THE POWER INEQUALITY ON NORMED SPACES

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Let X be a complex normed space, with dual space X'. Let T be a bounded linear operator on X. The numerical range $V(T)$ of T is defined as $\{f(Tx): x \in X, f \in X', \| x \| = \| f \| = f(x) = 1\}$, and the numerical radius $v(T)$ of T is defined as $\sup\{|z|: z \in V(T)\}$. For a unital Banach algebra A, the numerical range $V(a)$ of $a \in A$ is defined as $V(T_a)$, where T_a is the operator on A defined by $T_a b = ab$. It is shown in (**2**, Chapter 1.2, Lemma 2) that $V(a) = \{f(a): f \in D(1)\}$, where $D(1) = \{f \in A': \| f \| = f(1) = 1\}$.

For X a Hilbert space, we have the power inequality $v(T^n) \leqq v(T)^n$ [see (**1**)]. In (**3**) it is shown that, for a normed space X,

$$\| T^n \| \leqq n!(e/n)^n v(T)^n \quad (n = 1, 2, \ldots) \tag{1}$$

and that $\{\| T^n \|/v(T)^n\}$ is bounded when X has finite dimension. Glickfeld (**4**) has given an example of an operator T for which $\| T \| = ev(T)$. The purpose of this paper is to prove the following theorem.

Theorem. *There exists a Banach space X and a non-zero bounded linear operator T on X such that*

$$\| T^n \| = n!(e/n)^n v(T)^n \quad (n = 1, 2, \ldots).$$

Corollary. *For the operator of the theorem,*

$$v(T)^n \geqq n!(e^n - 2)/n^n v(T)^n > v(T)^n \quad (n = 2, 3, \ldots).$$

Hence the constants in equality (1) are best possible, and $\| T^n \|/v(T)^n$ need not be bounded. Also, the power inequality does not extend to normed spaces.

Proof of Theorem. Let n be a positive integer. Let A_n be the algebra of elements

$$\alpha_0 + \alpha_1 u + \ldots + \alpha_n u^n \quad (\alpha_0, \ldots, \alpha_n \in \boldsymbol{C})$$

where $u^{n+1} = 0$. For $a \in A_n$, define

$$p(a) = \inf\left\{\sum_{k=1}^{m} | c_k | e^{| z_k |}: \sum_{k=1}^{m} c_k e^{z_k u} = a,\ c_k, z_k \in \boldsymbol{C}, m \in \boldsymbol{P}\right\}.$$

Clearly p is subadditive. To see that p is an algebra-norm, let $a, a' \in A$. For any $\varepsilon > 0$, there exist a positive integer m, and $c_i, z_i \in \boldsymbol{C}$ ($i = 1, 2, \ldots, m$) such that

$$\sum_{i=1}^{m} c_i e^{z_i u} = a \quad \text{and} \quad \sum_{i=1}^{m} | c_i | e^{| z_i |} < p(a) + \varepsilon. \tag{2}$$

Similarly,

$$\sum_{j=1}^{m'} c'_j e^{z'_j u} = a' \quad \text{and} \quad \sum_{j=1}^{m'} |c'_j| e^{|z'_j|} < p(a') + \varepsilon.$$

These give

$$\sum_{i=1}^{m} \sum_{j=1}^{m'} c_i c'_j e^{(z_i + z'_j)u} = aa'$$

so that

$$p(aa') \leqq \sum_{i=1}^{m} \sum_{j=1}^{m'} |c_i c'_j| e^{|z_i|+|z'_j|} < (p(a)+\varepsilon)(p(a')+\varepsilon).$$

Since ε is arbitrary, $p(aa') \leqq p(a)p(a')$. Now assume that $p(a) = 0$, where $a = \alpha_0 + \ldots + \alpha_n u^n$. From (2),

$$|\alpha_r| = \left| \sum_{i=1}^{m} c_i z_i^r / r! \right| \leqq \sum_{i=1}^{m} |c_i| e^{|z_i|} < \varepsilon \quad (r = 0, 1, \ldots, n).$$

Since ε is arbitrary, $a = 0$.

Suppose that $u^n = \sum_{k=1}^{m} c_k e^{z_k u}$. Then, using the fact that $e^t \geqq (e/n)^n t^n$ $(t \geqq 0)$, we have

$$\sum_{k=1}^{m} |c_k| e^{|z_k|} \geqq \sum_{k=1}^{m} |c_k| (e/n)^n |z_k|^n \geqq (e/n)^n \left| \sum_{k=1}^{m} c_k z_k^n \right| = n!(e/n)^n.$$

Hence $p(u^n) \geqq n!(e/n)^n$. Also, $v(u) = \sup_{z \neq 0} |z|^{-1} \log p(e^{zu})$ by (**2**, Chapter 1.3, Theorem 4). Since $p(e^{zu}) \leqq e^{|z|}$, $v(u) \leqq 1$. From (1), we must in fact have $v(u) = 1$ and $p(u^n) = n!(e/n)^n$.

Now let A be the algebra of sequences $(a_1, a_2, \ldots)$, where $a_n \in A_n$ and $\{p(a_n)\}$ is bounded, with pointwise multiplication. For $a \in A$, let

$$\| a \| = \sup \{ p(a_n) : n = 1, 2, \ldots \}.$$

It may be proved that A is complete, and so is a Banach algebra. Let a be the element $(u_1, u_2, \ldots)$, where u_n is the element u of the algebra A_n above. Then $\| e^{za} \| = \sup \{ p(e^{zu_n}) : n = 1, 2, \ldots \} \leqq e^{|z|}$, so that $v(a) \leqq 1$. Also

$$\| a^n \| = n!(e/n)^n \quad (n = 1, 2, \ldots).$$

If we define, in the algebra A_n, a functional f by

$$f(\alpha_0 + \alpha_1 u + \ldots + \alpha_n u^n) = \alpha_0 + \lambda_1 \alpha_1 + \ldots + \lambda_n \alpha_n,$$

then it is easily seen that $f \in D(1)$ if and only if

$$|1 + \lambda_1 z + \ldots + \lambda_n z^n / n!| \leqq e^{|z|} \quad (z \in \boldsymbol{C}).$$

For $r = 1, 2, \ldots, n$, $f(u^r) = \lambda_r \in V(u^r)$, and so

$$v(u^r) = \sup \{ |\lambda_r| : |1 + \lambda_1 z + \ldots + \lambda_n z^n / n!| \leqq e^{|z|} \quad (z \in \boldsymbol{C}) \}.$$

It may be verified that, for $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = 0$, and $\lambda_n = n!(e^n - 2)/n^n$, $f \in D(1)$, so that $v(u^n) \geqq n!(e^n-2)/n^n$. Since $v(a^n) \geqq v(u^n)$, the corollary is established.

There remains the question of the best constants k_r in $v(a^r) \leqq k_r v(a)^r$. From the above, we have $k_r \geqq r!p_r$, where

$$p_r = \sup\{|\lambda_r| : |1+\lambda_1 z+\ldots+\lambda_n z^n| \leqq e^{|z|} (z \in C), n \geqq r\}.$$

Also, for any unital Banach algebra A, $a \in A$ with $v(a) = 1$, and $f \in D(1)$, we have

$$|f(e^{za})| = |1+\ldots+z^n f(a^n)/n!+\ldots| \leqq e^{|z|} \quad (z \in C).$$

Hence $f(a^r) \leqq r!q_r$, where

$$q_r = \sup\{|\lambda_r| : |1+\ldots+\lambda_r z^r+\ldots| \leqq e^{|z|} \quad (z \in C)\}.$$

Since this holds for any $f \in D(1)$, $v(a^r) \leqq r!q_r$. Hence $k_r \leqq r!q_r$. To show that $k_r = r!q_r$, it is enough to show that $p_r = q_r$. I am grateful to Professor J. G. Clunie for permission to publish his proof of the latter fact.

Lemma. $p_n = q_n$ $(n = 1, 2, \ldots)$.

Proof. For $0<\varepsilon<1$, there exists a function $f(z) = \sum_{k=0}^{\infty} c_k z^k$ such that $|f(z)| \leqq e^{|z|} (z \in C)$, and $|c_n| > q_n - \varepsilon$. Then, by Cauchy's inequality and Parseval's theorem, for $N>n$,

$$\sum_{k=N+1}^{\infty} |c_k|(1-\varepsilon)^k r^k \leqq \left(\sum_{k=N+1}^{\infty} |c_k|^2 r^{2k}\right)^{\frac{1}{2}} \left(\sum_{k=N+1}^{\infty} (1-\varepsilon)^{2k}\right)^{\frac{1}{2}}$$
$$\leqq \begin{cases} K r^{N+1} \varepsilon^{-1} (1-\varepsilon)^N & (0 \leqq r \leqq 1); \\ e^r \varepsilon^{-1} (1-\varepsilon)^N & (r \geqq 1), \end{cases}$$

where K is a constant. Let $g_N(z) = \sum_{k=0}^{N} c_k (1-\varepsilon)^k z^k$. For $z \in C$,

$$|g_N(z)| \leqq |f((1-\varepsilon)z)| + \sum_{k=N+1}^{\infty} |c_k|(1-\varepsilon)^k |z|^k.$$

For $0 \leqq |z| = r \leqq 1$, provided $K\varepsilon^{-1}(1-\varepsilon)^N \leqq \varepsilon$, we have

$$e^r - e^{(1-\varepsilon)r} \geqq \varepsilon r \geqq K\varepsilon^{-1}(1-\varepsilon)^N r^{N+1},$$

so that

$$|g_N(z)| \leqq e^{(1-\varepsilon)r} + K\varepsilon^{-1}(1-\varepsilon)^N r^{N+1} \leqq e^r.$$

For $r \geqq 1$, provided $\varepsilon^{-1}(1-\varepsilon)^N \leqq (1-e^{-\varepsilon})$,

$$|g_N(z)| \leqq e^{(1-\varepsilon)r} + \varepsilon^{-1}(1-\varepsilon)^N e^r$$
$$\leqq e^{(1-\varepsilon)r} + (1-e^{-\varepsilon})e^r$$
$$\leqq e^r.$$

Hence, for N sufficiently large, $|g_N(z)| \leqq e^{|z|}$. Therefore

$$p_n \geqq (1-\varepsilon)^n |c_n| > (1-\varepsilon)^n (q_n - \varepsilon).$$

As this holds for any ε with $0<\varepsilon<1$, $p_n \geqq q_n$. As $p_n \leqq q_n$, we have $p_n = q_n$.

It is of course not necessary to show that there exists a function f for which $|c_n| = q_n$, but perhaps it is worth mentioning that Montel's theorem gives such an extremal function.

Corollary. $k_n = n!p_n$ $(n = 1, 2, \ldots)$.

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