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NEAT AND CONEAT SUBMODULES OF MODULES OVER COMMUTATIVE RINGS

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Abstract

We prove that neat and coneat submodules of a module coincide when R is a commutative ring such that every maximal ideal is principal, extending a recent result by Fuchs. We characterise absolutely neat (coneat) modules and study their closure properties. We show that a module is absolutely neat if and only if it is injective with respect to the Dickson torsion theory.

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1. Introduction

The classical notion of purity for abelian groups (see, for example, [10]) was generalised to neatness by Honda [12]. Thus a subgroup A of an abelian group B is called *pure* if $nA = A \cap nB$ for every integer n, whereas A is called *neat* if $pA = A \cap pB$ for every prime p. Neatness has been extended from abelian groups to modules over a commutative ring in the following ways. Let R be a commutative ring. A short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in the module category Mod(R) is called:

(i) *neat* if for every simple module S the functor $\mathrm{Hom}_R(S, -)$ preserves its exactness, that is, the following is a short exact sequence:

$$0 \to \mathrm{Hom}_R(S, A) \xrightarrow{\mathrm{Hom}_R(S,f)} \mathrm{Hom}_R(S, B) \xrightarrow{\mathrm{Hom}_R(S,g)} \mathrm{Hom}_R(S, C) \to 0;$$

(ii) *coneat* if for every simple module S the functor $S \otimes_R -$ preserves its exactness, that is, the following is a short exact sequence:

$$0 \to S \otimes_R A \xrightarrow{S\otimes_R f} S \otimes_R B \xrightarrow{S\otimes_R g} S \otimes_R C \to 0.$$

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If the short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is neat (coneat), then f is called a *neat (coneat) monomorphism* and g is called a *neat (coneat) epimorphism*. Usually A will be identified with a submodule of B, and we simply say that A is a *neat (coneat) submodule* of B. Neatness is the same as $\mathcal{P}$*-purity* in the sense of [17, 33.1] for the class $\mathcal{P}$ of all simple modules, and coneatness has also been called *s-purity* in [7].

For $R = \mathbb{Z}$ the concepts of neat and coneat submodule of a module (abelian group) coincide, but over an arbitrary ring R they are in general inequivalent, as one may see from [11, Examples 3.2, 3.3]. Hence it is interesting to determine conditions on the commutative ring R under which neatness and coneatness are the same. In particular, it would be useful to obtain solutions expressed in terms of ideals of rings; see [1, 4, 13] for examples of recent related results using ideals.

Fuchs has recently reconsidered the problem of comparing neatness and coneatness, and his main result states that for an integral domain R the two concepts coincide if and only if every maximal ideal of R is (finitely generated) projective, that is, invertible [11, Theorem 5.2]. He also notes that if coneatness implies neatness over an arbitrary commutative ring R, then every maximal ideal of R must be finitely generated [11, page 137]. But if R has every maximal ideal finitely generated, then the two notions are still inequivalent by [11, Example 3.2].

With these results as motivation, we prove that neatness and coneatness are the same over any commutative ring having every maximal ideal principal (Theorem 2.1). Then we continue the study of absolutely neat and absolutely coneat modules over commutative rings from [6, 7] by giving new characterisations and analysing their closure properties. We show that a module is absolutely neat if and only if it is injective with respect to the Dickson torsion theory (Theorem 3.4). We characterise some properties of commutative rings by using absolute neatness (coneatness).

Throughout the paper R will denote a commutative ring with identity.

2. Neat and coneat submodules

We begin with our main result connecting neatness and coneatness.

THEOREM 2.1. *Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence in Mod(R). *Consider the following statements.*

(i) *A is neat in B.*

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(ii) *For every maximal ideal M of R and for every commutative diagram in* Mod(R)

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "u"] \arrow[d, "\alpha"'] & R \arrow[r, "v"] \arrow[d, "\beta"] & R/M \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

with exact rows, there exists a homomorphism $w : R \to A$ such that $wu = \alpha$.

(iii) *For every maximal ideal $M = rR$ of R, the equation $rx = a \in A$ has a solution $x \in A$ whenever it has a solution in B.*

(iv) *For every maximal ideal M of R, $MA = A \cap MB$.*

(v) *A is coneat in B.*

Then (i)⇔(ii) and (iv)⇔(v). If every maximal ideal of R is principal, then all statements are equivalent.

Proof. (i)⇔(ii). Using the diagram from (ii), the equivalence of (i) and (ii) follows by the homotopy lemma [17, 7.16] (see also [5]).

(iv)⇔(v). This follows by [11, page 136] (see also [16, page 170]).

For the rest of the proof we suppose that every maximal ideal of R is principal.

(ii)⇔(iii). First suppose that (ii) holds. Let $M = rR$ be a maximal ideal of R, and assume that the equation $rx = a \in A$ has a solution $b \in B$. We may define two homomorphisms $\alpha : M \to A$ by $\alpha(rs) = as$ for every $s \in R$, and $\beta : R \to B$ by $\beta(1) = b$. It is easy to see that α is well defined. Let $u : M \to R$ be the inclusion homomorphism. Then $\beta u = f\alpha$, and we obtain in the canonical way a commutative diagram in Mod(R) with exact rows as in (ii). Hence there exists a homomorphism $w : R \to A$ such that $wu = \alpha$. If $a_0 = w(1)$, then $a = \alpha(r) = wu(r) = w(r) = ra_0 \in rA$. Therefore, the equation $rx = a \in A$ has a solution in A.

Conversely, suppose that (iii) holds. Let $M = rR$ be a maximal ideal of R, and consider a commutative diagram in Mod(R) with exact rows as in (ii). Let $b = \beta(1)$ and $a = \alpha(r)$. Then $a = f(a) = f\alpha(r) = \beta u(r) = \beta(r) = rb$. Now by hypothesis the equation $rx = a \in A$ also has a solution in A, say $a_0 \in A$. Then we may define a homomorphism $w : R \to A$ by $w(1) = a_0$. It follows that $wu = \alpha$.

(iii)⇔(iv). First suppose that (iii) holds. Let $M = rR$ be a maximal ideal of R. Note that $MA = A \cap MB$ is equivalent to $rA = A \cap rB$, and we only need to show the inclusion $A \cap rB \subseteq rA$. To this end, let $a \in A \cap rB$, hence $a = rb$ for some $b \in B$. Hence the equation $rx = a \in A$ has a solution $b \in B$. By hypothesis, it must also have a solution $a_0 \in A$. Then $a = ra_0 \in rA$, as required.

Conversely, suppose that (iv) holds. Let $M = rR$ be a maximal ideal of R and assume that the equation $rx = a \in A$ has a solution $b \in B$. Then $a = rb \in A \cap rB = A \cap MB = MA = rA$, so the equation $rx = a$ has a solution in A. □

Remark 2.2. (1) The linking property between neatness and coneatness is condition (iii) of Theorem 2.1. Now assume that every maximal ideal M of R is only

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finitely generated, say $M = r_1R + \cdots + r_nR$, instead of principal. Then neatness of a submodule A of a module B is equivalent to the condition that the system of equations

$$r_i x = a_i \in A \quad (i \in I)$$

with unknown x has a solution in A whenever it has a solution in B [11, Lemma 2.2]. On the other hand, by Theorem 2.1 (iv)⇔(v), coneatness of A in B is equivalent to the condition $r_1A + \cdots + r_nA = A \cap (r_1B + \cdots + r_nB)$, that is, the equation

$$r_1x_1 + \cdots + r_nx_n = a \in A$$

with unknowns $x_1, \ldots, x_n$ has a solution in A whenever it has a solution in B. Comparing these characterisations of neatness and coneatness, one should not expect their equivalence, even over a commutative ring such that every maximal ideal is finitely generated. Specific examples are given by Fuchs [11, Examples 3.2, 3.3].

(2) The condition that every maximal ideal of R is principal is not necessary in order to have the coincidence of the two concepts of neatness and coneatness. Indeed, if R is a Prüfer domain with all maximal ideals finitely generated, then it is always an *N-domain* (in the sense that neatness and coneatness are the same) [11, page 142].

Example 2.3. Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in commuting indeterminates $x_1, \ldots, x_n$ over a field K. Then R is a Noetherian domain (for example, [3, Theorem 7.5]). In general, $(x_1 - a_1, \ldots, x_n - a_n)$ with $a_1, \ldots, a_n \in R$ is a maximal ideal of R which is not principal. If $n = 1$ and K is algebraically closed, the maximal ideals of R are of the form $(x_1 - a_1)R$ with $a_1 \in R$, so they are principal.

3. Absolutely neat (coneat) modules

A module is called *absolutely neat (coneat)* if it is a neat (coneat) submodule of any module containing it. In this section we study closure properties of the classes of absolutely neat (coneat) submodules. They will have a certain degree of similarity, but one should also expect differences given by the fact that neatness and coneatness do not coincide in general.

Example 3.1. In [11, Example 3.3] it was shown that, if R is an integral domain having a simple module S with projective dimension $\mathrm{p.d.}(S) > 1$, then there exists a short exact sequence $0 \to D \to M \to S \to 0$ which is not neat such that $T \otimes_R D = 0$ for every simple module T. The latter clearly implies that D is absolutely coneat, but D is not absolutely neat.

The following characterisation of absolutely coneat modules is known.

Theorem 3.2 [7, Theorem 2.2]. *The following are equivalent for a module A.*

(i) *A is absolutely coneat.*
(ii) *A is a coneat submodule of an injective module.*
(iii) *A is a coneat submodule of an absolutely coneat module.*

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Now we give a similar characterisation for absolutely neat modules.

Theorem 3.3. *The following are equivalent for a module A.*

(i) *A is absolutely neat.*
(ii) *A is a neat submodule of an injective module.*
(iii) *A is a neat submodule of an absolutely neat module.*

Proof. (i)⇒(ii). If A is absolutely neat, then it is neat in its injective hull $E(A)$.

(ii)⇒(iii). This is clear.

(iii)⇒(i). Assume that A is a neat submodule of an absolutely neat module B, and consider the induced neat short exact sequence $0 \to A \xrightarrow{\beta} B \to C \to 0$ in Mod(R). Let $\alpha : A \to D$ be a monomorphism. Now consider the pushout of α and β to obtain the following commutative diagram in Mod(R) with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\gamma"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & D \arrow[r, "\delta"] & P \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Since B is absolutely neat, the monomorphism $\gamma : B \to P$ is neat. It follows that $\delta\alpha = \gamma\beta$ is a neat monomorphism, so α must be a neat monomorphism [17, 33.2]. Therefore, A is absolutely neat. □

We may add two more equivalent conditions to absolute neatness. Denote by τ_D the Dickson torsion theory, that is, the hereditary torsion theory generated by all simple modules [9].

Theorem 3.4. *The following are equivalent for a module A.*

(i) *A is absolutely neat.*
(ii) *For every maximal ideal M of R, A is injective with respect to the canonical short exact sequence*
$$0 \to M \to R \to R/M \to 0.$$
(iii) *A is τ_D-injective.*

Proof. (i)⇒(ii). Assume that A is absolutely neat. By Theorem 3.3, A is a neat submodule of some injective module E. Let M be a maximal ideal of R. Then we have an induced commutative diagram in Mod(R) with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "u"] \arrow[d, "\alpha"'] & R \arrow[r, "v"] \arrow[d, "\beta"] & R/M \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"'] & E \arrow[r, "g"'] & E/A \arrow[r] & 0 \end{tikzcd}$$

Since A is neat in E, it follows that A is injective with respect to the upper short exact sequence by Theorem 2.1.

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(ii)⇒(i). Assume that A is injective with respect to the canonical short exact sequence $0 \to M \to R \to R/M \to 0$ for every maximal ideal M of R. Let M be a maximal ideal of R, and consider a commutative diagram in Mod(R) with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "u"] \arrow[d, "\alpha"'] & R \arrow[r, "v"] \arrow[d, "\beta"] & R/M \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

By hypothesis, there exists a homomorphism $w : R \to A$ such that $wu = \alpha$. Then A is neat in B by Theorem 2.1.

(ii)⇔(iii). This follows by [2, Corollary 1.4.3]. □

Since direct limits commute with tensor product, it is straightforward to check that the class of absolutely coneat modules is closed under arbitrary direct limits. In the case of absolute neatness we have a weaker property. Note that neatness is an inductive property (in the sense that direct limits of neat short exact sequences are also neat short exact sequences) if and only if every maximal ideal of R is finitely generated [11, Lemma 2.4].

Proposition 3.5. *Let R be a commutative ring such that every maximal ideal of R is finitely generated. Let $(A_i, f_{ij})_I$ be a direct system of absolutely neat modules with direct limit $(\varinjlim A_i, f_i)$ such that each f_{ij} is a monomorphism. Then $\varinjlim A_i$ is an absolutely neat module.*

Proof. First note that, since each f_{ij} is a monomorphism, so is each f_i. Let

$$0 \to \varinjlim A_i \xrightarrow{u} B \xrightarrow{v} C \to 0$$

be a short exact sequence in Mod(R). For every $i \in I$, $u_i = uf_i$ is a monomorphism and we have an exact sequence $0 \to A_i \xrightarrow{u_i} B \xrightarrow{v_i} C_i \to 0$, which is neat because A_i is absolutely neat. Now let S be a simple module. For every $i \in I$, A_i is absolutely neat, so Theorem 2.1 yields the following induced short exact sequence:

$$0 \to \mathrm{Hom}_R(S, A_i) \to \mathrm{Hom}_R(S, B) \to \mathrm{Hom}_R(S, C_i) \to 0.$$

Since every maximal ideal of R is finitely generated, S must be finitely presented. Now using the exactness of direct limits and the isomorphisms $\varinjlim \mathrm{Hom}_R(S, A_i) \cong \mathrm{Hom}_R(S, \varinjlim A_i)$ and $\varinjlim \mathrm{Hom}_R(S, C_i) \cong \mathrm{Hom}_R(S, \varinjlim C_i) \cong \mathrm{Hom}_R(S, C)$, we have the following induced short exact sequence with canonical homomorphisms:

$$0 \to \mathrm{Hom}_R(S, \varinjlim A_i) \to \mathrm{Hom}_R(S, B) \to \mathrm{Hom}_R(S, C) \to 0,$$

which shows that the initial short exact sequence is neat by Theorem 2.1. Hence $\varinjlim A_i$ is an absolutely neat module. □

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Proposition 3.6. *The class of absolutely neat modules is closed under direct products and extensions. If every maximal ideal of R is finitely generated, then the class of absolutely neat modules is closed under direct sums.*

Proof. Let $(A_i)_{i\in I}$ be a family of absolutely neat modules. Let $u : \prod_{i\in I} A_i \to B$ be a monomorphism in $\mathrm{Mod}(R)$. For every $j \in I$, let $f_j : A_j \to \prod_{i\in I} A_i$ be the canonical embedding. Then $u_j = uf_j : A_j \to B$ is a monomorphism, which is neat because A_j is absolutely neat. Denote by C_j the cokernel of u_j. Now let S be a simple module. Then for every $j \in I$ we have an induced short exact sequence

$$0 \to \mathrm{Hom}_R(S, A_j) \to \mathrm{Hom}_R(S, B) \to \mathrm{Hom}_R(S, C_j) \to 0,$$

hence a short exact sequence

$$0 \to \prod_{j\in I} \mathrm{Hom}_R(S, A_j) \to (\mathrm{Hom}_R(S, B))^I \to \prod_{j\in I} \mathrm{Hom}_R(S, C_j) \to 0$$

and so a short exact sequence

$$0 \to \mathrm{Hom}_R\Big(S, \prod_{j\in I} A_j\Big) \to \mathrm{Hom}_R(S, B^I) \to \mathrm{Hom}_R\Big(S, \prod_{j\in I} C_j\Big) \to 0.$$

It follows that the canonical monomorphism $\prod_{j\in I} A_j \hookrightarrow B \hookrightarrow B^I$ is neat, and so the monomorphism $\prod_{j\in I} A_j \hookrightarrow B$ is neat [17, 33.2]. Hence $\prod_{i\in I} A_i$ is an absolutely neat module.

The closure of the class of absolutely neat modules under extensions follows by [17, 35.2]. If every maximal ideal of R is finitely generated, then the closure of the class of absolutely neat modules under direct sums follows by Proposition 3.5. □

Proposition 3.7. *The class of absolutely coneat modules is closed under direct sums and extensions. If every maximal ideal of R is finitely generated, then the class of absolutely coneat modules is closed under direct products.*

Proof. The closure of the class of absolutely coneat modules under direct sums follows easily, because direct sums commute with tensor products. The closure of the class of absolutely coneat modules under extensions follows by [7, Theorem 2.4].

Let $(A_i)_{i\in I}$ be a family of absolutely coneat modules. Let $u : \prod_{i\in I} A_i \to B$ be a monomorphism in $\mathrm{Mod}(R)$. For every $j \in I$, let $f_j : A_j \to \prod_{i\in I} A_i$ be the canonical embedding. Then $u_j = uf_j : A_j \to B$ is a monomorphism, which is coneat because A_j is absolutely coneat. Now let S be a simple module. Then for every $j \in I$ we have an induced monomorphism $S \otimes_R A_j \to S \otimes_R B$, hence a monomorphism $\prod_{j\in I}(S \otimes_R A_j) \to (S \otimes B)^I$. Since every maximal ideal of R is finitely generated, S is finitely presented, and so tensor products commute with direct products. Then we have a monomorphism $S \otimes_R (\prod_{j\in I} A_j) \to S \otimes_R B^I$. It follows that the canonical monomorphism $\prod_{j\in I} A_j \hookrightarrow B \hookrightarrow B^I$ is coneat, and so the monomorphism $u : \prod_{j\in I} A_j \to B$ is coneat. Hence $\prod_{i\in I} A_i$ is an absolutely coneat module. □

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4. Absolutely neat (coneat) modules over particular rings

Every injective module is absolutely neat and absolutely coneat by Theorems 3.2 and 3.3. In the case of a commutative domain we have the following result.

THEOREM 4.1. *Let R be a commutative domain such that every maximal ideal is principal. Then the following are equivalent.*

(i) *R is Noetherian.*

(ii) *Every absolutely neat (coneat) module is injective.*

PROOF. Under the hypotheses on R, neatness and coneatness coincide by Theorem 2.1.

(i)⇒(ii). Since R is a Noetherian domain such that every maximal ideal of R is principal, it follows that every τ_D-injective module is injective [8, Corollary 2.4.3]. Then every absolutely neat module is injective by Theorem 3.4.

(ii)⇒(i). Let $(A_i)_{i\in I}$ be a family of injective modules. Then each A_i is absolutely neat, and so $\bigoplus_{i\in I} A_i$ is absolutely neat by Proposition 3.6. Now $\bigoplus_{i\in I} A_i$ is injective, which shows that R is Noetherian by the Bass–Papp theorem (for example, [14, 3.46]). □

EXAMPLE 4.2. Let $(K_i)_{i\in I}$ be an infinite family of fields, and let $R = \prod_{i\in I} K_i$. Then R is a commutative von Neumann regular ring, and so a V-ring. Then every simple module is injective, whence it follows that every module is absolutely coneat [7, Example 3.1]. In particular, the ideal $J = \bigoplus_{i\in I} K_i$ of R is absolutely coneat. But J is not finitely generated, hence J is not injective. Therefore, there are absolutely coneat modules which are not injective.

A ring R will be called *max-hereditary* if every maximal ideal is projective.

EXAMPLE 4.3. Recall that a commutative ring R is called *Rickart* (or a PP-ring) if every principal ideal of R is projective [14, 7.48]. Then any commutative Rickart ring with every maximal ideal principal is max-hereditary. An integral domain R is max-hereditary if and only if it is an N-domain [11, Theorem 5.2], so every Prüfer domain with all maximal ideals finitely generated is max-hereditary.

THEOREM 4.4. *The following are equivalent.*

(i) *R is max-hereditary.*

(ii) *The class of absolutely neat modules is closed under homomorphic images.*

PROOF. (i)⇒(ii). Assume that R is max-hereditary. Let B be an absolutely neat module, and A a submodule of B. Also, let M be a maximal ideal of R. Since R is max-hereditary, M is projective. The short exact sequence $0 \to M \to R \to R/M \to 0$ induces an exact sequence

$$\mathrm{Ext}^1_R(M, A) \to \mathrm{Ext}^2_R(R/M, A) \to \mathrm{Ext}^1_R(R, A)$$

in which the first and the last terms are zero. Then $\mathrm{Ext}^2_R(R/M, A) = 0$. The short exact sequence $0 \to A \to B \to B/A \to 0$ induces an exact sequence

$$\mathrm{Ext}^1_R(R/M, B) \to \mathrm{Ext}^1_R(R/M, B/A) \to \mathrm{Ext}^2_R(R/M, A).$$

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The first term is zero by Theorem 3.4, because B is absolutely neat. It follows that $\mathrm{Ext}^1_R(R/M, B/A) = 0$, and so B/A is absolutely neat by Theorem 3.4. Hence (ii) holds.

(ii)⇒(i). Assume that the class of absolutely neat modules is closed under homomorphic images. Let M be a maximal ideal of R. Let N be a module, and $E(N)$ its injective hull. Then $E(N)/N$ is absolutely neat by hypothesis, so $\mathrm{Ext}^1_R(R/M, E(N)/N) = 0$ by Theorem 3.4. The short exact sequence $0 \to N \to E(N) \to E(N)/N \to 0$ induces an exact sequence

$$\mathrm{Ext}^1_R(R/M, E(N)/N) \to \mathrm{Ext}^1_R(R/M, N) \to \mathrm{Ext}^2_R(R/M, E(N))$$

in which the first and the last terms are zero. The short exact sequence $0 \to M \to R \to R/M \to 0$ induces an exact sequence

$$\mathrm{Ext}^1_R(R, N) \to \mathrm{Ext}^1_R(M, N) \to \mathrm{Ext}^2_R(R/M, N)$$

in which the first and the last terms are zero. It follows that $\mathrm{Ext}^1_R(M, N) = 0$, and so M is projective. Therefore, R is max-hereditary. □

Theorem 4.5. *The following are equivalent.*

(i) *R is semisimple.*
(ii) *R is absolutely neat and every maximal ideal of R is finitely generated projective.*

Proof. (i)⇒(ii). Assume that R is semisimple. Then clearly every maximal ideal of R is finitely generated projective. For every (maximal) ideal M of R, the short exact sequence $0 \to M \to R \to R/M \to 0$ splits, so every module is absolutely neat by Theorem 3.4. Thus R is absolutely neat.

(ii)⇒(i). Assume that R is absolutely neat and every maximal ideal of R is finitely generated projective. Let M be a maximal ideal of R. Then M is finitely generated projective, so it is a direct summand of a finite direct sum of copies of R. Since R is absolutely neat, so is M by Proposition 3.6. Then the short exact sequence $0 \to M \to R \to R/M \to 0$ splits. Hence every maximal ideal of R is a direct summand. Then R is semisimple by [15, Proposition 3.25]. □

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