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AUTOMORPHISMS OF G-AZUMAYA ALGEBRAS

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Let R be a commutative ring, G a finite abelian group of order n and exponent m, and assume n is a unit in R. In **[10]**, F. W. Long defined a generalized Brauer group, $BD(R, G)$, of algebras with a G-action and G-grading, whose elements are equivalence classes of G-Azumaya algebras. In this paper we investigate the automorphisms of a G-Azumaya algebra A and prove that if $\mathrm{Pic}_m(R)$ is trivial, then these automorphisms are all, in some sense, inner.

In fact, each of these "inner" automorphisms can be written as the composition of an inner automorphism in the usual sense and a "linear" automorphism, i.e., an automorphism of the type

$$a \to \sum_{\sigma \in G} r(\sigma) a_\sigma$$

with $r(\sigma)$ a unit in R. We then use these results to show that the group of gradings of the centre of a G-Azumaya algebra A is a direct summand of G, and thus if G is cyclic of order p^r, A is the (smash) product of a commutative and a central G-Azumaya algebra. This fact leads to a proof of the short exact sequence

$$1 \to (BM(R, G)/B(R)) \times (BC(R, G)/B(R))$$

$$\to BD(R, G)/B(R) \to \prod_{i=1}^{t} D_i \to 1$$

where

$$G = \prod_{i=1}^{t} C_i$$

with C_i cyclic of odd order $p_i^{r_i}$, $p_i \neq p_j$ for $i \neq j$, $BM(R, G)$ and $BC(R, G)$ are the Brauer groups of G-module algebras and G-comodule algebras respectively, $B(R)$ is the usual Brauer group of R, and D_i is the dihedral group of order $2(p_i^{r_i} - p_i^{r_i - 1})$.

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Preliminaries. Throughout, R will denote a connected commutative ring with unity and G a finite abelian group of order n and exponent m. We assume n is a unit in R, R contains a primitive m^{th} root of unity, and $\operatorname{Spec}(R)$ has finitely many irreducible components. For the definitions of G-dimodules, G-dimodule algebras, G-Azumaya algebras, and the groups $BD(R, G)$, $BM(R, G)$ and $BC(R, G)$, we refer the reader to [**10**].

All algebras and modules are understood to be R-algebras and R-modules. Hom, $\otimes$, $\#$, etc. are taken over R unless otherwise stated. Here $\#$ denotes the smash product ([**9**], [**10**], [**14**]); for example if A and B are G-graded algebras with G-action, then $A \# B$ is the R-module $A \otimes B$ but with multiplication given by

$$(a \# b)(c \# d) = a\beta(c) \# bd \quad \text{for } b \in B_\beta.$$

The group of units of an algebra A is written $U(A)$. All formulae defined only for homogeneous elements of a graded module are to be extended by linearity.

1. Automorphisms of G-Azumaya algebras. The usual Morita theory results for Azumaya algebras may be imitated exactly for G-Azumaya algebras. If A and B are G-dimodule algebras, let $(G, A - B)$-Mod denote the category of $A - B$ bimodules P which are also G-dimodules such that the $A - B$ bimodule structure map on P is a G-dimodule map. For A a G-dimodule algebra, A is G-Azumaya implies that the following functors are inverse equivalences of categories:

$$K: (G, R - R)\text{-Mod} \to (G, A \# \bar{A} - R)\text{-Mod} \qquad \text{where } K(X) = A \otimes X$$

$$L: (G, A \# \bar{A} - R)\text{-Mod} \to (G, R - R)\text{-Mod} \qquad \text{where } L(Y) = Y^A.$$

(Recall that $Y^A = \{y \in Y: (a \# \bar{1})y = (1 \# \bar{a})y \text{ for all } a \in A\}$.) Then for $M \in (G, A \# \bar{A} - R)$-Mod, $A \otimes M^A \cong M$ by the isomorphism $a \otimes m \to am$. The proof is straightforward if we note that

$$\operatorname{Hom}_{A \# \bar{A}}(A, M) \cong M^A$$

under the map $f \to f(1)$.

Further details on standard Morita theory may be found in [**1**] and details on Morita theory in the context of G-dimodules may be found in [**11**]. (Interpret the results in [**11**] with $H = RG$.)

Now suppose A is a G-Azumaya algebra and denote by G-Aut(A) the group of algebra automorphisms Ω of A which commute with the elements of G acting as automorphisms on A and preserve the G-grading. Note that $G \subset G$-Aut(A). As in [**7**], any element Ω of G-Aut(A) may be used to define a left $A \# \bar{A}$-structure on the G-dimodule A by

$$(a \# \bar{b})(c) = \sum_{\sigma \in G} a(\sigma c)\Omega(b_\sigma).$$

Call this left $A \# \bar{A}$-module A_Ω. Define a left $A \# \bar{A}$-module structure on $A_\Omega \otimes_A A_\Gamma$ by

$$(c \# \bar{d})(a \otimes_A b) = \sum_{\sigma \in G} c(\sigma a) \otimes (\sigma b)\Gamma(d_\sigma).$$

Then an argument similar to [**1**, p. 78] or [**7**, p. 68] shows that

$$A_\Omega \otimes_A A_\Gamma \cong A_{\Omega\Gamma} \quad \text{in } \mathrm{Mod}(G, A \# \bar{A} - R),$$

A_Ω^A is a finitely generated projective rank 1 R-module and

$$A_\Omega^A \otimes A_\Gamma^A \cong A_{\Omega\Gamma}^A.$$

Thus there is a group homomorphism α from G-Aut(A) to Pic(R) defined by $\alpha(\Omega) = A_\Omega^A$.

Definition 1.1. An element Ω of G-Aut(A) is called *G-inner* if for some u in $U(A)$ and all a in A,

$$\Omega(a) = a(\bar{u} \# u^{-1})$$

where A has the usual right $\bar{A} \# A$-structure. Then

$$\Omega(a) = \sum_{\sigma \in G} (\sigma u)(a_\sigma)(u^{-1}).$$

We denote this automorphism by I_u. Let G-Inn(A) denote the set of G-inner automorphisms in G-Aut(A).

PROPOSITION 1.2. *The following is an exact sequence*

$$1 \to G\text{-Inn}(A) \to G\text{-Aut}(A) \xrightarrow{\alpha} \mathrm{Pic}(R)$$

where $\alpha(\Omega) = A_\Omega^A$ *for* $\Omega \in G$-Aut(A).

Proof. We must show that G-Inn(A) = Ker α. First suppose that $\Omega = I_u \in G$-Inn(A) and let $a \in A_\Omega^A$. Then since $(b \# \bar{1})a = (1 \# \bar{b})a$, an easy computation shows that

$$(b \# \bar{1})(au) = (1 \# \bar{b})(au).$$

Therefore

$$au \in A^A = R \quad \text{and} \quad A_\Omega^A = Ru^{-1} \cong R.$$

Conversely, suppose that $\Omega \in$ Ker α, so that $\Gamma = \Omega^{-1} \in$ Ker α also. There is an R-module isomorphism $\gamma: R \to A_\Gamma^A$ with $\gamma(1) = v$. Then $A_\Gamma^A = Rv$ and $v \in U(A)$. Since, for all $b \in A$,

$$bv = \sum_{\sigma \in G} (\sigma v)\Gamma(b_\sigma),$$

then

$$b = \sum_{\sigma \in G} (\sigma v)\Gamma(b_\sigma)v^{-1}.$$

Therefore

$$\begin{aligned}\Omega(b) &= \sum_{\sigma \in G} \Omega(\sigma v)\Omega\Gamma(b_\sigma)\Omega(v^{-1}) \\ &= \sum_{\sigma \in G} (\sigma\Omega(v))b_\sigma(\Omega(v))^{-1} \\ &= I_{\Omega(v)}(b),\end{aligned}$$

and $\Omega \in G\text{-Inn}(A)$.

LEMMA 1.3. *Suppose A is a G-Azumaya algebra and $I_u \in G\text{-Inn}(A)$. Then*

$$u^{-1}\sigma(u) \in U(R) \quad \textit{for all } \sigma \in G.$$

Proof. We show that for all $a \in A$,

$$(1 \# \bar{a})u^{-1}\sigma(u) = (a \# \bar{1})u^{-1}\sigma(u),$$

and thus $u^{-1}\sigma(u) \in A^A = R$.

$$\begin{aligned}(1 \# \bar{a})u^{-1}\sigma(u) &= \sum_{\gamma \in G} \gamma(u^{-1})\gamma\sigma(u)a_\gamma \\ &= \sum_{\gamma \in G} \gamma(u^{-1})[\gamma\sigma(u)a_\gamma\sigma(u^{-1})]\sigma(u) \\ &= \sum_{\gamma \in G} \gamma(u^{-1})\sigma[\gamma(u)\sigma^{-1}(a_\gamma)u^{-1}]\sigma(u) \\ &= \sum_{\gamma \in G} \gamma(u^{-1})\sigma(I_u(\sigma^{-1}(a_\gamma)))\sigma(u) \\ &= \sum_{\gamma \in G} \gamma(u^{-1})I_u(a_\gamma)\sigma(u) \\ &= \sum_{\gamma \in G} \gamma(u^{-1})\gamma(u)a_\gamma u^{-1}\sigma(u) \\ &= \left(\sum_{\gamma \in G} a_\gamma\right)u^{-1}\sigma(u) \\ &= (a \# \bar{1})u^{-1}\sigma(u).\end{aligned}$$

From the preceding lemma, we can see that each element I_u of $G\text{-Inn}(A)$ is a composition of an inner automorphism in the usual sense and a "linear" automorphism. For

$$I_u(a) = \sum_{\sigma \in G} \sigma(u)a_\sigma u^{-1} = \sum_{\sigma \in G} uu^{-1}\sigma(u)a_\sigma u^{-1} = i_u \cdot L_u(a)$$

where

$$i_u(a) = uau^{-1} \quad \text{and} \quad L_u(a) = \sum_{\sigma \in G} r(\sigma)a_\sigma, \quad r(\sigma) \in U(R).$$

$I_u(1) = 1$ implies $r(1) = 1$. Because I_u and i_u are R-algebra automorphisms, so is L_u. Also since I_u and L_u commute with the elements of G and preserve G-grading, i_u does as well. Therefore i_u and L_u lie in G-Aut(A).

From now on, we assume $\mathrm{Pic}_m(R)$ is trivial. Then for $\sigma \in G$, $\sigma = I_u$ for some $u \in U(A)$, i.e., $\sigma = i_\sigma \cdot L_\sigma$ for i_σ an inner automorphism and

$$L_\sigma(a) = \sum_{\gamma \in G} r(\sigma, \gamma)a_\gamma.$$

From the above discussion, $r(\sigma, 1) = 1$, and since L_1 may be taken to be the identity, $r(1, \gamma) = 1$. Since G acts as a group of automorphisms, for σ, τ in H, the group of gradings of the centre of A,

$$r(\sigma\tau, \gamma) = r(\sigma, \gamma)r(\tau, \gamma) \quad \text{for all } \gamma \in G.$$

Suppose A is fully graded (i.e., $A_\alpha A_\beta = A_{\alpha\beta}$ for all α, $\beta \in G$); every class in $BD(R, G)$ contains a fully graded algebra [**5**, p. 309]. Then each A_σ is faithful and since for all $\sigma \in G$, $L_\sigma(ab) = L_\sigma(a)L_\sigma(b)$ for $a \in A_\alpha$, $b \in B_\beta$,

$$r(\sigma, \alpha\beta) = r(\sigma, \alpha)r(\sigma, \beta).$$

Thus $\phi(\sigma, \tau) = r(\sigma, \tau)$ is a bilinear map from $H \times G$ to $U(R)$.

2. Automorphisms of *H*-Azumaya algebras. The results of the previous section may be generalized to apply to H-Azumaya algebras for H any finitely-generated projective commutative cocommutative Hopf algebra over R. (In Section 1, the Hopf algebra H in question was the group ring RG.)

We refer the reader to [**9**] for the definitions and details of H-dimodule algebras, H-Azumaya algebras and the groups $BD(R, H)$, $BM(R, H)$ and $BC(R, H)$. The notation is that used in [**9**] and [**14**].

Let H-Aut(A) be the group of algebra automorphisms Ω of A such that for $a \in A$

i) $\quad \Omega(ha) = h\Omega(a) \quad$ for all h in H

and

ii) $$\sum_{(\Omega(h))} \Omega(h)_{(0)} \otimes \Omega(h)_{(1)} = \sum_{(h)} \Omega(h_{(0)}) \otimes h_{(1)},$$

i.e., Ω "does not disturb" the H-dimodule structure of A. Let H-Inn(A) be those elements of H-Aut(A) of the form

$$I_u(a) = a(\bar{u} \# u^{-1}) \quad \text{for some } u \in U(A)$$

where A is a right $\bar{A} \# A$-module in the usual way. Then there is an exact sequence

$$1 \rightarrow H\text{-Inn}(A) \rightarrow H\text{-Aut}(A) \xrightarrow{\alpha} \mathrm{Pic}(R)$$

where $\alpha(\Omega) = A_\Omega^A$. (As in Section 1, A_Ω is the H-dimodule A but with left $A \# \bar{A}$-module structure given by

$$(a \# \bar{b})(c) = \sum_{(b)} a(b_{(1)}c)\Omega(b_{(0)}).$$

An H-inner automorphism I_u also may be written as a composition of an inner and a "linear" automorphism in H-Aut(A), i.e.,

$$I_u = i_u \cdot L_u$$

where

$$i_u(a) = uau^{-1} \quad \text{and} \quad L_u(a) = \sum_{(a)} u^{-1}(a_{(1)}u)a_{(0)}.$$

Again $u^{-1}(hu) \in U(R)$ for all $h \in H$.

3. Applications to $BD(R, G)$. First we prove that for any finite abelian group G, the group of gradings of the centre of a G-Azumaya algebra is a direct summand of G.

PROPOSITION 3.1. *For any finite abelian group G, and G-Azumaya algebra A with centre Z, let $H = \{\sigma : Z_\sigma \neq 0\}$, the group of gradings of Z. Then H is a direct summand of G.*

Proof. Since every element of $BD(R, G)$ has a fully graded representative [**5**, p. 309], and equivalent G-Azumaya algebras have isomorphic centres, we assume throughout that A is fully graded.

By [**12**, Corollary 2.5], H is a subgroup of G. Let

$$K = \{\sigma \in G : \sigma(z) = z \text{ for all } z \in Z\},$$

the group of elements of G which leave Z fixed. Then K is the group of elements of G which act as inner automorphisms on A. By [**12**, Proposition 2.2], Z is a Galois $(G/K)R$-object so that for a left $Z \# R(G/K)$-module M, the multiplication map from $Z \otimes M^{R(G/K)}$ to M is an R-module isomorphism.

Define a G/K action on A in the following way. Suppose

$$G/K = \prod_{i=1}^{t} C_i$$

where C_i is a cyclic group of prime power order $p_i^{r_i}$. For each i, select a generator of C_i, say $[\gamma_i]$, and let $\gamma_i \in [\gamma_i]$. Then $\gamma_i = \kappa_i \cdot \pi_i$ where κ_i is an inner automorphism and π_i is a linear automorphism as in the discussion following Lemma 1.3. Note that although $\gamma_i \in G$, κ_i and π_i may not lie in G. However, γ_i and π_i have the same action on Z. Now define the action of C_i on A by $[\gamma_i]^s(a) = \pi_i^s(a)$. It is easily checked that A is a $Z \# R(G/K)$-module with this action so that the multiplication map

$$m:Z \otimes A^{R(G/K)} \to A$$

is an R-module isomorphism.

Suppose

$$\pi_i(a) = \sum_{\gamma \in G} r(i, \gamma)a_\gamma,$$

and let

$$J_i = \{\sigma \in G:r(i, \sigma) = 1\}.$$

Since $r(i, \gamma)r(i, \tau) = r(i, \gamma\tau)$ and $r(i, 1) = 1$, J_i is a subgroup of G. Let

$$J = \bigcap_{i=1}^{t} J_i.$$

Then

$$A^{R(G/K)} = \bigoplus_{\sigma \in J} A_\sigma.$$

The inclusion $\supseteq$ is clear. Conversely, suppose $a \in A^{R(G/K)}$, a of grade τ, $\tau \notin J$, so that $r(i, \tau)a = a$ for all i but $r(j, \tau) \neq 1$ for some j. Since $r(j, \tau)$ is a primitive k-th root of unity for some k, and since

$$a = r(j, \tau)a = r(j, \tau)^2 a = \ldots = r(j, \tau)^{k-1}a,$$

we have

$$a = (1/k)(1 + r(j, \tau) + r(j, \tau)^2 + \ldots + r(j, \tau)^{k-1})a = 0.$$

Since the multiplication map from $Z \otimes \left(\bigoplus_{\sigma \in J} A_\sigma\right) \to A$ is a G-dimodule algebra isomorphism and A is fully graded, H and J must generate G. Also $H \cap J = \{1\}$ since $Z^G = R$ **[12]** and thus $G \simeq H \times J$.

Remark. Note that in general $J \neq K$. In fact the construction of J in the proof above depends upon the particular choice of γ_i and π_i.

In the remainder of this paper, we apply the results above and compute $BD(R, G)$ for G cyclic.

Suppose G is cyclic of order

$$n = \prod_{i=1}^{t} p_i^{r_i}, \quad p_i \text{ odd}, \quad p_i \neq p_j.$$

In [**4**], L. N. Childs showed that $BD(R, G)$ may be described by the following short exact sequence

$$(*) \qquad 1 \to H^2(G, U(R)) \times H^2(G, U(R)) \to BD(R, G)/B(R)$$
$$\to \prod_{i=1}^{t} D_i \to 1$$

where D_i is the dihedral group of order $2(p_i^{r_i} - p_i^{r_i - 1})$.

In [**3**], it was shown that if all cocycles in $H^2(G, U(R))$ are abelian, the classes in $BD(R, G)$ containing a central separable algebra form a subgroup, $B(R, G)$. $B(R, G)/B(R)$ may be described by the following short exact sequence [**3**, Theorem 1.2]:

$$(**) \quad 1 \to T \to B(R, G)/B(R) \xrightarrow{\beta} \mathrm{Aut}(G) \to 1$$

where

$$T = (BM(R, G)/B(R)) \times (BC(R, G)/B(R))$$
$$\simeq H^2(G, U(R)) \times H^2(G, U(R))$$

if $\mathrm{Pic}_m(R)$ is trivial.

In fact, (**) splits since

$$B(R, G)/B(R) \simeq T \times BT(R, G).$$

(See [**6**] for the definition of $BT(R, G)$ and proof that $BT(R, G) \simeq \mathrm{Aut}(G)$.)

If $r_i = 1$ for all i, then an outline of another proof of (*) using the subgroup structure of $BD(R, G)$ was sketched in [**3**, p. 524] using the sequence (**). Using the results above, we extend this proof to cases where $r_i > 1$.

PROPOSITION 3.3. *For G cyclic of prime power order p^r, $B(R, G)$ is a normal subgroup of index* 2 *in $BD(R, G)$.*

Proof. By [**3**], $B(R, G)$ is a subgroup of $BD(R, G)$. Map $BD(R, G)$ to $\{+1, -1\}$ by mapping the elements of $B(R, G)$ to 1 and the rest to -1. We must show that for A, B in $BD(R, G) - B(R, G)$, $A \# B \in B(R, G)$.

By Proposition 3.1, Z and W, the centres of A and B respectively, both have group of gradings G. By [**12**, Proposition 2.2 and 2.11], Z is a Galois RG-object and W is a Galois GR-object, so that

$$A \simeq A^G \# Z \simeq A^G \otimes Z \quad \text{and} \quad B \simeq W \# B_1 \simeq W \otimes B_1$$

as G-dimodule algebras. Now we must show that $Z \# W$ is central. By [**12**, Proposition 2.8], $Z \simeq RG_f^\phi$ with basis z_σ, $\sigma \in G$, and $W = RG_g^\chi$ with basis w_τ, $\tau \in G, f, g \in H^2(G, U(R))$ and ϕ, χ nondegenerate bilinear maps from $G \times G \to U(R)$. Suppose $x = z_\sigma \# w_\tau$ for some $\sigma, \tau \in G$, σ, τ not 1, and suppose α and β are such that

$$\phi(\alpha, \sigma) \neq 1 \quad \text{and} \quad \phi(\tau, \beta) \neq 1.$$

Then $1 \# w_\alpha$ and $z_\beta \# 1$ do not commute with x. Since $Z \# W$ is a free module with generators $z_\sigma \# w_\tau$, this shows that $Z \# W$ is central.

Finally, note that since R contains a primitive p^r-th root of unity, non-central G-Azumaya algebras of the form RG^ϕ exist, so that $B(R, G)$ is a proper subgroup of $BD(R, G)$.

Recall that the map β in the sequence (**) was defined as follows. For A central separable, the elements of G act as inner automorphisms on A. Then for $\sigma \in G$, $\sigma(a) = xax^{-1}$ for some homogeneous $x \in U(A)$. $\beta(A)$ is the element of $\mathrm{Aut}(G)$ defined by

$$\beta(A)(\sigma) = \sigma(\text{grade } x)^{-1}.$$

Details of the proof that β is a well-defined group epimorphism may be found in [**3**, pp. 519-520].

PROPOSITION 3.4. *Let G be cyclic of prime power order p^r. Then*

$$(BM(R, G)/B(R)) \times (BC(R, G)/B(R))$$

is a normal subgroup of $BD(R, G)/B(R)$.

Proof. Let T denote $(BM(R, G)/B(R)) \times (BC(R, G)/B(R))$. By [**3**, Theorem 1.2], $T = \operatorname{Ker} \beta$ is normal in $B(R, G)/B(R)$. Let $A \in BD(R, G)/B(R)$, A not central, $B \in BM(R, G)/B(R)$, $C \in BC(R, G)/B(R)$. By Proposition 3.3, $A \# B \# \bar{A}$ and $A \# C \# \bar{A}$ are in $B(R, G)/B(R)$. We show that they are in $\operatorname{Ker} \beta = T$.

By Proposition 3.1, $A \simeq D \# Z$ where Z is the centre of A, $Z \simeq RG_f^\phi, f \in H^2(G, U(R))$, ϕ bilinear, and $D = A^G$ is central. Then

$$\beta(A \# B \# \bar{A}) = \beta(D \# Z \# B \# \bar{Z} \# \bar{D}) = \beta(Z \# B \# \bar{Z}).$$

By [**12**], $\bar{Z} \simeq RG_{\phi f}^\phi$ and for $\sigma \in G$, σ-action on $Z \# B \# \bar{Z}$ is given by conjugation by $u_\sigma^{-1} \# 1 \# u_\sigma$ (where the u_σ are a basis for Z). Thus

$$\beta(Z \# B \# \bar{Z}) = 1.$$

Now consider

$$\beta(A \# C \# \bar{A}) = \beta(D \# Z \# C \# \bar{Z} \# \bar{D}) = \beta(Z \# C \# \bar{Z}).$$

The map $\alpha: C \to C$ defined by

$$\alpha(c) = \phi(\sigma, \gamma)^{-1}c$$

for c homogeneous of grade γ, is an algebra automorphism of C with $\alpha^m = 1$, so that α is an inner automorphism, i.e., there is a $w \in U(C)$ such that

$$\alpha(c) = wcw^{-1} \quad \text{for all } c \in C.$$

By [**12**, Proposition 4.2], w is homogeneous. By [**2**], $C \sim S \# RG$ for S a (commutative) Galois GR-object, and with grading on C given by the usual G-grading on RG. Hence $w = s \# u_\tau$ for some $s \in S$, $\tau \in G$, and

$$w^{-1} = \tau^{-1}(s^{-1}) \# u_{\tau^{-1}}.$$

Then

$$\begin{aligned}\alpha(1 \# u_\beta) &= (s \# u_\tau)(1 \# u_\beta)(\tau^{-1}(s^{-1}) \# u_{\tau^{-1}}) = s\beta(s^{-1}) \# u_\beta \\ &= \phi(\sigma, \beta)^{-1}1 \# u_\beta.\end{aligned}$$

Therefore, for all $\beta \in G$,

$$s\beta(s^{-1}) = \phi(\sigma, \beta)^{-1}$$

and, since S is commutative, α-action is given by conjugation by $s \# u_1 \in C_1$. Now we note that σ-action on $Z \# C \# \bar{Z}$ is given by conjugation by $u_\sigma^{-1} \# w \# u_\sigma$ so that $Z \# C \# \bar{Z} \in \operatorname{Ker} \beta = T$ also.

Again assume G is cyclic of prime power order p^r. From the above we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & T & \rightarrow & B(R, G)/B(R) \simeq T \times BT(R, G) & \xrightarrow{\beta} & \operatorname{Aut}(G) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & T & \rightarrow & BD(R, G)/B(R) & \rightarrow & N & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & \rightarrow & C_2 & \rightarrow & C_2 & \rightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

If G is cyclic of order p^r, $p \neq 2$, then $\operatorname{Aut}(G)$ is a cyclic group C of order $s = p^r - p^{r-1}$ and N is a group of order $2s$ containing C as a normal subgroup. Let Y be a generator of $BT(R, G)$ so that $Y^s = 1$. Let $X = RG_f^\phi$ with basis u_σ, $\sigma \in G$, and ϕ a nondegenerate bilinear map from $G \times G$ to $U(R)$. Since X is commutative,

$$X \# X \in B(R, G)/B(R).$$

The inner action of σ on $X \# X$ is given by conjugation by $u_\sigma^{-1} \# u_\sigma$ so that $\beta(X \# X)$ is trivial and $X \# X = 1$ in N. Then N is generated by X

and Y; $XY \in BD(R, G)/B(R)$ but is not in $B(R, G)/B(R)$ so that

$$XY \simeq (XY)^G \# RG_f^\chi \simeq RG_f^\chi \# (XY)_1,$$

$f \in H^2(G, U(R))$, χ a nondegenerate bilinear map from $G \times G \to U(R)$ as usual. As above $(XY)^2 = 1$ in N so that $XY = Y^{s-1}X$ and $N = D_s$, the dihedral group of order $2s$. If $p = 2$ and $r \geqq 2$ then $\mathrm{Aut}(G)$ is the direct product of C_2, the cyclic group of order 2, and C_s the cyclic group of order $s = 2^{r-2}$. In this case $N = C_2 \times D_s$ where D_s is the dihedral group of order 2^{r-1}.

Suppose G is cyclic of order $\prod_{i=1}^{t} p_i^{r_i}$; Corollary 1.4 of [**3**] shows that

$$B(R, G)/B(R) = \prod_{i=1}^{t} B(R, G_i)/B(R)$$

where G_i is cyclic of order $p_i^{r_i}$. The arguments of [**10**, Theorem 2.7] show that

$$\prod_{i=1}^{t} BD(R, G_i)/B(R) \simeq BD(R, G)/B(R)$$

and thus (*) is proved.

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