Compositio Math. 144 (2008) 774–786
doi:10.1112/S0010437X07003272

On the local structure of Dirac manifolds

Jean-Paul Dufour and Aïssa Wade

Abstract

We give a local normal form for Dirac structures. As a consequence, we show that the dimensions of the pre-symplectic leaves of a Dirac manifold have the same parity. We also show that, given a point m of a Dirac manifold M, there is a well-defined transverse Poisson structure to the pre-symplectic leaf P through m. Finally, we describe the neighborhood of a pre-symplectic leaf in terms of geometric data. This description agrees with that given by Vorobjev for the Poisson case.

1. Introduction

A *Dirac manifold* is a smooth manifold M equipped with a vector sub-bundle L of the Whitney sum $TM \oplus T^*M$ which is maximal isotropic with respect to the natural pairing on $TM \oplus T^*M$ and integrable in the sense that the smooth sections of L are closed under the Courant bracket (see § 2). The vector bundle L is then called a *Dirac structure* on M.

Dirac structures on manifolds were first introduced by Courant and Weinstein in the mid-1980s [CW86]. A few years later, further investigations were undertaken in [Cou90]. Recently, the theory of Dirac structures has been extensively developed in connection with various topics in mathematics and physics (see, for instance, [BC97, Gua04, LWX97, BW04]). Specific examples of Dirac manifolds include pre-symplectic and Poisson manifolds. Thus, it is important to understand the local structure of a Dirac manifold. The main goal of this paper is to provide a description of the local structure of such a manifold.

Every Dirac manifold admits a foliation by pre-symplectic leaves. The local structure of general Dirac manifolds was only studied in neighborhoods of *regular* points [Cou90] (see also [Gua04] for the case of complex Dirac structures). By a regular point, we mean a point for which there is an open neighborhood where the foliation is regular. It is natural to ask about the local structure around non-regular points. In [AB06], the special case of generalized complex manifolds is considered.

In § 3, we give a normal form for a Dirac structure L on a smooth manifold M near an arbitrary point $m \in M$ (see Theorem 3.2). This normal form allows us to conclude that the dimensions of the pre-symplectic leaves have the same parity.

We show in § 4 that, given a point m in a Dirac manifold M, there is a well-defined transverse Poisson structure whose rank at m is zero. This extends facts from the classical case of Poisson structures (see [Wei83]). In § 5, we describe the neighborhood of a pre-symplectic leaf of a Dirac manifold using the concept of geometric data (see [Vor00]). Dirac structures on manifolds are constructed from given geometric data. We prove that, conversely, one can construct geometric data from a Dirac manifold M with a fixed tubular neighborhood of a symplectic leaf P.

This paper is divided into five sections. Section 2 contains some basic definitions and results. Our main theorems are given in §§ 3–5 (see Theorems 3.2, 4.5, 5.1, and 5.4).

Received 25 March 2007, accepted in final form 24 August 2007, published online 14 March 2008.
2000 Mathematics Subject Classification 53D17, 58xx.
Keywords: normal forms, Poisson structure, Dirac structure, foliations.

2. Preliminaries

Let M be a smooth n-dimensional manifold. We denote by $\langle\cdot,\cdot\rangle$ the canonical symmetric bilinear operation on the vector bundle $TM \oplus T^*M \to M$. This induces a symmetric C^∞-bilinear operation on the space of smooth sections of $TM \oplus T^*M$ given by:

$$\langle (X_1,\alpha_1),(X_2,\alpha_2)\rangle = \tfrac{1}{2}(i_{X_2}\alpha_1 + i_{X_1}\alpha_2), \quad \text{for all } (X_1,\alpha_1),(X_2,\alpha_2) \in \Gamma(TM \oplus T^*M).$$

An *almost Dirac structure* on M is a sub-bundle of $TM \oplus T^*M \to M$ which is maximal isotropic with respect to $\langle\cdot,\cdot\rangle$.

The *non-skew symmetric Courant bracket* on $\Gamma(TM \oplus T^*M)$ is defined by

$$[(X_1,\alpha_1),(X_2,\alpha_2)]_C = ([X_1,X_2], \mathcal{L}_{X_1}\alpha_2 - i_{X_2}d\alpha_1),$$

where $\mathcal{L}_X = d \circ i_X + i_X \circ d$ is the Lie derivation by X. It is sometimes called the *Dorfman bracket*.

A *Dirac structure* L on M is an almost Dirac structure which is integrable (i.e. $\Gamma(L)$ is closed under the Courant bracket). In this case, the pair (M,L) is called a *Dirac manifold*.

Examples. We give the following examples.

(i) Let Ω be a 2-form on M. Consider the graph

$$L_\Omega = \{(X, i_X\Omega) \mid X \in TM\}.$$

Then L_Ω is a Dirac structure if and only if $d\Omega = 0$. Furthermore, a Dirac structure is the graph of a 2-form if and only if $L \cap (\{0\} \oplus T^*M) = \{0\}$ at every point.

(ii) Let π be a bivector field on M. We use the notation

$$L_\pi = \{(\pi^\sharp\alpha, \alpha) \mid \alpha \in T^*M\}.$$

Then L_π is Dirac if and only if π is a Poisson tensor. Furthermore, a Dirac structure is the graph of a bivector field if and only if $L \cap (TM \oplus \{0\}) = \{0\}$ at every point.

Let L be an almost Dirac structure on M. Consider the distribution

$$(\mathcal{D}_L)_x = pr_1(L_x) \quad \text{for all } x \in M,$$

where pr_1 is the canonical projection of L_x onto T_xM. The distribution $\mathcal{D}_L$ is involutive when L is integrable. Hence, in this case, L gives rise to a singular foliation. Furthermore, there is a skew-symmetric bilinear map $\Omega_L : \mathcal{D}_L \times \mathcal{D}_L \to C^\infty(M)$ given by

$$\Omega_L(X,Y) = \alpha(Y), \quad \text{for any } (X,\alpha),(Y,\beta) \in \Gamma(L). \tag{1}$$

We have the following proposition.

Proposition 2.1 (Courant [Cou90]). *If L is a Dirac structure on M then $d\Omega_L = 0$.*

Remark. As an immediate consequence of this proposition, one sees that a Dirac structure on M gives rise to a singular foliation by pre-symplectic leaves (i.e. on each leaf, there is a closed 2-form).

3. A local normal form for Dirac manifolds

3.1 Proposition

Let L be a Dirac structure on a smooth manifold M of dimension n, and let $m_0 \in M$. If the pre-symplectic leaf through m_0 is a single point then there is a neighborhood U of m_0 such that $L_{|U}$ is the graph of a Poisson structure Π.

https://doi.org/10.1112/S0010437X07003272 Published online by Cambridge University Press

Proof. Assume that the pre-symplectic leaf through m_0 is $P = \{m_0\}$. There are vector fields $X_1, \dots, X_n$, and 1-forms $\alpha_1, \dots, \alpha_n$ defined in an open neighborhood U of m_0 such that $L_{|U}$ is determined by the local sections $e_i = (X_i, \alpha_i)$, where $X_i(m_0) = 0$. In local coordinates $(y_1, \dots, y_n)$ such that $y_i(m_0) = 0$, we have the expressions

$$X_i = \sum_{j=1}^{n} X_{ij}(y) \frac{\partial}{\partial y_j}, \quad \alpha_i = \sum_{j=1}^{n} \alpha_{ij}\, dy_j.$$

The sections $e_i(m)$ give the following matrix

$$\begin{pmatrix} X_{11} & \dots & X_{1n} & \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & & & & \\ X_{n1} & \dots & X_{nn} & \alpha_{n1} & \dots & \alpha_{nn} \end{pmatrix} \quad \text{with } X_{ij}(m_0) = 0.$$

The sub-matrix $(\alpha_{ij}(m_0))$ is invertible since $\dim(L_{m_0}) = n$. Therefore, $(\alpha_{ij}(m))$ remains invertible at all points in a small neighborhood of m_0. Let $(\alpha^{ij}(m))$ be the inverse of $(\alpha_{ij}(m))$. Define

$$e'_i = \sum_{j=1}^{n} \alpha^{ij} e_j, \quad \text{for all } i = 1, \dots, n.$$

This can be written as

$$e'_i = \left(\sum_{i=1}^{n} X'_{ij} \frac{\partial}{\partial y_j}, dy_i \right) \quad \text{with } X'_{ij} = -X'_{ji}.$$

Define

$$\Pi = \sum_{i<j} X'_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}.$$

Then, the Schouten bracket $[\Pi, \Pi]$ is zero (this is due to the fact that $\Gamma(L)$ is closed under the Courant bracket). Furthermore, $L_{|U}$ is the graph of Π. $\square$

Now, we assume that the pre-symplectic leaf P through $m_0 \in M$ is not a single point. Let U be a neighborhood of m_0 with local coordinates $(x_1, \dots, x_r, y_1, \dots, y_s)$ such that $x_i(m_0) = y_j(m_0) = 0$, for all $i \in \{1, \dots, r\}$, $j \in \{1, \dots, s\}$, and such that the pre-symplectic leaf through m_0 has equations $y_1 = 0, \dots, y_s = 0$. In what follows, we use the notation

$$x = (x_1, \dots, x_r), \quad y = (y_1, \dots, y_s).$$

Without loss of generality, we can assume that there are vector fields $Y_i(x, y)$, $Z_j(x, y)$ defined on U and 1-forms $\alpha_i(x, y)$, $\beta_j(x, y)$ such that $\Gamma(L_{|U})$ is spanned by

$$\mathcal{S}_i = \left(\frac{\partial}{\partial x_i} + Y_i(x, y), \alpha_i(x, y) \right), \quad \mathcal{T}_j = (Z_j(x, y), \beta_j(x, y)),$$

with $Y_i(x, 0) = 0$, $Z_j(x, 0) = 0$, for all $i = 1, \dots, r$, $j = 1, \dots, s$. We want to find a new spanning set of local sections $\{\mathcal{S}'_i, \mathcal{T}'_j\}$ defined around m_0 which have very simple expressions. In other words, we want to find a normal form of L at m_0. We use the notation

$$X_i = \frac{\partial}{\partial x_i} + Y_i(x, y) = \frac{\partial}{\partial x_i} + \sum_{j=1}^{r} \widehat{Y}_{ij}(x, y) \frac{\partial}{\partial x_j} + \sum_{j=1}^{s} \widetilde{Y}_{ij}(x, y) \frac{\partial}{\partial y_j}.$$

One can note that there are smooth functions $f_{ij}(x, y)$ such that

$$\sum f_{ij}(x, y) X_j = \frac{\partial}{\partial x_i} + \sum_{j=1}^{s} Y'_{ij}(x, y) \frac{\partial}{\partial y_j}.$$

https://doi.org/10.1112/S0010437X07003272 Published online by Cambridge University Press

Equivalently, there is a matrix $(f_{ij}(x,y))$ whose coefficients are smooth functions and such that

$$(I + (\widehat{Y}_{ij}))(f_{ij}) = I.$$

Indeed, $(I + (\widehat{Y}_{ij}))$ is invertible at all points in U (up to a shrinking of U). Hence, there are smooth functions $f_{ij}(x,y)$ satisfying the above matrix equation. It follows that $\Gamma(L_{|U})$ is spanned by smooth sections of the form

$$\mathcal{S}'_i = \left(\frac{\partial}{\partial x_i} + \sum_{j=1}^{s} Y'_{ij}(x,y)\frac{\partial}{\partial y_j}, \alpha'_i(x,y)\right), \quad \mathcal{T}_j = (Z_j(x,y), \beta_j(x,y)),$$

for $i = 1,\dots,r$ and $j = 1,\dots,s$. We write

$$Z_i = \sum_{j=1}^{r} \widehat{Z}_{ij}(x,y)\frac{\partial}{\partial x_j} + \sum_{j=1}^{s} \widetilde{Z}_{ij}(x,y)\frac{\partial}{\partial y_j}.$$

Define

$$\mathcal{T}'_i = \mathcal{T}_i - \sum_{j=1}^{r} \widehat{Z}_{ij}(x,y)\mathcal{S}'_j.$$

Then, we see that $\Gamma(L_{|U})$ is spanned by smooth sections of the form

$$\mathcal{S}'_i = \left(\frac{\partial}{\partial x_i} + \sum_{j=1}^{s} Y'_{ij}(x,y)\frac{\partial}{\partial y_j}, \alpha'_i(x,y)\right), \quad \mathcal{T}'_j = \left(\sum_{k=1}^{s} Z'_{jk}(x,y)\frac{\partial}{\partial y_k}, \beta'_j(x,y)\right),$$

where $Y'_i(x,0) = Z_j(x,0) = 0$, for all $i = 1,\dots,r$ and $j = 1,\dots,s$. Using the fact that L is isotropic, we obtain

$$\alpha'_i\left(\frac{\partial}{\partial x_j}\right) + \alpha'_j\left(\frac{\partial}{\partial x_i}\right) = 0 \quad \text{and} \quad \beta'_j\left(\frac{\partial}{\partial x_i}\right) = 0 \quad \text{at every point } p \in P.$$

Moreover, a basis for the fiber L_{m_0} is given by the elements

$$\mathcal{S}'_i(m_0) = \left(\frac{\partial}{\partial x_i}, \alpha'_i\right), \quad \mathcal{T}'_j(m_0) = \left(0, \sum_{k=1}^{s} \beta'_{jk}\frac{\partial}{\partial y_k}\right),$$

for $i = 1,\dots,r$ and $j = 1,\dots,s$. Using matrix notation, we can put the $\mathcal{S}'_i(m_0)$ and $\mathcal{T}'_j(m_0)$ into row vectors which give the following rectangular matrix:

$$\begin{pmatrix} I & 0 & * & * \\ 0 & 0 & 0 & (\beta'_{ij}) \end{pmatrix}.$$

The sub-matrix $(\beta'_{ij}(m_0))$ is invertible since $\dim(L_{m_0}) = r + s = \dim M$. Hence, $(\beta'_{ij}(x,y))$ is invertible at all points $m = (x,y)$ in a small neighborhood of m_0. Let $(g_{ij}(x,y))$ be the inverse of the matrix $(\beta'_{ij}(x,y))$. We use the notation

$$\mathcal{T}''_i = \sum_{j=1}^{s} g_{ij}(x,y)\mathcal{T}'_j.$$

Hence, $\mathcal{T}''_i$ has the form

$$\mathcal{T}''_i = \left(\sum_{k=1}^{s} Z''_{ik}(x,y)\frac{\partial}{\partial y_k}, dy_i + \sum_{k=1}^{r} \beta''_{ik}(x,y)\, dx_k\right).$$

https://doi.org/10.1112/S0010437X07003272 Published online by Cambridge University Press

Now, we replace

$$\mathcal{S}'_i = \left(\frac{\partial}{\partial x_i} + \sum_{j=1}^{s} Y'_{ij}(x,y)\frac{\partial}{\partial y_j}, \sum_{j=1}^{r} \overline{\alpha}'_{ij}(x,y)\,dx_j + \sum_{k=1}^{s} \widetilde{\alpha}'_{ik}(x,y)\,dy_k\right)$$

by the following

$$\mathcal{S}''_i = \mathcal{S}'_i - \sum_{k=1}^{s} \widetilde{\alpha}'_{ik}(x,y)\mathcal{T}''_k.$$

We then obtain a new spanning set $\{\mathcal{S}''_i, \mathcal{T}''_j\}$ of local sections of $L_{|U}$. We summarize the above discussion in the following theorem.

Theorem 3.2. *Let L be a Dirac structure on a smooth manifold M. Given any point $m_0 \in M$, there is a coordinate system $(x_1,\dots,x_r,y_1,\dots,y_s)$ defined on an open neighborhood U of m_0 such that the intersection of U with the pre-symplectic leaf through m_0 is the set $\{y_1 = \cdots = y_s = 0\}$, and $\Gamma(L_{|U})$ is spanned by sections of the form*

$$\mathcal{H}_i = \left(\frac{\partial}{\partial x_i} + \sum_{k=1}^{s} X_{ik}(x,y)\frac{\partial}{\partial y_k},\ \sum_{k=1}^{r} \alpha_{ik}(x,y)\,dx_k\right)$$

and

$$\mathcal{V}_j = \left(\sum_{k=1}^{s} Z_{jk}(x,y)\frac{\partial}{\partial y_k},\ dy_j + \sum_{k=1}^{r} \beta_{jk}(x,y)\,dx_k\right),$$

where $X_{ik}(m_0) = 0$, $Z_{jk}(m_0) = 0$, for all $i \in \{1,\dots,r\}$ and $j \in \{1,\dots,s\}$.

We use the following notation:

$$X_i = \frac{\partial}{\partial x_i} + \sum_{k=1}^{s} X_{ik}(x,y)\frac{\partial}{\partial y_k}, \quad \alpha_i = \sum_{k=1}^{r} \alpha_{ik}(x,y)\,dx_k,$$

$$Z_j = \sum_{k=1}^{s} Z_{jk}(x,y)\frac{\partial}{\partial y_k}, \quad \beta_j = dy_j + \sum_{k=1}^{r} \beta_{jk}(x,y)\,dx_k.$$

Remarks.

(a) The normal form $(\mathcal{H}_i, \mathcal{V}_j)$ persists when a change of coordinates of the type $\Phi(x,y) = (x, \Phi_2(x,y))$ is performed.

(b) Using the fact that L is isotropic, one obtains

$$2\langle \mathcal{H}_i, \mathcal{V}_j\rangle = X_{ij} + \beta_{ji} = 0,$$

for all $i \in \{1,\dots,r\}$ and $j \in \{1,\dots,s\}$. Hence, $X_{ij} = -\beta_{ji}$. Furthermore,

$$2\langle \mathcal{V}_i, \mathcal{V}_j\rangle = Z_{ij} + Z_{ji} = 0.$$

This shows that the matrix $(Z_{ij}(x,y))$ is skew-symmetric. An analogous calculation shows that the matrix $(\alpha_{ij}(x,y))$ is also skew-symmetric.

(c) If Ω_L is the 2-form associated with the Dirac structure L (it is defined as in (1)) then we have

$$\Omega_L(X_i, X_j) = \alpha_{ij}, \quad \Omega_L(X_i, Z_j) = 0, \quad \Omega_L(Z_i, Z_j) = Z_{ij}.$$

Corollary 3.3. *Given a Dirac structure L on a smooth manifold M, the dimensions of the leaves of its associated pre-symplectic foliation have the same parity.*

Proof. It is sufficient to work in a small neighborhood of an arbitrary point $m_0 \in M$.

Case 1. Suppose that the leaf through m_0 is a single point. By Proposition 3.1, L is the graph of a Poisson structure in a neighborhood of m_0. Hence, the dimensions of the pre-symplectic leaves are even.

Case 2. Suppose that the leaf through m_0 is not a single point. Then, Theorem 3.2 provides a coordinate system $(x_1, \dots, x_r, y_1, \dots, y_s)$ defined on an open neighborhood U of m_0 and a spanning set $\{\mathcal{H}_i, \mathcal{V}_j\}$ of local sections of $L_{|U}$ such that

$$pr_1(\mathcal{H}_i) = X_i = \frac{\partial}{\partial x_i} + \sum_{k=1}^{s} X_{ik}(x,y)\frac{\partial}{\partial y_k}, \quad pr_1(\mathcal{V}_j) = Z_j = \sum_{k=1}^{s} Z_{jk}(x,y)\frac{\partial}{\partial y_k},$$

where the functions X_{ik}, Z_{ik} vanish at m_0. Let $(\mathcal{D}_L)_m$ be the tangent space to the leaf through $m = (x,y)$. It is spanned by the set of vectors $\{X_i(x,y), Z_j(x,y) \mid i = 1, \dots, rj = 1, \dots, s\}$ which corresponds to the matrix

$$\begin{pmatrix} I & * \\ 0 & (Z_{jk}(x,y)) \end{pmatrix}.$$

The sub-matrix $(Z_{jk}(x,y))$ is skew-symmetric, hence its rank is even. Since

$$\dim(\mathcal{D}_L)_m = r + \mathrm{rank}(Z_{jk}(x,y)),$$

we conclude that $\dim(\mathcal{D}_L)_m$ and $\dim(\mathcal{D}_L)_{m_0} = r$ have the same parity. □

We should mention this phenomenon of jumping dimensions that appeared in the study of generalized complex structures on even-dimensional manifolds (see [Gua04]). Moreover, as Weinstein pointed out in a private communication, this corollary can be obtained from [TW03, Lemma 2.2].

4. Transverse Poisson structures

4.1 Induced Dirac structures on submanifolds

Let L be a Dirac structure on a manifold M. Let Q be a submanifold of M. In this section, we review a result established in [Cou90] which says that, under certain regularity conditions, L induces a Dirac structure on Q. At every point $q \in Q$, we obtain a maximal isotropic vector space

$$(L_Q)_q = \frac{L_q \cap (T_qQ \oplus T_q^*M)}{L_q \cap (\{0\} \oplus T_qQ^\circ)},$$

where $T_qQ^\circ = \{v \in T_q^*M \mid v_{|T_qQ} = 0\}$. Using the map $(L_Q)_q \to T_qQ \oplus T_q^*Q$ given by

$$(u,v) \mapsto (u, v_{|T_qQ}),$$

one can identify $(L_Q)_q$ with a subspace of $T_qQ \oplus T_q^*Q$. In fact, L_Q defines a smooth sub-bundle of $TQ \oplus T^*Q$ if and only if $L_q \cap (T_qQ \oplus T_q^*M)$ has constant dimension. Moreover, one has the following result.

Proposition 4.1 (Courant [Cou90]). *If $L_q \cap (T_qQ \oplus T_q^*M)$ has constant dimension, then L_Q is a Dirac structure on Q.*

4.2 Existence of a transverse Dirac structure

Let L be a Dirac structure on M, m_0 a point of M, and Q a submanifold of M which contains m_0 and is transversal to the pre-symplectic leaf of m_0 in the sense that the tangent space of M at m_0 is the direct sum of the tangent spaces of Q and of the pre-symplectic leaf P. Choose coordinates (x_i, y_j) defined on an open neighborhood U of m_0 as in Theorem 3.2, but with the additional condition that Q is given by equations $x_1 = 0, \dots, x_r = 0$. We adopt the notation of the previous

https://doi.org/10.1112/S0010437X07003272 Published online by Cambridge University Press

section, i.e. $\Gamma(L_{|U})$ is spanned by sections of the form

$$\mathcal{H}_i = \left(\frac{\partial}{\partial x_i} + \sum_{k=1}^{s} X_{ik}(x,y)\frac{\partial}{\partial y_k}, \sum_{k=1}^{r} \alpha_{ik}(x,y)\,dx_k\right)$$

and

$$\mathcal{V}_j = \left(\sum_{k=1}^{s} Z_{jk}(x,y)\frac{\partial}{\partial y_k}, dy_j + \sum_{k=1}^{r} \beta_{jk}(x,y)\,dx_k\right),$$

where $X_{ik}(m_0) = 0$, $Z_{jk}(m_0) = 0$, for all $i \in \{1,\dots,r\}$ and $j \in \{1,\dots,s\}$.

Lemma 4.2. *The vector spaces $L_q \cap (T_qQ \oplus T_q^*M)$ have the same dimension for all $q \in Q$.*

Proof. Suppose that $(u(q), v(q))$ is a vector in $L_q \cap (T_qQ \oplus T_q^*M)$. We write

$$(u(q), v(q)) = \sum_{i=1}^{r} \lambda_i \mathcal{H}_i(q) + \sum_{j=1}^{s} \mu_j \mathcal{V}_j(q).$$

Then,

$$dx_k\left(\sum_{i=1}^{r} \lambda_i\left(\frac{\partial}{\partial x_i} + Y_i\right) + \sum_{j=1}^{s} \mu_j Z_{jk}\frac{\partial}{\partial y_k}\right) = \lambda_k = 0.$$

Consequently, $\dim(L_q \cap (T_qQ \oplus T_q^*M)) \leqslant s$. However, the vectors $(\mathcal{V}(q))_{j=1,\dots,s}$ are linearly independent at $q = m_0$. Therefore, they are linearly independent for all $q \in Q$ (we can suppose that the open neighborhood U of m_0 is small enough). This shows that $L_q \cap (\mathrm{Vert}_q \oplus T_q^*M)$ has constant dimension. $\square$

Now, applying Proposition 4.1, one can conclude that L_Q is a Dirac structure on Q. In fact, L_Q is spanned by the sections

$$\mathcal{V}_j(q) = (Z_j, dy_j)(q), \quad \text{for all } q \in Q,\ j \in \{1,\dots,s\},$$

where we use here the notation of the previous section.

Since the pre-symplectic leaf of L_Q at m_0 reduces to a point, Proposition 3.1 shows that L_Q is the graph of a Poisson structure. The corresponding Poisson tensor is given by

$$\Pi_Q(dy_i, dy_j) = -Z_j(0,y)\cdot y_i,$$

where $Z_j \cdot y_i$ is the directional derivative of y_i along Z_i. We have then proved the following result.

Theorem 4.3. *Let Q be a submanifold transversal to a pre-symplectic leaf P of the Dirac manifold M at a point m_0 ($T_{m_0}M = T_{m_0}P \oplus T_{m_0}Q$). Then the Dirac structure induces a Poisson structure Π_Q on a neighborhood of m_0 in Q, with $\Pi_Q(m_0) = 0$.*

The above calculations also show that there is an induced Poisson structure on each submanifold given by equations $x =$ constant. These Poisson structures fit together to give a Poisson tensor Π^V defined on a whole neighborhood of m_0 in M by

$$\Pi^V(dy_i, dy_j) = -Z_j(x,y)\cdot y_i,$$

where $y = (y_1,\dots,y_s)$ (respectively $x = (x_1,\dots,x_r)$) are local coordinates of Q (respectively P) around m_0.

Lemma 4.4. *For any $i = 1,\dots,r$, we have*

$$[X_i, \Pi^V] = 0,$$

where $X_i = p_1(\mathcal{H}_i)$.

https://doi.org/10.1112/S0010437X07003272 Published online by Cambridge University Press

Proof. Recall that $\mathcal{V}_j = (Z_j, \beta_j)$. For simplicity, we write β_j in the form $dy_j + \beta_j^V$. Then, we have

$$[\mathcal{H}_i, \mathcal{V}_j] = ([X_i, Z_j],\ d(X_i \cdot y_j) + \mathcal{L}_{X_i}\beta_j^V - i_{Z_j}\, d\alpha_i).$$

The fact that $L_{|U}$ is isotropic implies

$$\begin{aligned} 0 &= 2\langle [\mathcal{H}_i, \mathcal{V}_j], \mathcal{V}_k\rangle \\ &= [X_i, Z_j] \cdot y_k + Z_k \cdot (X_i \cdot y_j) + Z_k \cdot (\beta_j^V(X_i)) + d\beta_j^V(X_i, Z_k) - d\alpha_i(Z_j, Z_k). \end{aligned}$$

However, $d\alpha_i(Z_j, Z_k) = 0$ because Z_j and Z_k have only terms in $\partial/\partial y$. Moreover, if we use the notation

$$\beta_j^V = \sum_{k=1}^{r} \beta_{jk}(x, y)\, dx_k,$$

then

$$Z_k \cdot (\beta_j^V(X_i)) + d\beta_j^V(X_i, Z_k) = \sum_{\ell=1}^{s} Z_k^\ell \frac{\partial \beta_{ji}}{\partial y_\ell} - \sum_{\ell=1}^{s} Z_k^\ell \frac{\partial \beta_{ji}}{\partial y_\ell} = 0.$$

There follows

$$2\langle [\mathcal{H}_i, \mathcal{V}_j], \mathcal{V}_k\rangle = [X_i, Z_j] \cdot y_k + Z_k \cdot (X_i \cdot y_j) = 0.$$

This equation can be written as

$$X_i \cdot (Z_j \cdot y_k) - Z_j \cdot (X_i \cdot y_k) + Z_k \cdot (X_i \cdot y_j) = 0.$$

This is equivalent to the equation

$$[X_i, \Pi^V](dy_j, dy_k) = 0,$$

for all indexes $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$. This completes the proof of the lemma. □

Theorem 4.5. *Let Q and Q' be two submanifolds transversal to the same pre-symplectic leaf P of a Dirac structure L. The Poisson structures induced by L on Q and Q' (near $Q \cap P$ and $Q' \cap P$ respectively) are locally isomorphic.*

Proof. By connexity of P, it is sufficient to construct the isomorphism in the case where Q and Q' are near enough. First of all, if $Q \cap P = Q' \cap P$ then one can conclude that the induced Poisson structures is the same. This follows from the above expressions for $\Pi_Q(dy_i, dy_j)$ and similar expressions for Q'. Now suppose that $Q \cap P$ and $Q' \cap P$ are different. Hence, it is enough to work in a domain with coordinates (x, y) as above, that is, P has equation $y = 0$, Q has equation $x = 0$, and, moreover, Q' has equation $x = x^0$ where x^0 is some constant different from zero. Now we use Lemma 4.4: because X_i has a component $\partial/\partial x_i$ we can go from zero to x^0 in P using a sequence of trajectories of the different fields X_i, moreover the flows of these fields preserve vertical directions $x =$ constant and Lemma 4.4 says that they preserve also Π^V, so they exchange the Poisson structures on the vertical directions. □

Remark. It follows from Theorem 4.5 that each pre-symplectic leaf of a Dirac structure has a well-defined, up to isomorphism, Poisson transversal structure. This extends a well-known result in the Poisson case.

We can also remark that, in the classical case of Poisson structures, the above method used to prove the uniqueness of the transverse Poisson structure is simpler than those in literature (see, for instance, [Wei83]).

https://doi.org/10.1112/S0010437X07003272 Published online by Cambridge University Press

5. Geometric data

In [Vor00], Vorobjev considered what he called *geometric data*. Here we use the same terminology for a slightly different situation.

Definition. Let $p : E \to P$ be a vector bundle and let $\mathrm{Vert} = \ker dp \subset TE$. Geometric data on the vector bundle (E, p, P) consist of the following.

- A connection $\gamma : TE \to \mathrm{Vert}$.
- A vertical bivector field Π^V.
- A 2-form $\mathbb{F} \in \Omega^2(P) \otimes C^\infty(E)$ such that:
  (i) $[\Pi^V, \Pi^V] = 0$;
  (ii) $[\mathrm{hor}(u), \Pi^V] = 0$, for all $u \in \chi(P)$;
  (iii) $\partial_\gamma \mathbb{F} = 0$;
  (iv) $\mathrm{Curv}_\gamma(u, v) = (\Pi^V)^\sharp(d\mathbb{F}(u, v))$, for all $u, v \in \chi(P)$.

Unlike in [Vor00], we include the conditions (i)–(iv) in the definition of geometric data since we will consider only triples $(\gamma, \Pi^V, \mathbb{F})$ satisfying those conditions. In fact, the main difference between Vorobjev's definition of geometric data and that given above is that the 2-form $\mathbb{F}$ is not necessarily nondegenerate. Now, let us explain the above notation. Here γ is an Ehresmann connection: at each point $e \in E$, $\gamma_e : T_eE \to \mathrm{Vert}_e$ is a projection map. So $\mathrm{Hor} := \ker \gamma$ gives a horizontal distribution. We have the splitting

$$T_eE = \mathrm{Hor}_e \oplus \mathrm{Vert}_e, \quad \text{for all } e \in E.$$

Consequently, for every vector field u on the base manifold P, there is an horizontal vector field $\mathrm{hor}(u)$ (tangent to Hor) which is obtained by lifting u. A 2-vector is 'vertical' if it is a section of $\Lambda^2 \mathrm{Vert}$. The curvature of γ is given by

$$\mathrm{Curv}_\gamma(u, v) = [\mathrm{hor}(u), \mathrm{hor}(v)] - \mathrm{hor}[u, v], \quad \text{for all } u, v \in \chi(P).$$

The operator $\partial_\gamma : \Omega^k(P) \otimes C^\infty(E) \to \Omega^{k+1}(P) \otimes C^\infty(E)$ is defined by

$$\begin{aligned}\partial_\gamma \mathbb{G}(u_0, \dots, u_k) = {} & \sum_{i=0}^{k} (-1)^i \mathcal{L}_{\mathrm{hor}(u)}(\mathbb{G}(u_0, \dots, \widehat{u_i}, \dots, u_k)) \\ & + \sum_{i<j} (-1)^{i+j} \mathbb{G}([u_i, u_j], u_0, \dots, \widehat{u_i}, \dots, \widehat{u_j}, \dots, u_k).\end{aligned}$$

We have the following theorem.

Theorem 5.1. *Fix a tubular neighborhood of a submanifold P of a manifold M, it defines a vector bundle structure $p : E \to P$ on an open neighborhood E of P (P is identified to the zero section). Any Dirac structure on M which has P as a pre-symplectic leaf determines geometric data on E, up to a shrinking.*

To prove Theorem 5.1, we need to establish a couple of lemmas. We first introduce some notation. Consider a point $m_0 \in P$ and a neighborhood U of m_0 in E with coordinates (x_i, y_j) as in Theorem 3.2 but with the additional condition that $x =$ constant are the fibers of $p : E \to P$. Then, Vert is generated by the vector fields $\partial/\partial y_j$. Using the notation of § 3, we have the following lemmas.

Lemma 5.2. *For any $i, j, k \in \{1, \dots, r\}$, we have*

$$X_i \cdot \alpha_{jk} + X_j \cdot \alpha_{ki} + X_k \cdot \alpha_{ij} = 0.$$

Proof. We have the Courant bracket

$$[\mathcal{H}_i, \mathcal{H}_j] = ([X_i, X_j], \mathcal{L}_{X_i}\alpha_j - i_{X_j}\, d\alpha_i).$$

https://doi.org/10.1112/S0010437X07003272 Published online by Cambridge University Press

Since L is isotropic, we obtain

$$\langle [\mathcal{H}_i, \mathcal{H}_j], \mathcal{H}_k \rangle = 0.$$

This gives

$$0 = \alpha_k([X_i, X_j]) + X_k \cdot (\alpha_j(X_i)) + (i_{X_i}\, d\alpha_j - i_{X_j}\, d\alpha_i)(X_k).$$

However, $\alpha_k([X_i, X_j]) = 0$, for all $i, j, k \in \{1, \dots, r\}$. Moreover,

$$d\alpha_j(X_i, X_k) - d\alpha_i(X_j, X_k) = X_i \cdot \alpha_{jk} - X_k \cdot \alpha_{ji} - X_j \cdot \alpha_{ik} + X_k \cdot \alpha_{ij}.$$

It follows that

$$0 = \langle [\mathcal{H}_i, \mathcal{H}_j], \mathcal{H}_k \rangle = X_i \cdot \alpha_{jk} - X_j \cdot \alpha_{ik} + X_k \cdot \alpha_{ij}.$$

This completes the proof of the lemma. □

Lemma 5.3. *For any $i, j \in \{1, \dots, r\}$, we have*

$$[X_i, X_j] = (\Pi^V)^\sharp\, d\alpha_{ij}.$$

Proof. Since L is isotropic, we have

$$\begin{aligned} 0 &= 2\langle [\mathcal{H}_i, \mathcal{H}_j], \mathcal{V}_k \rangle \\ &= dy_k([X_i, X_j]) + Z_k \cdot \alpha_{ji} + (i_{X_i}\, d\alpha_j - i_{X_j}\, d\alpha_i)(Z_k). \end{aligned}$$

However,

$$d\alpha_j(X_i, Z_k) - d\alpha_i(X_j, Z_k) = 2 \sum_{\ell=1}^{s} Z_k^\ell \frac{\partial \alpha_{ij}}{\partial y_\ell} = 2 Z_k \cdot \alpha_{ij}.$$

It follows that

$$2\langle [\mathcal{H}_i, \mathcal{H}_j], \mathcal{V}_k \rangle = dy_k([X_i, X_j]) + Z_k \cdot \alpha_{ji} + 2 Z_k \cdot \alpha_{ij} = dy_k([X_i, X_j]) + Z_k \cdot \alpha_{ij}.$$

We conclude that

$$dy_k([X_i, X_j]) = -\Pi^V(dy_k, d\alpha_{ij}).$$

The lemma follows. □

Proof of Theorem 5.1. We first construct γ or, equivalently, the horizontal sub-bundle Hor. Define

$$\mathrm{Hor}_e = \mathrm{pr}_1(L_e \cap (T_e M \oplus \mathrm{Vert}_e^\circ)),$$

where Vert_e° is the annihilator of Vert_e. In local coordinates as above, Vert is generated by the vector fields X_i and we have

$$\mathrm{hor}\left(\frac{\partial}{\partial x_i}\right) = X_i.$$

By definition, Π^V is the bivector field given by putting together the Poisson 2-vectors we have on each fiber of E. Its local expressions are given by

$$\Pi^V(dy_i, dy_j) = -Z_j(x, y) \cdot y_i.$$

We define $\mathbb{F}$ by the formula

$$\mathbb{F}(u, v) = \Omega_L(\mathrm{hor}(u), \mathrm{hor}(v)).$$

Locally, we have the components

$$\mathbb{F}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \alpha_i(X_j) = \alpha_{ij} = -\alpha_{ji}.$$

Note that we may have to shrink E in order for γ and Π^V to be well defined. Now, we have to show that properties (i)–(iv) of the definition of geometric data hold. However, it is sufficient to work in local coordinates. Moreover, these properties are exactly equivalent to the fact that Π^V is Poisson, Lemmas 4.4, 5.2, and 5.3, respectively. Therefore, any Dirac structure on M having P as a pre-symplectic leaf determines geometric data on the vector bundle $p : E \to P$ (up to a shrinking of the total space E). $\square$

Conversely, we have the following theorem.

Theorem 5.4. *Any geometric data $(\gamma, \Pi^V, \mathbb{F})$ on a vector bundle $p : E \to P$ induce a Dirac structure on the total space E.*

We divide the proof of Theorem 5.4 into lemmas. Let $(\gamma, \Pi^V, \mathbb{F})$ be geometric data on a vector bundle (E, p, P). From now on, if u is a vector field on P, we simply denote by $\overline{u}$ the horizontal lift of u (instead of $\mathrm{hor}(u)$). Define

$$\begin{aligned} L^H_x &= \mathrm{Span}\{(\overline{u}, \alpha_u)_x \mid u \in \chi(P), \alpha_{u|\mathrm{Vert}} = 0, \alpha_u(\overline{v}) = \mathbb{F}(u, v)\}, \\ L^V_x &= \mathrm{Span}\{((\Pi^V)^\sharp\beta, \beta)_x \mid \beta_{|\mathrm{Hor}} = 0\}. \end{aligned}$$

Clearly, the sub-bundles L^H and L^V of $TM \oplus T^*M$ are isotropic with respect to $\langle\cdot,\cdot\rangle$. Moreover, we have the following lemma.

Lemma 5.5. *Both spaces $\Gamma(L^H)$ and $\Gamma(L^V)$ are closed under the Courant bracket.*

Proof. Let $\mathcal{V}_u = (\overline{u}, \alpha_u)$, $\mathcal{V}_v = (\overline{v}, \alpha_v)$, and $\mathcal{V}_w = (\overline{w}, \alpha_w)$ be elements of $\Gamma(L^H)$. We have

$$\begin{aligned} 2\langle[\mathcal{V}_u, \mathcal{V}_v], \mathcal{V}_w\rangle &= 2\langle([\overline{u}, \overline{v}], \mathcal{L}_{\overline{u}}\alpha_v - i_{\overline{v}}d\alpha_u), \mathcal{V}_w\rangle \\ &= (\alpha_w([\overline{u}, \overline{v}]) + \mathcal{L}_{\overline{u}}(\alpha_v(\overline{w}))) + \text{c.p.}, \end{aligned}$$

where the symbol c.p. stands for the two other terms obtained by cyclic permutation of the indexes. Using the definition of α_v and the fact that

$$[\overline{u}, \overline{v}] = \overline{[u, v]} + (\Pi^V)^\sharp(d\mathbb{F}(u, v)),$$

we obtain

$$2\langle[\mathcal{V}_u, \mathcal{V}_v], \mathcal{V}_w\rangle = (\mathbb{F}(w, [u, v]) + \mathcal{L}_{\overline{u}}(\mathbb{F}(v, w))) + \text{c.p.} = \partial_\gamma\mathbb{F}(u, v, w) = 0.$$

Similarly, one can show that the closedness of the space $\Gamma(L^V)$ under the Courant bracket follows from the fact that Π^V is a Poisson bivector field. $\square$

Lemma 5.6. *For any $u, v \in \chi(M)$, $\beta \in (\mathrm{Hor})^\circ$, we have*

$$d\alpha_v(\overline{u}, (\Pi^V)^\sharp\beta) = -\Pi^V(d\mathbb{F}(u, v), \beta).$$

The proof of this lemma is straightforward. It is left to the reader.

Lemma 5.7. *For any $\mathcal{H}_1, \mathcal{H}_2 \in \Gamma(L^H)$, and for any $\mathcal{V} \in \Gamma(L^V)$, we have*

$$\langle[\mathcal{H}_1, \mathcal{H}_2], \mathcal{V}\rangle = 0.$$

Proof. Let

$$\mathcal{H}_1 = (\overline{u}, \alpha_u), \quad \mathcal{H}_2 = (\overline{v}, \alpha_v) \quad \text{and} \quad \mathcal{V} = ((\Pi^V)^\sharp\beta, \beta).$$

Then,

$$2\langle[\mathcal{H}_1, \mathcal{H}_2], \mathcal{V}\rangle = \beta([\overline{u}, \overline{v}]) + \Pi^V(\beta, d\mathbb{F}(u, v))) + d\alpha_v(\overline{u}, (\Pi^V)^\sharp\beta) - d\alpha_u(\overline{v}, (\Pi^V)^\sharp\beta).$$

Using Lemma 5.6, we obtain

$$2\langle[\mathcal{H}_1, \mathcal{H}_2], \mathcal{V}\rangle = \beta([\overline{u}, \overline{v}]) - \Pi^V(d\mathbb{F}(u, v), \beta) = 0.$$ $\square$

LEMMA 5.8. *For any $u \in \chi(P)$, and for any 1-forms β_1, β_2 on M such that $\beta_{i_{|\mathrm{Hor}}} = 0$, $i = 1, 2$, we have*

$$d\beta_1((\Pi^V)^\sharp \beta_2, \overline{u}) = \mathcal{L}_{\overline{u}}((\Pi^V)^\sharp \beta_1, \beta_2) - \Pi^V(\beta_1, \mathcal{L}_{\overline{u}}\beta_2).$$

Proof. Indeed, we have

$$d\beta_1((\Pi^V)^\sharp \beta_2, \overline{u}) = -\mathcal{L}_{\overline{u}}(\Pi^V(\beta_1, \beta_2)) + \beta_1([\overline{u}, (\Pi^V)^\sharp \beta_2]).$$

Using the fact that $[\overline{u}, \Pi^V] = 0$, we obtain the formula of this lemma. □

LEMMA 5.9. *For any $\mathcal{V}_1, \mathcal{V}_2 \in \Gamma(L^V)$, and for any $\mathcal{H} \in \Gamma(L^H)$, we have*

$$\langle [\mathcal{V}_1, \mathcal{V}_2], \mathcal{H} \rangle = 0.$$

Proof. Let $\mathcal{V}_i = ((\Pi^V)^\sharp \beta_i, \beta_i)$, $i = 1, 2$, and $\mathcal{H} = (\overline{u}, \alpha_u)$. By definition,

$$2\langle [\mathcal{V}_1, \mathcal{V}_2], \mathcal{H} \rangle = \alpha_u([(\Pi^V)^\sharp \beta_1, (\Pi^V)^\sharp \beta_2]) + \mathcal{L}_{\overline{u}}(\Pi^V(\beta_1, \beta_2)) + d\beta_2((\Pi^V)^\sharp \beta_1, \overline{u}) - d\beta_1((\Pi^V)^\sharp \beta_2, \overline{u}).$$

Now, using Lemma 5.8 and the fact that $\alpha_{u_{|\mathrm{Vert}}} = 0$, we obtain

$$2\langle [\mathcal{V}_1, \mathcal{V}_2], \mathcal{H} \rangle = -\mathcal{L}_{\overline{u}}(\Pi^V(\beta_1, \beta_2)) + \Pi^V(\beta_1, \mathcal{L}_{\overline{u}}\beta_2) + \Pi^V(\mathcal{L}_{\overline{u}}\beta_1, \beta_2) = 0.$$ □

LEMMA 5.10. *For any $\mathcal{V}_1, \mathcal{V}_2 \in \Gamma(L^V)$, and for any $\mathcal{H}_1, \mathcal{H}_2 \in \Gamma(L^H)$, we have*

$$\langle [\mathcal{V}_1, \mathcal{H}_1], \mathcal{V}_2 \rangle = 0 \quad \text{and} \quad \langle [\mathcal{V}_1, \mathcal{H}_1], \mathcal{H}_2 \rangle = 0.$$

Proof. Let $\mathcal{H}_i = (\overline{u}_i, \alpha_{u_i})$ and $\mathcal{V}_i = ((\Pi^V)^\sharp \beta_i, \beta_i)$, for $i = 1, 2$. On the one hand,

$$2\langle [\mathcal{H}_1, \mathcal{V}_1], \mathcal{H}_2 \rangle = \alpha_{u_2}([\overline{u}_1, (\Pi^V)^\sharp \beta_1]) + d\beta_1(\overline{u}_1, \overline{u}_2) - d\alpha_{u_1}(\Pi^V(\beta_1), \overline{u}_2).$$

Since $[\overline{u}_1, \Pi^V] = 0$, we obtain

$$\alpha_{u_2}([\overline{u}_1, (\Pi^V)^\sharp \beta_1]) = \alpha_{u_2}((\Pi^V)^\sharp \mathcal{L}_{\overline{u}_1}\beta_1) = 0.$$

It follows that

$$\begin{aligned} 2\langle [\mathcal{H}_1, \mathcal{V}_1], \mathcal{H}_2 \rangle &= d\beta_1(\overline{u}_1, \overline{u}_2) - d\alpha_{u_1}(\Pi^V(\beta_1), \overline{u}_2) \\ &= -\beta_1([\overline{u}_1, \overline{u}_2]) + \Pi^V(d\mathbb{F}(u_1, u_2), \beta_1) \text{ by Lemma 5.6} \\ &= 0 \end{aligned}$$

since $[\overline{u}_1, \overline{u}_2] = \overline{[u_1, u_2]} + (\Pi^V)^\sharp(d\mathbb{F}(u_1, u_2))$. On the other hand,

$$\begin{aligned} 2\langle [\mathcal{H}_1, \mathcal{V}_1], \mathcal{V}_2 \rangle &= \beta_2([\overline{u}_1, (\Pi^V)^\sharp \beta_1]) + d\beta_1(\overline{u}_1, (\Pi^V)^\sharp \beta_2) \\ &= \Pi^V(\mathcal{L}_{\overline{u}_1}\beta_1, \beta_2) + \mathcal{L}_{\overline{u}_1}(\Pi^V(\beta_2, \beta_1)) - \Pi^V(\mathcal{L}_{\overline{u}_1}\beta_2, \beta_1) \\ &= 0. \end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 5.4. Let L be the vector bundle over E whose fibre at $e \in E$ is $L_e = L_e^H + L_e^V$. It follows immediately from Lemmas 5.5, 5.7, 5.9, and 5.10 that L is a Dirac structure on E. □

COROLLARY 5.11. *All geometric data $(\gamma, \Pi^V, \mathbb{F})$ on a vector bundle (E, p, P) whose associated 2-form $\mathbb{F}$ is nondegenerate determine a Poisson structure on the total space E.*

Proof. Consider the geometric data $(\gamma, \Pi^V, \mathbb{F})$ on E, where $\mathbb{F}$ is nondegenerate. Then, on each leaf of the foliation associated to the Dirac structure L obtained from Theorem 5.4, the pre-symplectic 2-form is nondegenerate. However, we know that all of the leaves of a Dirac manifold are symplectic if and only if the associated Dirac structure is the graph of a Poisson bivector field. Hence, one obtains the corollary. □

Remark. This corollary was given in [Vor00] without proof. It has been established in [Bra04] and [DW04] by methods which are different from that used here.

Acknowledgement

The work was partially supported by the Shapiro visitor program at Penn State.

References

AB06 M. Abouzaid and M. Boyarchenko, *Local structure of generalized complex manifolds*, J. Symplectic Geom. **4** (2006), 43–62.

BC97 A. Bloch and P. Crouch, *Representations of Dirac structures on vector spaces and nonlinear L-C circuits*, in *Differential geometry and control*, Boulder, CO, 1997, Proceedings of Symposia in Pure Mathematics, vol. 64 (American Mathematical Society, Providence, RI, 1999), 103–117.

Bra04 O. Brahic, Thesis, Université Montpellier 2 (2004).

BW04 H. Bursztyn and A. Weinstein, *Poisson geometry and Morita equivalence*, in *Poisson geometry, deformation quantisation and group representations*, London Mathematical Society Lecture Note Series, vol. 323 (Cambridge University Press, Cambridge, 2005), 1–78.

Cou90 T. Courant, *Dirac structures*, Trans. Amer. Math. Soc. **319** (1990), 631–661.

CW86 T. Courant and A. Weinstein, *Beyond Poisson structures*, in *Action hamiltoniennes de groupes. Troisième théorème de Lie*, Lyon, 1986, Travaux en cours, vol. 27 (Hermann, Paris, 1988), 39–49.

DW04 B. Davis and A. Wade, *Nonlinearizability of certain Poisson structures near a symplectic leaf*, Trav. Math. **16** (2005), 69–85.

Gua04 M. Gualtieri, *Generalized complex geometry*. PhD thesis, Oxford (2004), math.DG/0401221.

LWX97 Z.-J. Lu, A. Weinstein and P. Xu, *Manin triples for Lie bialgebroids*, J. Differential Geom. **45** (1997), 547–574.

TW03 X. Tang and A. Weinstein, *Quantization and Morita equivalence for constant Dirac structures on tori*, Ann. Inst. Fourier (Grenoble) **54** (2004), 1565–1580.

Vor00 Y. Vorobjev, *Coupling tensors and Poisson geometry near a single symplectic leaf*, in *Lie algebroids and related topics in differential geometry*, Banach Center Publication, vol. 54 (Polish Academy of Science, Warsaw, 2001), 249–274.

Wei83 A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geom. **18** (1983), 523–557.

Jean-Paul Dufour dufourj@math.univ-montp2.fr
Département de Mathématiques, CNRS-UMR 5030, Université Montpellier 2, 34095 Montpellier cedex 05, France

Aïssa Wade wade@math.psu.edu
Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA