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REGULARITY OF POWERS OF BIPARTITE GRAPHS

AJAY KUMAR✉ and RAJIV KUMAR

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Abstract

For a simple bipartite graph G, we give an upper bound for the regularity of powers of the edge ideal $I(G)$ in terms of its vertex domination number. Consequently, we explicitly compute the regularity of powers of the edge ideal of a bipartite Kneser graph. Further, we compute the induced matching number of a bipartite Kneser graph.

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1. Introduction

Let $S = \mathsf{k}[x_1, \ldots, x_n]$ be a polynomial ring, where k is a field. For a homogeneous ideal I, Cutkosky *et al.* [5] and independently Kodiyalam [11] proved that $\operatorname{reg}(S/I^s) = as + b$ for some $a, b \in \mathbb{Z}$ and $s \gg 0$. The value of a can be determined by the degrees of generators of I but the value of b is quite mysterious. During the last few decades, many researchers have studied the problem of understanding the value of b for some special classes of ideals, for example, edge ideals and cover ideals. In this paper, we consider the edge ideal $I(G)$ of a bipartite graph G and find an upper bound for the value of b in terms of a combinatorial invariant of G.

For any graph G, it is known that

$$\nu(G) \le \operatorname{reg}(S/I(G)) \le \text{co-chord}(G),$$

where $\nu(G)$ denotes the induced matching number of G and co-chord(G) denotes the co-chordal number of G (see [10, 14]). Bıyıkoğlu and Civan in [4] proved that for any graph G, $\operatorname{reg}(S/I(G)) \le \beta(G)$, where $\beta(G)$ is called the upper independent vertex-wise domination set of G (see Definition 2.1(vi)). Beyarslan *et al.* in [3] proved that for any graph G,

$$\operatorname{reg}(S/I(G)^s) \ge 2s + \nu(G) - 2 \quad \text{for } s \ge 1.$$

Moreover, they proved that in the special cases of forests (for $s \geq 1$) and cycles ($s \geq 2$), the equality holds. In [8], it is shown that for bipartite graphs,

$$\operatorname{reg}(S/I(G)^s) \leq 2s + \text{co-chord}(G) - 2 \quad \text{for } s \geq 1.$$

Recently, Herzog and Hibi [7] obtained a new upper bound for the regularity of powers of the ideal of a graph G. They proved that

$$\operatorname{reg}(S/I(G)^s) \leq 2s + c - 1 \quad \text{for } s \geq 1,$$

where c is the dimension of the independence complex $\Delta(G)$ of G.

In Section 3, we prove the main result of this paper, which gives a new upper bound for $\operatorname{reg}(S/I(G)^s)$ for any bipartite graph G.

THEOREM 1.1 (Theorem 3.11). *Let G be a bipartite graph and $I(G)$ be its edge ideal. Then $\operatorname{reg}(S/I(G)^{s+1}) \leq 2s + \beta(G)$ for all $s \geq 0$.*

To prove Theorem 3.11, we use the technique of even-connection with respect to the s-fold product $e_1 \cdots e_s$ of edges (see Definition 2.5), which was introduced by Banerjee in [2]. Alilooee and Banerjee [1] proved that if G is a bipartite graph, then the colon ideal $I(G)^{s+1} : e_1 \cdots e_s$ is a quadratic square-free monomial ideal. Further, the graph G' associated to $I(G)^{s+1} : e_1 \cdots e_s$ is also a bipartite graph on the same partition and G' is the union of G with all the even-connections with respect to the s-fold product $e_1 \cdots e_s$ (see Remark 3.9).

In Section 4, we study the regularity of powers of edge ideals of the bipartite Kneser graph $\mathcal{H}(m,k)$ for $k \geq 1$ and $m \geq 2k$ (see Definition 2.2). Bipartite Kneser graphs are of great interest because they are Hamiltonian, as shown by Mütze and Su [13]. We are interested in finding the regularity of powers of edge ideals of bipartite Kneser graphs. In [12], it is shown that

$$2(s-1) + \binom{2k}{k} \leq \operatorname{reg}(S/I(\mathcal{H}(m,k))^s) \leq 2(s-1) + \binom{m}{k},$$

and the lower bound is attained if $m = 2k+1$. It is known that the problem of finding the induced matching number of the graph is an NP-hard problem. Given $k \geq 1$ and $m \geq 2k+1$, we compute the induced matching number of the bipartite Kneser graph $\mathcal{H}(m,k)$.

THEOREM 1.2 (Corollary 4.3). *For $m \geq 2k+1$, let $G = \mathcal{H}(m,k)$ be the bipartite Kneser graph. Then the induced matching number of G is given by $\nu(G) = \binom{2k}{k}$.*

The following question is posed in [3]: for which graphs G does

$$\operatorname{reg}(S/I(G)^s) = 2s + \nu(G) - 2 \quad \text{for } s \gg 0?$$

For certain classes of graphs, for example, the bipartite P_6-free graph and very well-covered, unmixed bipartite, weakly chordal bipartite, forest graphs, it is known that $\operatorname{reg}(S/I(G)^s) = 2s + \nu(G) - 2$ for $s \gg 0$ (see [3, 8, 9]). Using Theorem 3.11, we prove that the regularity of powers of edge ideals of $\mathcal{H}(m,k)$ attains the lower bound.

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THEOREM 1.3 (Corollary 4.4). *For $m \geq 2k+1$, let $G = \mathcal{H}(m,k)$ be the bipartite Kneser graph. Then, for all $s > 0$,* $\operatorname{reg}(S/I(G)^s) = 2(s-1) + \binom{2k}{k}$.

2. Preliminaries

For a positive integer n, we write $[n] = \{1, 2, \dots, n\}$. For a finite set Y, the family of all subsets of Y of size s is denoted by $Y^{(s)}$.

DEFINITION 2.1. Let G be a simple graph with vertex set $V(G) = \{x_1, \dots, x_n\}$ and edge set $E(G)$.

(i) For a pair of vertices $x_i, x_j \in V(G)$, we say x_i is *adjacent* to x_j if and only if $x_i x_j \in E(G)$.

(ii) A subset W of V is called an *independent set* if none of the edges of G has both endpoints in W.

(iii) For a vertex $v \in V$, the *open neighbourhood* of v is $N_G(v) = \{x : xv \in E(G)\}$ and the *closed neighbourhood* of v is $N_G[v] = N_G(v) \cup \{v\}$.

(iv) For an edge $e = x_i x_j$, we define $N_G[e] = N_G[x_i] \cup N_G[x_j]$.

(v) An independent set W is called a *vertex dominant set* if $N_G[e] \cap W \neq \emptyset$ for any edge e in G. It is called a *minimal vertex dominant* set if any proper subset of W is not a vertex-wise dominant set of G.

(vi) The upper independent vertex-wise domination number of a graph G is defined by $\beta(G) = \max\{|W| : W$ is an independent minimal vertex dominating set of $G\}$.

(vii) A graph G is called *bipartite* if $V(G) = X \sqcup Y$ for two independent subsets X and Y of $V(G)$.

(viii) A subgraph G' of G is called *induced* if for every pair of vertices $x_i, x_j \in V(G')$, $x_i x_j \in E(G')$ if and only if $x_i x_j \in E(G)$.

(ix) A *matching* of G is a subgraph of G consisting of pairwise disjoint edges. If the subgraph is an induced subgraph, then the matching is called an *induced matching*. The largest size of an induced matching in G is called the *induced matching number*, denoted by $\nu(G)$.

(x) The graph G is a *cycle* of length n if after relabelling the vertices of G, the edge set is $E(G) = \{x_1x_2, \dots, x_{n-1}x_n, x_nx_1\}$.

(xi) A finite sequence of vertices $x_{i_1}, \dots, x_{i_r}$ is called a *path* from x_{i_1} to x_{i_r} in G if $x_{i_j} x_{i_{j+1}} \in E(G)$ for $1 \leq j \leq r-1$.

(xii) A graph is called *co-chordal* if its complement graph G^c does not have any induced cycle of length greater than or equal to 4. The co-chordal number, denoted by co-chord(G), is the minimum number of co-chordal subgraphs required to cover the edges of G.

DEFINITION 2.2. The bipartite Kneser graph $\mathcal{H}(m,k)$ is a graph with vertex set $V(G) = [m]^{(k)} \cup [m]^{(m-k)}$ and two distinct vertices A, B are adjacent if and only if $A \subset B$ or $B \subset A$. For $m = 2k$, $\mathcal{H}(m,k)$ does not have any edges, so we assume that $m \geq 2k+1$.

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DEFINITION 2.3. Let k be a field and $S = \mathsf{k}[x_1, x_2, \ldots, x_n]$ be a standard graded polynomial ring over k. The *Castelnuovo–Mumford regularity* of a finitely generated graded S-Module M is given by $\operatorname{reg}(M) = \max_{i,j}\{j - i : \operatorname{Tor}_i(M, \mathsf{k})_j \neq 0\}$.

DEFINITION 2.4. Let G be a simple graph with the vertex set $\{x_1, \ldots, x_k\}$ (without isolated vertices). Then the *edge ideal* of G is defined as

$$I(G) = \langle x_i x_j : x_i x_j \text{ is an edge of } G \text{ for some } i, j\rangle.$$

DEFINITION 2.5 [2, Definition 6.2]. Let G be a graph on the vertex set V. Then vertices $x, y \in V$ are called *even-connected* with respect to the s-fold product $e_1 \cdots e_s$ of edges in G if there exists a path $p_0 p_1 \ldots p_{2k+1}$ in G such that:

(a) $p_0 = x$ and $p_{2k+1} = y$;
(b) $p_{2l+1} p_{2l+2} = e_i$ for some i for all l with $0 \leq l \leq k - 1$;
(c) $|\{l \geq 0 \mid p_{2l+1} p_{2l+2} = e_i\}| \leq |\{j \mid e_j = e_i\}|$ for all i.

THEOREM 2.6 [2, Theorem 5.2]. *Let G be a simple graph and the set of minimal monomial generators of $I(G)^s$ be $\{m_1, \ldots, m_k\}$, where $s > 0$. Then,*

$$\operatorname{reg}(S/I(G)^{s+1}) \leq \max\{\operatorname{reg}(S/I(G)^{s+1} : m_t) + 2s \text{ for } 1 \leq t \leq k, \operatorname{reg}(S/I(G)^s)\}.$$

3. Vertex-wise domination number

In general, there is no relation between $\beta(G)$ and co-chord(G), for a simple graph G. For example, if P_4 is a simple path on 4 vertices, one can check that $\beta(P_4) = 2$, but P_4 is a co-chordal graph. However, in [4], it is shown that $\beta(C_7) = 2$ and co-chord$(C_7) = 3$, where C_7 denotes the cycle of length 7.

REMARK 3.1. Let W be a minimal vertex dominant set of G and $w \in W$. Then there exists an edge $e \in G$ such that $N_G[e] \cap W = \{w\}$.

NOTATION 3.2. Let G be a triangle-free graph and $I(G)$ its edge ideal. For $x_1 x_2 \in E(G)$, let G' be the graph associated to the monomial ideal $I(G)^2 : x_1 x_2$. Denote by $N_G(x_1) \setminus \{x_2\} = \{x_{1,1}, \ldots, x_{1,r}\} = X_1$ and $N_G(x_2) \setminus \{x_1\} = \{x_{2,1}, \ldots, x_{2,s}\} = X_2$. To illustrate the notation, we consider a graph G on the vertex set $\{x_1, x_2, x_3, x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}\}$ and the edge set $E(G) = \{x_1 x_2, x_1 x_{1,1}, x_1 x_{1,2}, x_2 x_{2,1}, x_2 x_{2,2}, x_{1,1} x_3\}$, as shown in Figure 1. Then $I(G)^2 : x_1 x_2 = I(G) + \langle x_{1,1} x_{2,1}, x_{1,1} x_{2,2}, x_{1,2} x_{2,1}, x_{1,2} x_{2,2}\rangle$, that is, G' is obtained from the graph G by connecting all vertices of X_1 with vertices of X_2.

PROPOSITION 3.3. *Let G be a triangle-free graph and $I(G)$ be its edge ideal. Let $e \in E(G)$ and G' be the graph associated to the monomial ideal $I(G)^2 : e$. Then $\beta(G') \leq \beta(G)$.*

We prove this proposition in the following sequence of lemmas.

LEMMA 3.4. *With notation as in Notation 3.2, let W be a minimal vertex dominant set in G' such that $W \cap (X_1 \cup X_2) = \varnothing$. Then W is a minimal vertex dominant set in G.*

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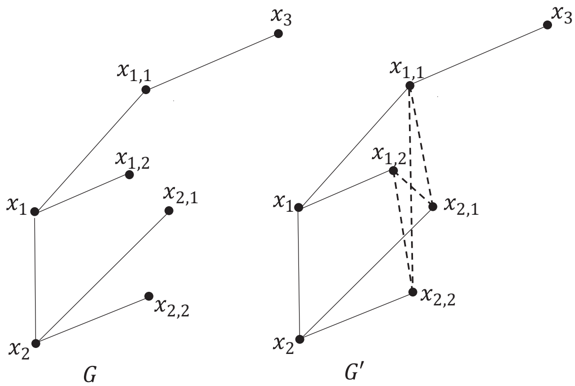

FIGURE 1. Illustrative example for Notation 3.2.

PROOF. Since $N_G[e] \subset N_{G'}[e] \subset N_G[e] \cup X_1 \cup X_2$ for any $e \in E(G)$, we have $N_G[e] \cap W = N_{G'}[e] \cap W$. Hence, W is a vertex dominant set in G. We claim that W is a minimal vertex dominant set in G. In contrast, assume that W is not a minimal vertex dominant set in G. Then there exists a vertex $v \in W$ such that $W_1 = W \setminus \{v\}$ is a vertex dominant set in G. Since W is a minimal vertex dominant set in G', W_1 is not a vertex dominant set in G'. There exists an edge $f \in E(G')$ such that $W_1 \cap N_{G'}[f] = \emptyset$. However, $N_G[f] \cap W_1 = N_{G'}[f] \cap W_1 = \emptyset$, so $f \notin E(G)$ and hence $f = x_{1,i}x_{2,j}$ for some i, j.

However, note that $v \in N_{G'}[f]$. Since $v \notin X_1 \cup X_2$, then $v \notin \{x_{1,i}, x_{2,j}\}$ and $v \in N_G[f]$. Without loss of generality, assume that $vx_{1,i} \in E(G)$. Since $N_G[f] \cap W_1 = \emptyset$, we have $N_G[x_{1,i}] \cap W_1 = \emptyset$ and so $N_G[v] \cap W_1 \neq \emptyset$. This implies that v and some of its adjacent vertices are in W, contradicting the hypothesis that W is an independent set. □

LEMMA 3.5. *With notation as in Notation 3.2, let W be a minimal vertex dominant set in G' such that $W \cap X_1 \neq \emptyset$. Then $W \cup \{x_2\}$ is a vertex dominant set in G.*

PROOF. First of all, note that since W is an independent set in G' and $W \cap X_1 \neq \emptyset$, we get $W \cap X_2 = \emptyset$. Let f be an edge in G. If $x_2 \in N_G[f]$, then we are through. Suppose $x_2 \notin N_G[f]$. This implies that $x_{2,j}$ is not an endpoint of the edge f for any j. Hence, $N_G[f] \subset N_{G'}[f] \subset N_G[f] \cup X_2$. Since $W \cap X_2 = \emptyset$, we get $N_G[f] \cap W = N_{G'}[f] \cap W \neq \emptyset$, which proves the lemma. □

LEMMA 3.6. *With notation as in Notation 3.2, let W be a minimal vertex dominant set in G' such that $W \cap X_1 \neq \emptyset$. Let $W_1 = W \cup \{x_2\}$. Suppose $W_1 \setminus \{v\}$ is a vertex dominant set in G for some $v \in W_1$. Then $v \in X_1 \cup \{x_2\}$.*

PROOF. On the contrary, assume that $v \notin X_1 \cup \{x_2\}$. Since $W \setminus \{v\}$ is not a vertex dominant set in G', there is an edge $f \in E(G')$ such that $N_{G'}[f] \cap (W \setminus \{v\}) = \emptyset$. If $f = x_{1,i}x_{2,j}$ for some i, j, then $X_1 \subset N_{G'}[f]$. Hence, $X_1 \cap (W \setminus \{v\}) \subset N_{G'}[f] \cap (W \setminus \{v\}) \neq \emptyset$, which is a contradiction to our hypothesis. Therefore, $f \in E(G)$. Since $N_G[f] \subset N_{G'}[f]$

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and $N_{G'}[f] \cap W \setminus \{v\} = \emptyset$, $N_G[f] \cap W \setminus \{v\} = \emptyset$. Also, we have $N_G[f] \cap W_1 \setminus \{v\} = N_G[f] \cap (W \cup \{x_2\}) \setminus \{v\} \neq \emptyset$. Because, $v \neq x_2$, we get $x_2 \in N_G[f]$. Since $W_1 \setminus \{v\}$ is a vertex dominant set in G, we get $X_1 \subset N_{G'}[f]$, reaching the same contradiction. □

LEMMA 3.7. *With notation as in Notation 3.2, let W be a minimal vertex dominant set in G' with $W \cap X_1 \neq \emptyset$. Let $v \in W \cap X_1$. Then $\widehat{W} = W \setminus \{v\}$ is not a vertex dominant set in G.*

PROOF. On the contrary, assume that $\widehat{W} = W \setminus \{v\}$ is a vertex dominant set in G. Since $\widehat{W}$ is not a vertex dominant set in G', there exists $f \in E(G')$ such that $N_{G'}[f] \cap \widehat{W} = \emptyset$. This implies that $v \in N_{G'}[f]$. Note that $f \notin E(G)$. As for $f \in E(G)$, we have $N_G[f] \cap \widehat{W} \subset N_{G'}[f] \cap \widehat{W} = \emptyset$, which is a contradiction to the fact that $\widehat{W} = W \setminus \{v\}$ is a vertex dominant set of G. Hence, $f = x_{1,i}x_{2,j}$ for some i, j and $N_{G'}[f] = N_G[x_{1,i}] \cup N_G[x_{2,j}] \cup \{X_1 \cup X_2\}$. This implies that $N_{G'}[f] \cap \widehat{W} = ((N_G[x_{1,i}] \cup N_G[x_{2,j}]) \cap \widehat{W}) \cup (\widehat{W} \cap X_1) = \emptyset$. Consider an edge $f' = x_1, x_{1,i} \in E(G)$. Then

$$N_G[f'] \cap \widehat{W} = (N_G[x_1] \cup N_G[x_{1,i}]) \cap \widehat{W} = (\{x_1\} \cup X_1 \cup N_G[x_{1,i}]) \cap \widehat{W}. \tag{3.1}$$

Note that $x_1 \notin \widehat{W}$, because otherwise $N_{G'}[f] \cap \widehat{W} \neq \emptyset$, which is a contradiction. Since $X_1 \cap \widehat{W} = \emptyset$ and $N_G[x_{1,i}] \cap \widehat{W} \subset (N_G[x_{1,i}] \cup N_G[x_{2,j}]) \cap \widehat{W} = \emptyset$, Equation (3.1) gives $N_G[f'] \cap \widehat{W} = \emptyset$, which is a contradiction. Hence, $\widehat{W} = W \setminus \{v\}$ is not a vertex dominant set in G. □

LEMMA 3.8. *With notation as in Notation 3.2, let W be a minimal vertex dominant set in G' such that $W \cap X_1 \neq \emptyset$ and $W_1 = W \cup \{x_2\}$. Let $\emptyset \neq T \subset W_1$. If $W_1 \setminus T$ is a vertex dominant set in G, then $|T| \leq 1$.*

PROOF. On the contrary, suppose that $|T| \geq 2$. First we show that $x_2 \notin T$. Using Lemma 3.7, we can see that if $x_2 \in T$, then $W_1 \setminus T = W \setminus (T \setminus \{x_2\})$ is not a vertex dominant set in G. Thus, $x_2 \notin T$.

Let $y \in T \subset W$. Since W is a minimal vertex dominant set of G', there exists an edge $f \in E(G')$ such that $N_{G'}[f] \cap W = \{y\}$. Therefore, $N_{G'}[f] \cap (W \setminus T) = \emptyset$. If $f \in E(G)$, then $\emptyset \neq N_G[f] \cap (W_1 \setminus T) \subset (N_{G'}[f] \cap (W \setminus T)) \cup (N_G[f] \cap \{x_2\})$. This implies that $(N_G[f] \cap \{x_2\}) \neq \emptyset$, and hence $x_2 \in N_G[f]$, which means that $X_1 \subset N_{G'}[f]$. Thus, $W \cap X_1 \subset N_{G'}[f] \cap W = \{y\}$. Since $W \cap X_1 \neq \emptyset$, we have $W \cap X_1 = \{y\}$. Let $y' \in T \setminus \{y\}$. Then $y' \notin X_1$. Now the fact that $W_1 \setminus T$ is a vertex dominant set in G implies that $W_1 \setminus \{y'\}$ is a vertex dominant set in G, which gives a contradiction to Lemma 3.6. If $f \in E(G') \setminus E(G)$, then $X_1 \subset N_{G'}[f]$. Now proceeding as before, $W_1 \setminus \{v\}$ is a vertex dominant set in G for some $v \notin X_1 \cap \{x_2\}$, which is a contradiction by Lemma 3.6. □

PROOF OF PROPOSITION 3.3. Let W be a minimal vertex dominating set of G'. If we have $W \cap \{X_1 \cup X_2\} = \emptyset$, then by Lemma 3.4, W is a minimal vertex dominating set of G. Otherwise, using Lemma 3.5, $W_1 = W \cup \{x_2\}$ is a vertex dominating set of G. Further, by Lemma 3.8, either $W_1 = W \cup \{x_2\}$ is a minimal vertex dominating set of G or $W_1 \setminus \{v\}$ is a minimal vertex dominating set of G for some $v \in W_1$. It follows from the definition of $\beta(G)$ that $\beta(G') \leq \beta(G)$. □

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To prove our main theorem, we shall use the following remark.

REMARK 3.9. Let G be a bipartite graph and $s \geq 1$ be an integer. Then for every s-fold product $e_1 \cdots e_s$, the following statements hold.

(a) The ideal $(I(G)^{s+1} : e_1 \cdots e_s)$ is a quadratic square-free monomial ideal. Moreover, the graph G' associated to $(I(G)^{s+1} : e_1 \cdots e_s)$ is bipartite on the same vertex set and the same bipartition as G (see [1, Proposition 3.5]).
(b) The ideal $I(G)^{s+1} : e_1 \cdots e_s = (I(G)^2 : e_1)^s : e_2 \cdots e_s$ (see [1, Lemma 3.7]).

Note that if G is a triangle-free graph, then the graph H associated to $I(G)^2 : e$ need not be a triangle-free graph, for $e \in E(G)$. Thus, in view of Remark 3.9(a), we prove the following result for bipartite graphs.

COROLLARY 3.10. *Let G be a bipartite graph and u be a minimal monomial generator of $I(G)^s$. Then $\beta(G') \leq \beta(G)$, where G' is the graph associated to $I(G)^{s+1} : u$.*

PROOF. We use induction on s. For $s = 1$, the result follows from Proposition 3.3. Assume that $s > 1$. Let $u = e_1 \cdots e_s$ for some edges $e_1, \ldots, e_s$ in the edge set $E(G)$. If H is the graph associated to $I(G)^2 : e_1$, then by Proposition 3.3, $\beta(H) \leq \beta(G)$. By Remark 3.9, the graph H is a bipartite graph and $I(G)^{s+1} : e_1 \cdots e_s = I(H)^s : e_2 \cdots e_s$. Hence, by induction, we get $\beta(G') \leq \beta(H) \leq \beta(G)$. □

Now we are ready to prove our main theorem.

THEOREM 3.11. *Let G be a bipartite graph and $I(G)$ be its edge ideal. Then $\operatorname{reg}(S/I(G)^{s+1}) \leq 2s + \beta(G)$ for all $s \geq 0$.*

PROOF. We use induction on s. For $s = 0$, the result follows from [4, Theorem 3.19]. Now assume that $s \geq 1$. In view of Theorem 2.6, it is enough to prove that

$$\operatorname{reg}(S/I(G)^{s+1} : u) \leq \beta(G)$$

for all minimal monomial generators u of $I(G)^s$. Let G' be the graph associated to $(I(G)^{s+1} : u)$. Now, the proof follows from Corollary 3.10 and [4, Theorem 3.19]. □

4. Bipartite Kneser graphs

THEOREM 4.1 (Frankl, [6]). *Suppose $\mathcal{A} = \{A_1, \ldots, A_l\}$ is a family of r-sets and $\mathcal{B} = \{B_1, \ldots, B_l\}$ is a family of s-sets such that:*

(i) $A_i \cap B_i = \emptyset$ *for* $1 \leq i \leq m$;
(ii) $A_i \cap B_j \neq \emptyset$ *for* $1 \leq i < j \leq m$.

Then

$$l \leq \binom{r+s}{s}.$$

PROPOSITION 4.2. *Let $G = \mathcal{H}(m, k)$ be the bipartite Kneser graph. Then $\beta(G) \leq \binom{2k}{k}$.*

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PROOF. Let $W = \{C_1, \dots, C_t, C_{t+1}, \dots, C_m\}$ be a minimal vertex dominant set in G, where $C_i \in [n]^{(k)}, 1 \le i \le t$, and $C_i \in [n]^{(n-k)}, t+1 \le i \le m$. Since W is a minimal vertex dominant set in G, for each vertex $C_i \in W$, there exists a vertex D_i such that $N_G(D_i) \cap W = \{C_i\}$. This implies that

$$\begin{aligned} C_i \subset D_j \quad &\text{if and only if } i = j, \quad 1 \le i,j \le t \\ C_j \supset D_i \quad &\text{if and only if } i = j, \quad t+1 \le i,j \le m. \end{aligned}$$

Therefore,

$$\begin{aligned} C_i \cap D_j^c = \phi \quad &\text{if and only if } i = j, \quad 1 \le i,j \le t \\ C_j^c \cap D_i = \phi \quad &\text{if and only if } i = j, \quad t+1 \le i,j \le m. \end{aligned}$$

Consider the collection $W' = \{(X_1, Y_1), \dots, (X_m, Y_m)\}$ of ordered pairs, where $X_i = C_i, Y_i = D_i^c$ for $1 \le i \le t$ and $X_i = D_i, Y_i = C_i^c$ for $t+1 \le i \le m$. By the choice of the collection W', it is clear that $X_i \cap Y_i = \varnothing$ for all i, and $X_i \cap Y_j \ne \varnothing$ for $1 \le i < j \le t$ and $t+1 \le i < j \le m$. Now, since W is an independent set, $C_i \not\subset C_j$ and hence $C_i \cap C_j^c \ne \varnothing$ for all $i \ne j$. Therefore, $X_i \cap Y_j \ne \varnothing$ for $1 \le i \le t$ and $t+1 \le j \le m$. This implies that $X_i \cap Y_j \ne \varnothing$ for $1 \le i < j \le m$ and $X_i \cap Y_i = \varnothing$ for $1 \le i \le m$. Since $|X_i| = |Y_i| = k$ for all i, in view of Theorem 4.1, we get $m \le \binom{2k}{k}$. □

COROLLARY 4.3. *For $m \ge 2k+1$, let $G = \mathcal{H}(m,k)$ be the bipartite Kneser graph. Then the induced matching number of G is given by $\nu(G) = \binom{2k}{k}$.*

PROOF. In view of [10, Lemma 2.2] and [4, Theorem 3.19],

$$\nu(G) \le \operatorname{reg}(S/I(G)) \le \beta(G).$$

Using [12, Lemma 4.2], $\nu(G) \ge \binom{2k}{k}$. Now, by Proposition 4.2, $\nu(G) = \binom{2k}{k}$. □

COROLLARY 4.4. *For $m \ge 2k+1$, let $G = \mathcal{H}(m,k)$ be the bipartite Kneser graph. Then, for all $s > 0$, $\operatorname{reg}(S/I(G)^s) = 2(s-1) + \binom{2k}{k}$.*

PROOF. From [3, Theorem 4.5] and Corollary 4.3, $\operatorname{reg}(S/I(G)^s) \ge 2(s-1) + \binom{2k}{k}$. Now, by Theorem 3.11 and Proposition 4.2, $\operatorname{reg}(S/I(G)^s) \le 2(s-1) + \binom{2k}{k}$, and hence we get the desired result. □

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AJAY KUMAR, Department of Mathematics,
Indian Institute of Technology Jammu, Jammu, India
e-mail: ajay.kumar@iitjammu.ac.in

RAJIV KUMAR, Department of Mathematics,
Indian Institute of Technology Jammu, Jammu, India
e-mail: gargrajiv00@gmail.com