proof of the theorem.

Theorem (4.4) (c) expresses the non-triviality condition $J(\lambda, \mathbf{k}) \neq U(L)$ as an explicit set of $|\pi|^2$ algebraic relations (which need not be independent) on the images in K of the mappings λ and $\mathbf{k}$.

We now turn to *condition* (B):

The set π contains the s primitive cycles $q_j\, p_j (j = 1, \ldots, s)$ of L'. For any couple $(\lambda, \mathbf{k})$ we may formulate the following condition: that the $\mathbf{k}(q_j\, p_j)$ $(j = 1, \ldots, s)$ satisfy

$$\mathbf{k}(q_j\, p_j) \neq r(r + 1 + \lambda(d_j)) \tag{4.6}$$

for every integer r. We shall show that (4.6) (together with the non-triviality condition) implies condition (B).

Let $\mathbf{J}'(\Delta)$ denote the set of all maximal left ideals of $U(L)$ that contain a left ideal of the form $J(\lambda, \mathbf{k})$, where λ satisfies (iv) of Definition (3.1) and where (4.6) and Theorem (4.4) (c) hold for $(\lambda, \mathbf{k})$.

(4.7) THEOREM. *If $J \in \mathbf{J}'(\Delta)$, then $\rho_{U/J} \in \mathbf{S}(\Delta)$.*

Proof. Since Theorem (4.4) (c) holds for $(\lambda, \mathbf{k})$, $\rho_{U/J}$ is not the zero representation. Thus we need only prove that (4.6) implies condition (B). Now for any $\alpha \in \Delta$, the first relation in Lemma (2.7) implies that

$$q_\alpha{}^t p_\alpha{}^t \equiv \prod_{i=1}^{t} \left(\mathbf{k}(q_\alpha\, p_\alpha) - (t - i)(t - i + 1 + \lambda(d_\alpha))\right) . 1$$

mod J for any integer $t > 0$. From (4.6) it follows that the right-hand side is a non-zero K-multiple of 1 modulo J. Thus since $1 \notin J$, we have that $p_\alpha{}^t \notin J$ for any integer $t > 0$. By (2.4), $p_\alpha{}^t + J$ has weight $\lambda + t\alpha$ so that it follows that, with λ, $\lambda + t\alpha$ is a weight of $\rho_{U/J}$ for each integer $t > 0$. Similarly, using the second relation in Lemma (2.7), it follows that (4.6) implies that, with λ, $\lambda - t\alpha$ is a weight of $\rho_{U/J}$ for each integer $t > 0$. This proves the theorem.

Thus, for any given L and Δ, we may state that those couples $(\lambda, \mathbf{k})$ that satisfy (c) of Theorem (4.4), (iv) of Definition (3.1), and (4.6) characterize a subset, denoted by $\mathbf{S}'(\Delta)$, of the set $\mathbf{S}(\Delta)$ of all Δ-standard representations of L. We have not been able to resolve the question as to whether $\mathbf{S}'(\Delta)$ is the whole of $\mathbf{S}(\Delta)$, that is, whether the scalars $\mathbf{k}(q_j p_j)(j = 1, \ldots, s)$ associated with any Δ-standard representation of L must of necessity satisfy (4.6).

(4.8) CONJECTURE. $\mathbf{S}'(\Delta) = \mathbf{S}(\Delta)$.

Also, although a method has now been provided to characterize explicitly, for any given L and Δ, a set $\mathbf{S}'(\Delta)$ of Δ-standard representations of L, there is no assurance that a simultaneous solution $(\lambda, \mathbf{k})$ to the above three conditions on $(\lambda, \mathbf{k})$ exists, so that the existence problem in general remains unsolved.

5. The standard order 1 representations of L. Let L be arbitrary. Let Δ be any subset of S consisting of a single simple root α of L. (The analysis will be independent of the particular simple root α used.) Then $\Delta^+ = \Delta = \{\alpha\}$ while $\Delta^0 = P - \{\alpha\}$. The Lie algebra L', being the Lie algebra spanned by $\{d_\alpha, q_\alpha, p_\alpha\}$, is isomorphic to the simple Lie algebra A_1 (**8**, p. 14–01). A basis for $U(L')$ is provided by the set of all elements of the form

$$u = d_\alpha^{m(1)} q_\alpha^{m(2)} p_\alpha^{m(3)}$$

where the exponents $m(1)$, $m(2)$, $m(3)$ are integers $\geqslant 0$. The mass $\mu\{u\}$ of u is $(m(3) - m(2))\alpha$. The basis cycles in $U(L')$ are accordingly all of the form u with $m(2) = m(3)$. An easy inspection yields that *there exist exactly two primitive cycles of L', namely the elements d_α and $q_\alpha p_\alpha$.* The set π consists of the single cycle $q_\alpha p_\alpha$. Let $\mathbf{k}$ be any mapping from π to K. Then $\mathbf{k}$ is specified by the single scalar $k = \mathbf{k}(q_\alpha p_\alpha)$.

Since (c) of Theorem (4.4) is trivially satisfied, it follows (from §4) that *the elements of* $\mathbf{S}'(\Delta)$ *are characterized by all couples of the form* (λ, k), *where* λ *is any element*

$$\lambda = \sum_{i=1}^{n} k(\alpha_i)\alpha_i$$

of D^* satisfying:

$$(5.1) \qquad 0 \leqslant (\textit{real part of } k(\alpha)) < 1$$

and where k is any element of K satisfying:

$$(5.2) \qquad k \neq r(r + 1 + \lambda(d_\alpha))$$

for every integer r. In particular, we note that $\mathbf{S}'(\Delta) \neq \emptyset$.

We proceed to show that the unique scalar $\mathbf{k}(q_\alpha p_\alpha)$ associated with any given $\{\alpha\}$-standard representation of $U(L)$ must of necessity satisfy (5.2), that is, that Conjecture (4.8) is true for the case $\Delta = \{\alpha\}$.

Let ρ denote any $\{\alpha\}$-standard representation of $U(L)$ and let λ be the characteristic weight of ρ.

(5.3) LEMMA. *The weight spaces of ρ belonging to the doubly infinite α-ladder through λ are all one-dimensional.*

Proof. We use the proof of Theorem (3.6). An element v of the form (3.7) has mass $t\alpha$, where t is any integer, if and only if $m(\gamma) = 0$ for each $\gamma \in \Delta^0 = P - \{\alpha\}$. For any integer t, it is easily seen that there is only one primitive element of L' having mass equal to $t\alpha$, namely $p_\alpha{}^t$ if $t \geqslant 0$ and $q_\alpha{}^t$ if $t \leqslant 0$. Thus for $\zeta = t\alpha$, (3.9) has exactly one solution $p(x)$. This establishes the lemma.

(5.4) LEMMA. *The unique scalar $k = \mathbf{k}(q_\alpha p_\alpha)$ associated with ρ satisfies* (5.2).

Proof. Let $k = r(r + 1 + \lambda(d_\alpha))$ for some integer r. Suppose $r \geqslant 0$. Then by Lemma (2.7) and (3.3)

$$q_\alpha{}^{r+j}p_\alpha{}^{r+j} \equiv 0 \mod J(\lambda, \mathbf{k}) \tag{5.5}$$

for each integer $j \geqslant 1$. We now consider all basis elements x of $U(L)$ of the form (3.2) such that the product $xp_\alpha{}^{r+1}$ has mass equal to zero. If x does not contain a factor of the form $p_\gamma(\gamma \in \Delta^0 = P - \{\alpha\})$, then it is easily seen that apart from factors $d_\alpha(\alpha \in S)$, x has the form $q_\alpha{}^{r+1+i}p_\alpha{}^i$ for $i =$ an integer $\geqslant 0$. Using (5.5) this implies that $xp_\alpha{}^{r+1} \equiv 0 \bmod J(\lambda, \mathbf{k})$. If x contains a factor of the form $p_\gamma(\gamma \in \Delta^0)$, then since $[p_\gamma, p_\alpha]$ is either zero or a multiple of $p_{\alpha+\gamma}$ (where $\alpha + \gamma \in \Delta^0$), it follows that each summand in the decomposition of $xp_\alpha{}^{r+1}$ with respect to the basis (3.2) contains at least one factor of the form $p_\delta(\delta \in \Delta^0)$, so that again $xp_\alpha{}^{r+1} \equiv 0 \bmod J(\lambda, \mathbf{k})$. There thus exists no element y in $U(L)$ such that $yp_\alpha{}^{r+1} \equiv c.\, 1 \bmod J(\lambda, \mathbf{k})$, where $c \in K$, $c \neq 0$. With respect to the left regular representation of $U(L)$ modulo $J(\lambda, \mathbf{k})$, the annihilator ideal J of the class $1 + J(\lambda, \mathbf{k})$ is accordingly a proper left ideal of $U(L)$ containing not only the ideal $J(\lambda, \mathbf{k})$ but also the element $p_\alpha{}^{r+1}$. But by the proof of Lemma (5.4), the $(\lambda + (r + 1)\alpha)$ weight space of $\rho(\simeq\rho_{U/J})$ is spanned by $p_\alpha{}^{r+1} + J$, and since this weight space is *one-dimensional*, $p_\alpha{}^{r+1} \notin J$. This is a contradiction. To prove the lemma for the case when $r \leqslant 0$, it is convenient to use the basis of $U(L)$ which, by Lemma (2.2), can be derived from the set of elements (3.2) by interchanging the two products involving, respectively, the factors q_α and p_α. The argument is then similar to the above, with $q_\alpha{}^{|r|+1}$ substituted for $p_\alpha{}^{r+1}$ and with the second relation in Lemma (2.7) used. (We need also to note that, for $\gamma \in \Delta^0$, $[p_\gamma, q_\alpha]$ is either zero or a multiple of $p_{\gamma-\alpha}$, $\gamma - \alpha \in \Delta^0$.) This concludes the proof of the lemma.

Thus for $\Delta = \{\alpha\}$, it holds that $\mathbf{S}(\Delta) = \mathbf{S}'(\Delta)$, and by §4 and (3.13) we may state:

(5.6) THEOREM. *The standard order* 1 *representations of $U(L)$ are characterized by all the triples of the form (α, λ, k), where α is any simple root of L, where λ is any element of D^* satisfying* (5.1), *and where k is any element of K satisfying* (5.2).

6. Standard order 2 representations of A_2. The simple Lie algebra A_2 (**8**, p. 14–01) has two simple roots which we denote by α_{01} and α_{10}. It has one other positive root, namely $\alpha_{11} = \alpha_{01} + \alpha_{10}$. A basis for A_2 is given by eight elements, $d_{01}, d_{10}, p_{01}, p_{10}, p_{11}, q_{01}, q_{10}$, and q_{11}, that satisfy the Lie multiplication table as given in Fig. 1. (The elements p_{ij} (or q_{ij}), for $ij = 01, 10, 11$, belong to the roots α_{ij} (or, respectively, $-\alpha_{ij}$).)

	d_{10}	d_{01}	p_{10}	p_{01}	p_{11}	q_{10}	q_{01}	q_{11}
d_{10}	0	0	$2p_{10}$	$-p_{01}$	p_{11}	$-2q_{10}$	q_{01}	$-q_{11}$
d_{01}	0	0	$-p_{10}$	$2p_{01}$	p_{11}	q_{10}	$-2q_{01}$	$-q_{11}$
p_{10}	$-2p_{10}$	p_{10}	0	p_{11}	0	d_{10}	0	$-q_{01}$
p_{01}	p_{01}	$-2p_{01}$	$-p_{11}$	0	0	0	d_{01}	q_{10}
p_{11}	$-p_{11}$	$-p_{11}$	0	0	0	$-p_{01}$	p_{10}	$d_{10} + d_{01}$
q_{10}	$2q_{10}$	$-q_{10}$	$-d_{10}$	0	p_{01}	0	$-q_{11}$	0
q_{01}	$-q_{01}$	$2q_{01}$	0	$-d_{01}$	$-p_{10}$	q_{11}	0	0
q_{11}	q_{11}	q_{11}	q_{01}	$-q_{10}$	$-d_{10} - d_{01}$	0	0	0

FIGURE 1 (The entry $[x, y]$ appears in the row of x and the column of y.)

If a Δ-standard representation of A_2 is of order 2, then there is only one choice for Δ, namely $\Delta = S = \{\alpha_{01}, \alpha_{10}\}$. Thus, $\Delta^+ = P = \{\alpha_{01}, \alpha_{10}, \alpha_{11}\}$ while $\Delta^0 = \emptyset$. The Lie algebra $L'(\Delta)$ is accordingly the whole of A_2. A basis for $U(A_2)$ is given by the set of all the elements of the form:

$$u \equiv q_{11}{}^{m(1)} q_{01}{}^{m(2)} q_{10}{}^{m(3)} p_{10}{}^{m(4)} p_{01}{}^{m(5)} p_{11}{}^{m(6)} d_{10}{}^{m(7)} d_{01}{}^{m(8)} \tag{6.1}$$

where the exponents are integers $\geqslant 0$. The mass $\mu\{u\}$ of u is equal to

$$(-m(1) - m(3) + m(4) + m(6))\alpha_{10} + (-m(1) - m(2) + m(5) + m(6))\alpha_{01}.$$

Thus, u is a cycle in $U(A_2)$ if and only if

$$m(6) - m(1) = m(3) - m(4) = m(2) - m(5).$$

By an inspection of possibilities we find that *there exist exactly seven primitive cycles of* A_2, *namely*: $d_{01}, d_{10}, c_1 = q_{01}\, p_{01}, c_2 = q_{10}\, p_{10}, c_3 = q_{11}\, p_{11}, c_4 = q_{11}\, p_{10}\, p_{01}$, *and* $c_5 = q_{01}\, q_{10}\, p_{11}$. The set π consists of the last five of these, so that any mapping $\mathbf{k}$ from π to K is specified by the five scalars $k_i = \mathbf{k}(c_i)(i = 1, \ldots, 5)$.

By allowing the factors in the elements (6.1) to commute, it is clear that a *c-basis* of $U(A_2)$ is provided by the set of all elements u of the form:

$$u = x(u) c_j{}^{n(6)} c_3{}^{n(5)} c_2{}^{n(4)} c_1{}^{n(3)} d_{10}{}^{n(2)} d_{01}{}^{n(1)} \tag{6.2}$$

where $j \in \{4, 5\}$, where the exponents are integers $\geqslant 0$, where $x(u)$ is a primitive element of A_2 of the form (6.1), and where priority in the construction of the elements (6.2) is given to cycles that appear to the right (for example, the element $q_{11}\, p_{10}\, p_{01}\, p_{11}$ of the form (6.1) determines the element $p_{10}\, p_{01}\, (q_{11}\, p_{11})$ (and not $p_{11}\, (q_{11}\, p_{10}\, p_{01})$) as the corresponding element (6.2)).

Using the table in Fig. 1, the c-basis decompositions of the pairwise products of the primitive cycles of A_2 may be found. It turns out that the following four of these:

$$\begin{aligned} c_1 c_2 &= c_4 - c_5 + c_2 c_1, \\ c_1 c_4 &= -c_4 d_{01} + (c_2 - c_3)c_1 + c_4 c_1, \\ c_2 c_4 &= -c_4 d_{10} - c_2 c_1 + c_3(d_{10} + c_2) + c_4 c_2, \\ c_4 c_5 &= c_3(c_2 + d_{10})(c_1 + d_{01}) + c_4(d_{01} + c_1 - c_2 + c_3), \end{aligned}$$

are sufficient for the application of condition (c) of Theorem (4.4). We have verified that this condition becomes equivalent to the condition that the scalars $k_1, \ldots, k_5$ have the form

$$\begin{aligned} k_1 &= z(z - 1 - \lambda(d_{01})), \\ k_2 &= (z - 1)(z + \lambda(d_{10})), \\ zk_3 &= k_4 = k_5 = z(z - 1 - \lambda(d_{01}))(z + \lambda(d_{10})), \end{aligned} \tag{6.3}$$

where $z \in K, \lambda \in D^*$. For $z \neq 0, -\lambda(d_{10}), 1 + \lambda(d_{01})$, the couples of the form (λ, z) are in one-to-one correspondence with the 5-tuples $(k_1, \ldots, k_5)$ determined by them.

It is readily seen that the conditions (4.6) on k_1 and k_2 become equivalent to

$$z \neq t, t + \lambda(d_{01}), t - \lambda(d_{10}) \tag{6.4}$$

for every integer t.

Thus, by §4, we may state:

(6.5) THEOREM. *For $L = A_2$ and $\Delta = \{\alpha_{01}, \alpha_{10}\}$, the elements of $\mathbf{S}'(\Delta)$ are characterized by all couples of the form (λ, z), where $\lambda \equiv k(01)\alpha_{01} + k(10)\alpha_{10}$ $(k(01), k(10) \in K)$ satisfies*

$$0 \leqslant (\text{real part of } k(ij)) < 1 \tag{6.6}$$

for $ij = 01, 10$, and where $z \in K$ satisfies (6.4).

We note that again $\mathbf{S}'(\Delta) \neq \emptyset$.

We now turn to a description of the weight spaces of any $\rho \in \mathbf{S}'(\Delta)$.

We recall that any couple of the form (λ, z), with λ (in D^*) and z (in K) satisfying (6.6) and (6.4), respectively, determines an element ρ of $\mathbf{S}'(\Delta)$ as follows: Let $J(\lambda, \mathbf{k})$ be the left ideal of $U(A_2)$ generated by the seven elements $d_{01} - \lambda(d_{01})$, $d_{10} - \lambda(d_{10})$, and $c_i - k_i (i = 1, \ldots, 5)$, where the k_i are expressed in terms of λ and z by (6.3). (By Theorem (4.4), $J(\lambda, \mathbf{k})$ is proper.) Let J be the unique maximal left ideal of $U(A_2)$ containing $J(\lambda, \mathbf{k})$. Then ρ is equivalent to the left regular representation of $U(A_2)$ modulo J.

Let J^* be the left ideal of $U(A_2)$ generated by the above seven generators for $J(\lambda, \mathbf{k})$ as well as by two elements of the form

$$h_1 = p_{10} p_{01} - k_6 p_{11} \quad \text{and} \quad h_2 = q_{01} q_{10} - k_7 q_{11},$$

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where $k_6, k_7 \in K$.

(6.7) LEMMA. *The following three conditions are equivalent:*
(1) $1 \notin J^*$,
(2) $q_{11}\,h_1,\, q_{01}\,q_{10}\,h_1,\, p_{11}\,h_2$, *and* $p_{10}\,p_{01}\,h_2$ *belong to* $J(\lambda, \mathbf{k})$,
(3) $k_6 = k_7 = z$.

Proof. (1) ⇔ (2): Given (1). Each of the terms in (2) has mass equal to zero so that each reduces modulo $J(\lambda, \mathbf{k})$ to a K-multiple of 1. If one of them reduces to a non-zero multiple of 1, then $1 \in J^*$, contradicting (1). Conversely, let (2) hold. Let $1 = j + z_1 h_1 + z_2 h_2$, where $j \in J(\lambda, \mathbf{k})$ and $z_1, z_2 \in U(A_2)$. By the mass-graded structure of $U(A_2)$, each summand on the right-hand side may be assumed to have mass equal to zero. Then $\mu\{z_1\} = -\alpha_{11}$ and $\mu\{z_2\} = \alpha_{11}$. Using Lemma (2.2), a basis of $U(A_2)$ may be chosen such that each basis element of mass equal to $-\alpha_{11}$ (or α_{11}) is expressed as a product terminating in a factor q_{11} or $q_{01}\,q_{10}$ (or, respectively, in a factor p_{11} or $p_{01}\,p_{10}$) with all other factors being primitive cycles of A_2. Let z_1 and z_2 be so expressed. Then (2) implies that $1 \in J(\lambda, \mathbf{k})$, which is a contradiction.

(2) ⇔ (3): By reducing each term in (2) modulo $J(\lambda, \mathbf{k})$ and applying (6.3) and (6.4), this equivalence is easily verified. This concludes the proof of the lemma.

(6.8) THEOREM. *For* $L = A_2$ *and* $\Delta = \{\alpha_{01}, \alpha_{10}\}$, *let* ρ *in* $\mathbf{S}'(\Delta)$ *be determined by the couple* (λ, z). *Then each weight space of* ρ *is one-dimensional and the weights of* ρ *are given by* all *elements of the form* $\lambda + m_1\,\alpha_{01} + m_2\,\alpha_{10}$ *where* m_1, m_2 *are integers.*

Proof. We choose $k_6 = k_7 = z$. Then the unique maximal left ideal J of $U(A_2)$ containing $J(\lambda, \mathbf{k})$ also contains the ideal J^* defined above. Taking into account the explicit forms of the elements $x(u)$ appearing in (6.2), we have verified that, modulo J^*, each element of the form (6.2) reduces, up to a K-multiple, to one of the following forms:

$$p_{10}{}^r p_{01}{}^t, \quad p_{10}{}^r q_{01}{}^t, \quad p_{01}{}^r q_{10}{}^t, \quad q_{01}{}^r q_{10}{}^t \tag{6.9}$$

where the exponents r, t range through all integers $\geqslant 0$. (The computations involved are laborious and are omitted.) Since two distinct elements of the form (6.9) have distinct masses, (2.4) implies that the weight spaces of ρ_{U/J^*}, and thus of ρ, are one-dimensional. The inequalities (6.4) now ensure that for each of the classes (6.9) there exists an operator of ρ_{U/J^*} that transforms the class into a *non-zero* K-multiple of the class $1 + J^*$. (Again we omit the details.) It thus follows that J^* *is maximal*, that is $J^* = J$. The second statement of the theorem now follows by computing the masses of the elements (6.9).

7. Conjectures. The results of Lemma (5.3) and Theorem (6.8) lead us to state, for arbitrary L and Δ, the following two conjectures:

Let $\rho \in \mathbf{S}'(\Delta)$. Let λ be the characteristic weight of ρ and let W be the set of weights of ρ. Let $B(\lambda, \Delta)$ denote the set of *all* elements in D^* of the form $\lambda + \sum_{\alpha \in \Delta} m(\alpha)\alpha$, where the $m(\alpha)$ are integers. The set $B(\lambda, \Delta) \cap W$ may be called the *boundary* of W.

(7.1) CONJECTURE. *Each boundary weight space of ρ has dimension equal to one.*

(7.2) CONJECTURE. $B(\lambda, \Delta) \subseteq W$.

Note added in proof. It has come to the author's attention that a proof of the result of Lemma (2.6) is also given by F. W. Lemire in his doctoral dissertation: *Infinite dimensional irreducible representations of simple Lie algebras* (Queen's University, 1967), pp. 49–57.

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