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COSET ENUMERATION IN A FINITELY PRESENTED SEMIGROUP

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1. **Introduction**. The enumeration method for finite groups, the so-called Todd-Coxeter process, has been described in [2], [3]. Leech [4] and Trotter [5] carried out the process of coset enumeration for groups on a computer. However Mendelsohn [1] was the first to present a formal proof of the fact that this process ends after a finite number of steps and that it actually enumerates cosets in a group. Dietze and Schaps [7] used Todd-Coxeter's method to find all subgroups of a given finite index in a finitely presented group. B. H. Neumann [8] modified Todd-Coxeter's method to enumerate cosets in a semigroup, giving however no proofs of the effectiveness of this method nor that it actually enumerates cosets in a semigroup.

The present paper presents a proof of the fact that given a finite and finitely presented semigroup, the coset enumeration process in that semigroup ends after a finite number of steps and that it actually does enumerate the semigroup cosets. In this proof we make use of the lemma on stabilization of r initial rows in all tables after a finite number of steps (Mendelsohn, [1a]) as well as some fragments of Schreier's theory—appropriately modified for semigroups—which has been described e.g. in [6] and used by Mendelsohn in his proof. The final result in this paper is the description of an effective way of determining the semigroup P_S of transformations of a set Z into itself, where $Z=\{1, 2, \ldots, k\}$, isomorphic to a given finite and finitely presented semigroup S of order $k-1$.

The paper is self-consistent, however Mendelsohn's terminology is widely used.

2. **Description of the method.** Let $\pi=(a_1, a_2, \ldots, a_n;\ u_1=v_1, \ldots, u_r=v_r)$ be a presentation of a semigroup S. Suppose that $u_i \equiv a_{i_1} a_{i_2} \cdots a_{i_k}$ and $v_i \equiv a_{p_1} a_{p_2} \cdots a_{p_t}$. For each relation $u_i=v_i$ we build a double-table with $k+t+2$ columns. In the i-th row, the first and $(k+2)$-nd entries are to be the integer i. The $(k+1)$-st and last columns of this table are called final columns. At the top of this table between columns, the consecutive letters of the word u will appear followed by—one position being left empty—the letters of the word v_i. (See Table I). The q-th column will be called the "l-column" of the letter a_{i_q}, while the $(q+1)$-st column will be called the "r-column" of a_{i_q}.

For the generators we build a table with $n+1$ columns. At the top of this table between i-th and $(i+1)$-st columns we place the gererator a_i. In the i-th

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row of the table of generators, the first entry is the integer i. The unfilled places in the table of generators are ordered from left to right and each place in the i-th row precedes any place in the j-th row if $i<j$. If in the i-th row and $(k+1)$-st column of the table of generators appears the integer c, then we say, that from the table of generators follows the equation $i \cdot a_k = c$.

TABLE I

$a_{i_1}\, a_{i_2} \cdots a_{i_k}$	$a_{p_1}\, a_{p_2} \cdots a_{p_t}$
1	1
2	2
3	3
4	4
5	5
.	.
.	.
.	.
.	.

TABLE OF GENERATORS

	$a_1\, a_2 \cdots a_n$
1	
2	
3	
4	
5	
.	
.	
.	
.	

We build one table of the semigroup S, by placing side by side the double-tables corresponding to the relations $u_i = v_i$ $i = 1, 2, \ldots, r$.

At the end of the table of the semigroup S we place the table of generators (see B. H. Neumann [8]).

The process of coset enumeration is based on the principle of mutual adjustment of final columns in each double-table. Let i be the number ("name") of some element $s \in S$. The equation $i \cdot t = j$ where $t \in S$ denotes that the element $s \cdot t$ has number j. From equations $i \cdot u_k = q$ and $i \cdot v_k = z$ where $u_k = v_k$ is the relation from π, follows $q = z$ since $sv_k = su_k$ in S (q is the number of su_k while z is the number of sv_k).

The above idea is exactly that of Neumann [8]. Let us formulate a procedure, according to the above idea.

2.1 If from the table of generators follows $k \cdot a_i = m$ and in the "l-column" of a_i in any double-table is entered the integer k while the "r-column" of a_i is the final one of this table, then we enter the integer m into both final columns of this table, in the same row.

The process of coset enumeration of the semigroup S proceeds as follows:

At the first unfilled place in the table of generators we enter the integer 2. This will yield the equation $1 \cdot a_1 = 2$. If in any double-table in the "l-column" of a_1 is entered the integer 1, then in "r-column" of a_1 we place the integer 2. Then we proceed according to (2.1).

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2.2 Now, let us suppose that no rows of a double-table are completely filled. Then we enter into the first unfilled place of the table of generators the next integer i.e. 3. This will yield the equation $1 \cdot a_2 = 3$ (or $2 \cdot a_1 = 3$ if there is one generator in π only). We proceed as before, we enter the integer 3 into all appropriate places of the double-tables and we apply the principle (2.1). If the assumption (2.2) holds, then we continue the process for the integers $4, 5, \ldots, i-1$. The integer i is then entered into the first available place of the table of generators. This induces an equation $k \cdot a_j = i$. Next we proceed as before:

If in any double-table in the "l-column" of a_j is entered the integer k, then in "r-column" of a_j, in the same row we place the integer i. Then we proceed according to (2.1).

At this stage two possible situations may be observed:

1° No rows of the double-tables are completely filled, then at the first unfilled place of the table of generators we enter the integer $i+1$ and we proceed as before.

2° A certain row of a double-table is completely filled—we have a maybe new equation $i \cdot a_l = d$ or $r \cdot a_j = i$. Then, in the first case the integer d is placed in the i-th row, in the $(l+1)$-st column of the table of generators, while in the second case the integer i is placed in the r-th row, $j+1$-st column of the table of generators. If the place $[i, l+1]$ in the first case or the place $[r, j+1]$ of the table of generators in the second case, has been yet unfilled, then we say that creation occurs. These equations are entered in the double-tables as described above. If no new equations of type $r \cdot a_j = i$ or $i \cdot a_l = d$ are induced, then we proceed to the integer $i+1$.

There are two other situations in which one does not proceed to the integer $i+1$. The first is the closure (see Mendelsohn [1]). Closure occurs if after all possible entries of the integer i and smaller integers the first i rows of the table of the semigroup S are completely filled. In this case, if the other case does not occur, we stop the process of coset enumeration. The second case is redundancy. Redundancy may occur when proceeding as described in 2°. Then we may obtain a new equation $j \cdot a_d = k$ $(j < i)$ while from the table of generators follows that $j \cdot a_d = m$ and $m \neq k$. Suppose that $m < k$. Then we replace every appearance of k by m and we delete the k-th row from the table of semigroup S. Next, every integer n which is greater than k is replaced by $(n-1)$ at all its appearances, and we fill the double-tables in accordance with the new equations. If new redundancy appears, we alter the table of the semigroup S accordingly. If there are no further redundancies or possible entries following from the table of generators and if closure results, we stop. If closure does not result, then we enter into the first available place in the table of generators the smallest integer which does not appear in any previous place and continue as before.

3. **The proof of the finiteness of the enumeration process after a finite number of steps for a finite semigroup.** One can easily observe that the first appearance of the integer m in the k-th row, $(i+1)$-st column of the table of generators implies $k<m$. Indeed, if $k>m$, then the first $k-1$ rows of the table of generators are completely filled by integers smaller than m i.e. not greater than $k-1$. It follows that the first $k-1$ rows of each double-table are completely filled by integers not greater than $k-1$. Thus closure would appear before entering m.

In the sequel, we shall use the notation $u \equiv v$ to denote the fact that words u and v are identical, i.e. identical expressions.

Let $\pi=(a_1, a_2, \ldots, a_n;\ u_1=v_1,\ u_2=v_2, \ldots, u_r=v_r)$ where u_i, v_i, $(i=1,2,\ldots,r)$ are words in the generators $a_1,\ldots,a_n$ be a presentation of a semigroup S. The notation $\vdash_{\pi} w=u$ means that there exists a finite sequence of words $w\equiv C_0, C_1, \ldots, C_k \equiv u$ such that $C_i \equiv XT_iY$, $C_{i+1}\equiv XR_iY$, $i=0, 1,\ldots,k-1$, where X, Y are words in the generators $a_1,\ldots,a_n$ or empty words, while $T_i=R_i$ or $R_i=T_i$ is a defining relation from π. The moment of placing the k-th entry (regular or not) into the table of generators will be subsequently called the k-th step of the process.

For the k-th step of the process we define below partial functions N and ψ, which depend implicitly on k.

N maps the set of all words in generators $a_1, a_2, \ldots, a_n$ into the of all positive integers, as follows:

1° $N(\Lambda)=1$ $\quad(\Lambda-$ empty word)

2° If from the table of generators follows $k \cdot a_i = s$ and for a word w $N(w)=k$, then $N(wa_i)=s$.

It is clear, that if $w\equiv a_{i_1} a_{i_2} \cdots a_{i_m}$, u is a word in generators $a_1, a_2, \ldots, a_n$ or the empty word and $N(u)$, $N(uw)$ are defined, then $N(uw)=(\cdots((N(u)\cdot a_{i_1})\cdot a_{i_2})\cdots)\cdot a_{i_m}$. (If p is an integer and a_{i_k} is a generator then $p\cdot a_{i_k}$ denotes the integer which is placed in the p-th row, (i_k+1)-st column of the table of generators, while $p\cdot w$ denotes the integer $(\cdots((p\cdot a_{i_1})\cdot a_{i_2})\cdot\cdots)\cdot a_{i_m})$.

From this definition follows that if N is defined for the word $w\equiv a_{i_1} a_{i_2}\cdots a_{i_m}$ then N is also defined for the words $a_{i_1} a_{i_2}\cdots a_{i_t}$ for any t such that $1\le t\le m$. The integer $N(w)$ will be called the number of the word w.

Let the partial function ψ map integers into words in generators $a_1, a_2, \ldots, a_n$ or the empty word as follows:

1° $\psi(1)\equiv\Lambda$

2° If the first entry of s appears in the k-th row, $(i+1)$-st column of the table of generators and $\psi(k)$ is defined, then

$$\psi(s)\equiv\psi(k)a_i$$

From this definition follows that if ψ is defined for the integer s then ψ is defined for all the integers t where $1\le t\le s$.

COROLLARY 1. *If the function N is defined for the word* $w \equiv a_{i_1} a_{i_2} \cdots a_{i_m}$ *then the function* ψ *is defined for the integers* $1, 2, \ldots, s$ *where* $s = \max_{1 \le t \le m} \{N.(a_{i_1} \cdots a_{i_t})\}$.

Note that for given integers m and s, if the number of the completely filled rows in the table of generators is sufficiently large, then for words of length not greater than m, the values of N are defined and all the words $\psi(1), \psi(2), \ldots, \psi(s)$ are defined.

LEMMA 1. If $\vdash_{\pi} w = u$ *and a sufficiently large number* (*which depends on w and u*) *of rows of the table of the semigroup S are completely filled* i.e. *the number k of the step of the process is sufficiently large* (see Lemma 4) *then* $N(w) = N(u)$.

Proof. If $\vdash_{\pi} w = u$ then there exists a finite sequence of words $C_0, C_1, \ldots, C_k$ such that $w \equiv C_0$, $u \equiv C_k$, $C_i \equiv XT_iY$, $C_{i+1} \equiv XR_iY\ i = 0, 1, \ldots, k-1$ and $T_i = R_i$ or $R_i = T_i$ is a defining relation from π. For the proof, we need to consider a transition $\vdash_{\pi} XTY = XRY$ where $T \equiv a_{i_1} a_{i_2} \cdots a_{i_t}$, $R \equiv a_{j_1} a_{j_2} \cdots a_{j_r}$ and $T = R$ is a defining relation from π.

Let $N(X) = p$. Since in p-th of the double-table corresponding to the relation $T = R$ the same integer is entered in the final columns, let this integer be z, therefore $(\cdots((p \cdot a_{i_1}) \cdot a_{i_2}) \cdot \cdots) \cdot a_{i_t} = (\cdots((p \cdot a_{j_1}) \cdot a_{j_2}) \cdot \cdots) \cdot a_{j_r} = z$. If $Y \equiv a_{l_1} a_{l_2} \cdots a_{l_y}$ then

$$\begin{aligned} N(XTY) &= (\cdots(((\cdots((p \cdot a_{i_1}) \cdot a_{i_2}) \cdot \cdots) \cdot a_{i_t}) \cdot a_{l_1}) \cdot \cdots) \cdot a_{l_y} = \\ &= (\cdots((z \cdot a_{l_1}) \cdot a_{l_2}) \cdot \cdots) \cdot a_{l_y} = \\ &= (\cdots(((\cdots((p \cdot a_{j_1}) \cdot a_{j_2}) \cdot \cdots) \cdot a_{j_r}) \cdot a_{l_1}) \cdot \cdots) \cdot a_{l_y} = \\ &= N(XRY). \end{aligned}$$

Thus $N(C_i) = N(C_{i+1})$ for $i = 0, 1, \ldots, k-1$. Hence $N(w) = N(u)$.

Creation occurs, if after entering the integer s (which corresponds to the equation $k \cdot a_i = s$) into the table of generators in the k-th row, $(i+1)$-st column, from double-tables follows the equation $r \cdot a_j = s$ or $s \cdot a_l = d$ and this equation has not been entered into the table of generators. Redundancy occurs when in the table of generators the equation $k \cdot a_i = s$ is entered while from the double-tables follows an equation $k \cdot a_i = t$ and $s \neq t$. Entering equations which are obtained by creation ($r \cdot a_j = s$ or $s \cdot a_l = d$) into the table of generators as well as the substition of the integer t by s, if $t > s$ (and s by t otherwise) in case of redundancy, will be called non-regular entering.

LEMMA 2. *Every non-regular entering into the table of generators follows from the presentation* π *of the semigroup S* i.e.

1° *If equation* $k \cdot a_i = s$ *induces a new equation* $r \cdot a_j = s$ *then* $\vdash_{\pi} \psi(r)\, a_j = \psi(s)$.

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2° *If equation* $k \cdot a_i = s$ *induces a new equation* $s \cdot a_l = d$ *then* $\vdash_{\pi} \psi(s)\ a_l = \psi(d)$.

3° *If equations* $k \cdot a_i = s$ *and* $k \cdot a_i = t$ *appear, and* $s \neq t$ *then* $\vdash_{\pi} \psi(s) = \psi(t)$ *moreover, if* $N(w) = s$ *then* $\vdash_{\pi} w = \psi(s)$.

Proof. In proceeding with the enumeration process we enter non-regular enterings into the table of generators in some order. The proof is by induction on the succession of non-regular enterings.

Let us note that as long as non-regular enterings do not appear, each equation $k \cdot a_i = s$ corresponds to the identity $\psi(k)a_i \equiv \psi(s)$.

The induction step. A non-regular entering occurs. Assume that the lemma holds for the previous enterings.

1° In some double-table in the p-th row we have:

$$(\cdots((p \cdot a_\alpha) \cdot a_\beta) \cdot \cdots) \cdot a_\gamma = k, (\cdots((p \cdot a_\xi) \cdot a_\eta) \cdot \cdots) \cdot a_\omega = r$$

where the relation $a_\alpha a_\beta \cdots a_\gamma a_i = a_\xi a_n \cdots a_\omega a_j$ belongs to π. This case corresponds to creation which induces a new equation of the type $r \cdot a_\gamma = s$. Since previous non-regular enterings follow from the presentation π, therefore $\vdash_{\pi} \psi(p)a_\xi a_\eta \cdots a_\omega = \psi(r)$, $\vdash_{\pi} \psi(p)a_\alpha a_\beta \cdots a_\gamma = \psi(k)$ and if $N(u) = r$ then $\vdash_{\pi} u = \psi(r)$. Now, we enter into the table of generators the equation $k \cdot a_i = s$. Since the previous non-regular enterings follow from the presentation π, therefore $\vdash_{\pi} \psi(k)a_i = \psi(s)$. A new equation $r \cdot a_j = s$ is induced. Since $\vdash_{\pi} \psi(p)a_\xi a_\eta \cdots a_\omega a_j = \psi(p)a_\alpha a_\beta \cdots a_\gamma a_i$ therefore $\vdash_{\pi} \psi(r)a_j = \psi(k)a_i$. Hence $\vdash_{\pi} \psi(r)a_j = \psi(s)$. Since $N(ua_j) = s$ and $\vdash_{\pi} ua_j = \psi(r)a_j$, therefore $\vdash_{\pi} ua_j = \psi(s)$.

2° In some double-table in the p-th row we have:

$$(\cdots((p \cdot a_\alpha) \cdot a_\beta) \cdot \cdots) \cdot a_\gamma = k, \qquad p \cdot u(A) = d,$$

where $u(A)$ is a word in generators $a_1, \ldots, a_n$ and the relation $a_\alpha a_\beta \cdots a_\gamma a_i a_l = u(A)$ belongs to π. This case corresponds to creation which induces a new equation of the type $s \cdot a_l = d$. Since previous non-regular enterings follow from the presentation π, therefore $\vdash_{\pi} \psi(p)a_\alpha a_\beta \cdots a_\gamma = \psi(k)$, $\vdash_{\pi} \psi(p)u(A) = \psi(d)$. Now, we enter into the table of generators the equation $k \cdot a_i = s$. Since the previous non-regular enterings follow from the presentation π, therefore $\vdash_{\pi} \psi(k)a_i = \psi(s)$ and if $N(w) = s$ then $\vdash_{\pi} w = \psi(s)$. A new equation $s \cdot a_l = d$ is induced. Since $\vdash_{\pi} \psi(p)a_\alpha a_\beta \cdots a_\gamma a_i a_l = \psi(p)u(A)$ therefore $\vdash_{\pi} \psi(s)a_l = \psi(d)$. Since $N(wa_l) = d$ and $\vdash_{\pi} wa_l = \psi(s)a_l$ therefore $\vdash_{\pi} wa_l = \psi(d)$.

3° An equation $k \cdot a_i = s$ is entered into the table of generators. In some double-table in the p-th row we have:

$(\cdots((p \cdot a_\alpha) \cdot a_\beta) \cdot \cdots) \cdot a_\gamma = k$, $p \cdot u(A) = t$ where $u(A)$ is a word in generators $a_1, \ldots, a_n$ and the relation $a_\alpha a_\beta \cdots a_\gamma a_i = u(A)$ belongs to π. This case corresponds to redundancy. Since the previous non-regular enterings follow from π therefore $\vdash_{\overline{\pi}} \psi(p)a_\alpha a_\beta \cdots a_\gamma = \psi(k)$, $\vdash_{\overline{\pi}} \psi(p)u(A) = \psi(t)$, $\vdash_{\overline{\pi}} \psi(k)a_i = \psi(s)$ i.e. $\vdash_{\overline{\pi}} \psi(p)a_\alpha a_\beta \cdots a_\gamma a_i = \psi(s)$. We obtain the equation $k \cdot a_i = t$ as a result of the adjustment of final columns of the double-table. Since $\vdash_{\overline{\pi}} \psi(p)a_\alpha a_\beta \cdots a_\gamma a_i = \psi(p)u(A)$ therefore $\vdash_{\overline{\pi}} \psi(s) = \psi(t)$.

Suppose that $s < t$. In effect of a redundancy we delete the t-th row and we replace any integer greater than t by $t-1$. Now, we modify the function N and ψ for the next step.

$$\overline{N(w)} = \begin{cases} N(w) & \text{if } N(w) < t \\ s & \text{if } N(w) = t \\ N(w)-1 & \text{if } N(w) > t \end{cases}$$

$$\overline{\psi(x)} = \begin{cases} \psi(x) & \text{if } x < t \\ \psi(x+1) & \text{if } x > t \end{cases}$$

It is clear that any of the equations $\underline{k} \cdot a_i = s$ entered into the table of generators implies at present $\vdash_{\overline{\pi}} \overline{\psi(k)}a_t = \overline{\psi(s)}$ and if $\overline{N(w)} = s$ then $\vdash_{\overline{\pi}} w = \overline{\psi(s)}$.

LEMMA 3. If $N(u) = N(w)$ then $\vdash_{\overline{\pi}} u = w$.

Proof. By lemma 2 $N(u) = s$ implies $\vdash_{\overline{\pi}} u = \psi(s)$ and if $N(w) = s$ then $\vdash_{\overline{\pi}} w = \psi(s)$. Hence $\vdash_{\overline{\pi}} u = w$.

LEMMA 4. (*Mendelsohn* [1*a*]). *After a finite number of steps the first r rows of the table of the semigroup S are stabilized* i.e. *none of the entries are further altered because of redundancy.*

Proof. Use induction on the row number. For $i = 1$ in the first row of the table of the semigroup S there is only a finite number of places and each is to be occupied by a positive integer. The effect of a redundancy is to replace some of these entries by some smaller positive integer and this can happen only finitely often. Hence, after a finite number of steps the first row becomes stable.

If closure does not result for the first k rows then suppose that the first k rows are stabilized after a finite number of steps. Since the first appearance of $k+1$ is somewhere in the first k rows beyond this point no redundancy involves the replacement of $k+1$ by a smaller integer. The $k+1$ row will not be deleted.

The argument used for the first row is now valid for the $(k+1)$-st.

This proof is an exact copy of the proof of Mendelsohn.

If the process is finite for s rows, then the function N, which enumerates the cosets of S and maps the set of all words in the generators $a_1, \ldots, a_n$ onto the set$\{1, 2, \ldots, s\}$ as well as the function ψ leading from $\{1, 2, \ldots, s\}$ into the set of all words, are defined.

If the process continues indefinitely without closure then these functions are also defined. The function N maps the set of all words onto the set of all positive integers while the function ψ maps the set of all positive integers into the set of words. In such a case these functions, of course, need not be defined effectively. By lemma 1, 2, and 3 we obtain the following

THEOREM. The function N enumerates the "cosets" of a semigroup i.e. $N(u)=N(w)$ iff $\vdash_{\pi} u=w$. The set $\psi(1)$, $\psi(2),\cdots$ is a right-sided "Schreier system" for π i.e. if $\psi(j)\equiv a_{i_1}\cdots a_{i_j}$ then $a_{i_1}\cdots a_{i_{j-1}}\equiv\psi(k)$ for some integer $k<j$. The mapping $\phi_u : i\to N(\psi(i)\cdot u)$ is an embedding of a semigroup π in the semigroup of transformations of the set of all integers.

COROLLARY 2. A semigroup S is finite iff the process of its coset enumeration stops after a finite number of steps.

REMARK. In this paper, the role of an algorithmic set described by Mendelsohn in [1] is played by the table of generators. Note that each generator a_i induces a mapping of the set of all positive integers into itself, since we fill all the unfilled places of the table of generators.

4. EXAMPLES. To illustrate the enumeration process we consider the semigroup $\pi=(a, b;\ ab^2=b^2,\ b^2a=a)$. It is a particular case of a semigroup investigated by Neumann [8]. We build the table of the semigroup S and we fill the rows of the table until redundancy appears.

	a	b	b		b	b		b	b	a		a		a	b
1	2	5	7	1	3	7	1	3	7	2	1	2	1	2	3
2	4		7	2	5	7	2	5	$\underline{7}$	$\underline{4}$	2	4	2	4	5
3	6			3	7		3	7		6	3	6	3	6	7
4				4			4				4		4		
5				5	7		5	7			5		5		7
6				6			6				6		6		
7	2			7			7			2	7	2	7	2	
.				.			.				.		.		
.				.			.				.		.		
.				.			.				.		.		

Two equations $5\cdot b=7$ and $7\cdot a=2$ follow from the double-tables. These equations are entered into the table of generators and then all appropriate places in the double-tables are filled. We obtain redundancy. In the double-table the part underlined induces a new equation $7\cdot a=4$. Since from the table of generators follows $7\cdot a=2$, therefore we delete the 4-th. Next, we replace every appearance of 4 by 2 and we replace the numbers of the rows $5, 6, 7, \ldots$ by $4, 5, 6, \ldots$ respectively. We have now obtained the altered table.

$a\ b\ b$	$b\ b$	$b\ b\ a$	a	$a\ b$
1 2 4 6	1 3 6	1 3 6 2	1 2	1 2 3
2 2 4 6	2 4 6	2 4 6 2	2 2	2 2 4
3 5	3 6	·3 6 5	3 5	3 5 6
4	4 6	4 6	4	4 6
5	5	5	5	5
6 2	6	6 2	6 2	6 2
7	7	7	7	7
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

We then continue according to the previously described method and we have the following situation:

$a\ b\ b$	$b\ b$	$b\ b\ a$	a	$a\ b$
1 2 4 6	1 3 6	1 3 6 2	1 2	1 2 3
2 2 4 6	2 4 6	2 4 6 2	2 2	2 2 4
3 5 7 8	3 6 8	3 6 8 5	3 5	3 5 6
4 5 7 8	4 6 8	4 6 8 5	4 5	4 5 6
5 5 7 8	5 7 8	5 7 8 5	5 5	5 5 7
6 2 4 6	6 8 6	6 8 6 2	6 2	6 2 8
7 2 4 6	7 8 6	7 8 6 2	7 2	7 2 8
8 5 7 8	8 6 8	8 6 8 5	8 5	8 5 6

Closure occurs since 8 rows of the table of the semigroup S are completely filled with the integers which are not-greater than 8. We stop the enumeration process.

The semigroup S is isomorphic with the semigroup P_S of transformations of the set $\{1, 2, \ldots, 8\}$ generated by two transformations ϕ_a, ϕ_b such that

$$\phi_a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 2 & 5 & 5 & 5 & 2 & 2 & 5 \end{pmatrix}$$

$$\phi_b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 6 & 7 & 8 & 8 & 6 \end{pmatrix}$$

Moreover, we find the set of coset representatives

$$\psi(2) \equiv a,\ \psi(3) \equiv b,\ \psi(4) \equiv ab,\ \psi(5) \equiv ba,$$
$$\psi(6) \equiv bb,\ \psi(7) \equiv bab,\ \psi(8) \equiv bbb.$$

2° The cyclic semigroup $S = (a;\ a^n = a)$ is isomorphic with the semigroup P_S of transformations, generated by ϕ_a such that

$$\phi_a(k) = \begin{cases} k+1 & \text{for} \quad k = 1, 2, \ldots, n-1 \\ 2 & \text{for} \quad k = n \end{cases}$$

3° The free semigroup on n generators $a_1, \ldots, a_n$ is isomorphic with the semigroup P_S generated by the transformations $\phi a_1, \ldots, \phi a_n$ such that

$$\phi_{a_i}(k) = (k-1)n + (i+1) \quad \text{for} \quad i = 1, 2, \ldots, n \text{ and } k = 1, 2, \ldots$$

4° Our method allows to enumerate the cosets of a semigroup with identity represented by the empty word Λ.

Let $\pi = (a, b, \Lambda;\ ab^2 = \Lambda,\ a^2 = b^3,\ ba = b^2a)$ where Λ denote the empty word. We assign the number 1 to Λ. We then continue the enumeration in a well known fashion and we obtain the following results $N(a) = N(b) = N(\Lambda) = 1$. Thus π is the presentation of the trivial semigroup.

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