NULL TRIGONOMETRIC SERIES IN DIFFERENTIAL EQUATIONS

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1. Introduction. In this paper it is shown how trigonometric series which are Cesàro-summable to zero may be used to solve differential equations. The explicit solution of the general ordinary linear equation with constant coefficients is found in terms of trigonometric series and special cases are dealt with.

2. Null trigonometric series. By the Heine-Cantor and subsequent theorems, if the trigonometric series

2.1 $$\tfrac{1}{2} a_0 + \sum (a_n \cos nx + b_n \sin nx) = \sum c_n e^{inx},$$

where $\sum$ applied to the real form denotes summation for n from 1 to ∞ and applied to the complex form from $-\infty$ to ∞, is convergent to the sum zero for all values of x in the closed interval $(-\pi, \pi)$, or for all values except (possibly) those of a uniqueness (or unicity) set, then every one of the coefficients a_n, b_n, c_n must be zero [**6**, p. 103; **9**, pp. 274, 291].[1]

If convergence to zero is replaced by summability to zero, say by the Abel-Poisson or the Cesàro definition, the position is different. There are series with non-vanishing coefficients which have the sum zero for all values of x, with or without exceptional values [**9**, p. 297].

A trigonometric series whose coefficients are not all zero, whose sum by a method (T) is zero for all values of x, or for all values with specified exceptions, may be called a *null trigonometric series*, or briefly NTS, in the field T. In what follows T will be Cesàro summation to some positive integral order, specified or unspecified.

The simplest examples of NTS are

2.2 $$\tfrac{1}{2} + \sum \cos nx = \tfrac{1}{2} \sum e^{inx},$$

2.3 $$\sum n \sin nx = -\tfrac{1}{2} \sum ine^{inx},$$

which are respectively summable (C, 1) to zero for $x \not\equiv 0 \pmod{2\pi}$ and summable (C, 2) to zero for all values including $x \equiv 0$, the summability of 2.3 being non-uniform in the neighbourhood of $x \equiv 0$.

More generally, using D to denote d/dx, the series

2.4 $$D^r \tfrac{1}{2} + \sum D^r \cos n(x - \alpha) = \tfrac{1}{2} \sum (in)^r e^{in(x-\alpha)},$$

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[1]There appears to be a slight error of statement in [**6**, p. 104, Osservazione II].

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where r is any non-negative integer and α any real number, is null (C, $r+1$) for all real values of x except $x \equiv \alpha \pmod{2\pi}$ if r is even or zero, and for all values without exception if r is odd. The series 2.4 has a "singularity" at $x \equiv \alpha$, where it ceases to be finitely summable if r is even and the summability is non-uniform if r is odd [**2**, p. 2].

It may be observed that these series are the expansions of the "Dirac function" and its derivatives [**5**].

From the work of Verblunsky and others it follows that if the series 2.1 is summable (A) to zero for $x \not\equiv 0$ and the condition $|a_n| + |b_n| = o(n)$ is satisfied then the series is a constant multiple of series 2.2. Since this condition is necessary for (C, 1) summability, the same applies to (C, 1), and it follows that the only trigonometric series which are null (C, 1) with the single singularity at $x \equiv \alpha$ are constant multiples of the series 2.4 with $r = 0$. If there are a finite number of singularities, say $\alpha_1, \alpha_2, \ldots, \alpha_m$, in each period, the series are linear combinations of series of the same type with $\alpha = a_1, \alpha_2, \ldots, \alpha_m$.

For (C, k) summability the work of Wolf shows that the null series are linear combinations of series of type 2.4 where the index r has the values $0, 1, \ldots, k-1$ [**7**; **9**, p. 302; **4**, p. 92; **8**, p. 355; **1**].

If there is only one singularity α in a period and $\alpha \equiv \pi$, the null (C, k) series must be a linear combination of series of type 2.4 with $r = 0, 1, 2, \ldots, k-1$, and $\alpha \equiv \pi$, viz.

2.5
$$\Lambda_k(-\pi, \pi) = \sum_{r=1}^{k} A'_r[D^{r-1}\tfrac{1}{2} + \sum (-1)^n D^{r-1} \cos nx]$$
$$= \sum_{r=1}^{k} A_r \sum (-1)^n n^{r-1} e^{inx}$$

where the A'_r, A_r are arbitrary constants. This may be called the general NTS of order k for $(-\pi, \pi)$.

3. Solution of differential equations in trigonometric series.

Let

3.1
$$\Phi = 0$$

be a differential equation. In what follows only ordinary equations of a sufficiently simple type will be considered, so that Φ represents a function of x, y, and the derivatives of y. In the case of a linear equation of order m with constant coefficients we shall have $\Phi = F(D)y - f(x)$, where $D = d/dx$, $F(D)$ is a polynomial of degree m with constant coefficients and $f(x)$ is a function of x.

Our purpose is to solve 3.1 in periodic form, getting the general solution (complete primitive) in terms of trigonometric series and involving arbitrary constants. For special solutions incorporating given "initial conditions" it may be observed that for a problem of periodic type such conditions may involve the average value of y, or the general property of continuity and periodicity, or other conditions which lead easily to the evaluation of the constants. An example of such a special solution is 3.8(b) below.

Consider any trigonometric series

$$\tag{3.2} \tfrac{1}{2} a_0 + \sum (a_n \cos nx + b_n \sin nx) = \sum c_n e^{inx}$$

and its rth derived series

$$\tag{3.3} \sum n^r [a_n \cos (nx + \tfrac{1}{2} r\pi) + b_n \sin (nx + \tfrac{1}{2} r\pi)] = \sum (in)^r c_n e^{inx}.$$

Suppose 3.3, for $r = m$, is summable (C, k_1), where k_1 is a non-negative integer; and suppose that the summability is uniform over any closed interval $-\pi < -\omega \leqslant x \leqslant \omega < \pi$, interior to $(-\pi, \pi)$. It follows, by a theorem on convergence factors [**2**, p. 131, Theorem 76], that 3.3 is necessarily summable (C, $k_1 - m + r$) for $r = 0, 1, \ldots, m - 1$ provided $k_1 - m + r \geqslant 0$. Thus all the series 3.2, 3.3 for $r = 1, 2, \ldots, m$ are uniformly summable (C, k_1). Then, by a theorem on differentiation of summable series [**2**, p. 349, Theorem 249], if the sum of 3.2 is a function y having derivatives $D^r y$ for $r = 1, 2, \ldots, m$ then the (C, k_1) sums of the series 3.3 are equal to these derivatives; thus

$$\tag{3.4} D^r y = \sum n^r [a_n \cos (nx + \tfrac{1}{2} r\pi) + b_n \sin (nx + \tfrac{1}{2} r\pi)] = \sum (in)^r c_n e^{inx}, \quad (\mathrm{C}, k_1)$$

for $r = 1, 2, \ldots, m$.

If now $F(D)$ is a differential operator of order m, of suitable type, we shall have

$$\tag{3.5} \begin{aligned} F(D)y &= F(D)\tfrac{1}{2}a_0 + \sum F(D)(a_n \cos nx + b_n \sin nx) \\ &= \tfrac{1}{2}a'_0 + \sum (a'_n \cos nx + b'_n \sin nx) \\ &= \sum F(D)c_n e^{inx} = \sum c'_n e^{inx}, \qquad (\mathrm{C}, k_2), \end{aligned}$$

for some integral value of k_2. This will necessarily be the case if $F(D)$ is a polynomial with constant coefficients; then $c'_n = F(in)c_n$, with corresponding values for a'_n, b'_n; and $k_2 = k_1$.

If then $f(x)$ is a function which can be expanded in $(-\pi, \pi)$ as a trigonometric series, summable (C, k_3), say

$$\tag{3.6} f(x) = \tfrac{1}{2}\alpha_0 + \sum (\alpha_n \cos nx + \beta_n \sin nx) = \sum \gamma_n e^{inx}, \qquad (\mathrm{C}, k_3),$$

the differential equation 3.1 with $\Phi = F(D)y - f(x)$ will be equivalent to

$$\tag{3.7} \tfrac{1}{2}(a'_0 - \alpha_0) + \sum [(a'_n - \alpha_n) \cos nx + (b'_n - \beta_n) \sin nx] = 0$$

or

$$\sum (c'_n - \gamma_n) e^{inx} = 0, \qquad (\mathrm{C}, k),$$

where k is some integer, viz. the greater of k_2, k_3.

By the Heine-Cantor theory, described in §2, the equality 3.7 could be satisfied for all x in $(-\pi, \pi)$ with the possible exception of $x \equiv \pi$ in the field of convergence (i.e. with $k = 0$) by and only by equating to zero the coefficients $a'_n - \alpha_n$, $b'_n - \beta_n$, or $c'_n - \gamma_n$, for all integral n. The coefficients a_n, b_n, or c_n might thence be deduced and so give a solution of 3.1; but the general solution could not usually be found in this way.

By the Verblunsky-Wolf theory, however, the equality 3.7 will be satisfied in the field (C, k) where $k \geqslant 1$ by equating the coefficients to the corresponding coefficients of the general NTS of order k with singularity at $x \equiv \pi$, viz. $\Lambda_k(-\pi, \pi)$ of 2.5. The corresponding coefficient-equations are

$$a'_0 - \alpha_0 = A'_1, \quad a'_n - \alpha_n = (-1)^n(A'_1 - n^2 A'_3 + \ldots \pm n^{2s}A'_{2s+1}),$$

$$b'_n - \beta_n = (-1)^n(-nA'_2 + n^3 A'_4 - \ldots \pm n^{2t-1} A'_{2t});$$

$$c'_n - \gamma_n = (-1)^n(A_1 + n A_2 + \ldots + n^{k-1} A_k);$$

where s, t are the greatest integers for which $2s - 1 \leqslant k, 2t \leqslant k$. These equations will, in suitable cases, determine the constants a'_n, b'_n, or c'_n, and thence a_n, b_n, or c_n. The required solution will then be found. If the solution, when found, is such that its derived series of order m is Cesàro-summable, uniformly in $-\pi < -\omega \leqslant x \leqslant \omega < \pi$, the process is justified and the solution will be the true solution.

To elucidate the process, consider the equations:

3.8 (a) $(D^2 + q^2)\, y = 0$, (b) $(D^2 + q^2)\, y = \frac{1}{2} \cot \frac{1}{2}x$,

where q is real but not integral, finding for (b), besides the general solution, the special solution with initial conditions:

(i) y is continuous (including $x \equiv \pi$) and periodic 2π.

(ii) The mean value of y over a period is y_0.

Using the real form, let $y = \frac{1}{2}a_0 + \sum (a_n \cos nx + b_n \sin nx)$. The equation (a) then gives

$$q^2\, \tfrac{1}{2}\, a_0 + \sum\, (q^2 - n^2)(a_n \cos nx + b_n \sin nx) = 0.$$

If we equated the coefficients to zero, treating this series as a convergent trigonometric series, we should obtain only the useless particular solution $y = 0$. So equate the coefficients instead to those of a suitable NTS. Taking the NTS $\Lambda_k(-\pi, \pi)$ of 2.5 with $k = 2$, viz.

$$A_1[\tfrac{1}{2} + \sum\, (-1)^n \cos nx] + A_2 \sum\, (-1)^n\, n \sin nx,$$

the coefficient equations are

$$q^2\, \tfrac{1}{2}a_0 = \tfrac{1}{2}A_1, \quad (q^2 - n^2)\, a_n = A_1(-1)^n, \quad (q^2 - n^2)\, b_n = A_2 n(-1)^n.$$

The solution of (a) is therefore

$$y = A_1\left[\frac{1}{2q^2} + \sum \frac{(-1)^n \cos nx}{q^2 - n^2}\right] + A_2 \sum \frac{(-1)^n\, n \sin nx}{q^2 - n^2}.$$

This is convergent for all x, but non-uniformly so[2] in the neighbourhood of $x \equiv \pi$ unless $A_2 = 0$. To prove the validity of the process in this case it suffices

[2]The choice of Λ with k equal to the order of the equation is governed by the fact that any greater value of k would lead only to a solution Cesàro-equivalent to the one found.

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to observe that the second derived series of the series found for y is uniformly summable (C, 2) in $(-\pi < -\omega \leqslant x \leqslant \omega < \pi)$ in virtue of the equalities

$$\sum \frac{(-1)^n (-n^2) \cos nx}{q^2 - n^2} = \sum (-1)^n \cos nx - q^2 \sum \frac{(-1)^n \cos nx}{q^2 - n^2},$$

$$\sum \frac{(-1)^n (-n^3) \sin nx}{q^2 - n^2} = \sum (-1)^n n \sin nx - q^2 \sum \frac{(-1)^n n \sin nx}{q^2 - n^2}.$$

It is easy to verify the solution otherwise by finding the corresponding Fourier series for the usual form of solution in terms of $\cos qx$, $\sin qx$.

To solve (b) we have $\frac{1}{2} \cot \frac{1}{2}x = \sum \sin nx$, (C, 1) (this being the Cauchy principal-value Fourier series of $\frac{1}{2} \cot \frac{1}{2}x$) and we have only to subtract the coefficient 1 of $\sin nx$ in this series from b_n in the above coefficient-equations. The resulting general solution of (b) is

$$y = A_1 \left[\frac{1}{2q^2} + \sum \frac{(-1)^n \cos nx}{q^2 - n^2} \right] + A_2 \sum \frac{(-1)^n n \sin nx}{q^2 - n^2} + \sum \frac{\sin nx}{q^2 - n^2}.$$

This is convergent for all x, but non-uniformly about $x \equiv \pi$. In this case, in consequence of the point of non-uniform summability of $\sum \sin nx$ and infinite discontinuity of $\frac{1}{2} \cot \frac{1}{2}x$ at $x \equiv 0$, the second derived series is non-uniformly summable about $x \equiv 0$ as well as $x \equiv \pi$; but the justification applies to the two open intervals $(-\pi, 0)$, $(0, \pi)$ separately.

To find the required special solution, condition (i) shows that $A_2 = 0$ because the sum of the series $\sum (-1)^n n (q^2 - n^2)^{-1} \sin nx$ is discontinuous at $x \equiv \pi$ like $\sum (-1)^n n^{-1} \sin nx$, and (ii) shows that $A_1/2q^2 = y_0$. The special solution is therefore

$$y = y_0 \left[1 + \sum \frac{2q^2 \cos nx}{q^2 - n^2} \right] + \sum \frac{\sin nx}{q^2 - n^2}.$$

This is absolutely and uniformly convergent for all x.

In 3.8 if q is replaced by a positive integer N (or zero) the solution fails owing to the zero denominator $q^2 - n^2$ for $n = N$. Supposing $N > 0$, the equation (a) then has the simple solution $y = A_1 \cos Nx + A_2 \sin Nx$, which is the complementary function for (b). To find a particular integral for (b) the procedure is to solve

3.9 $$(D^2 + N^2)\, y = \tfrac{1}{2} \cot \tfrac{1}{2}x$$

in a trigonometric series suitable for $(-\pi, \pi)$. As above, replace $\frac{1}{2} \cot \frac{1}{2}x$ by the series $\sum \sin nx$, substitute $y = \frac{1}{2}a_0 + \sum (a_n \cos nx + b_n \sin nx)$ and equate the difference of the two sides to the NTS $\Lambda_2(-\pi, \pi)$. Thus

$$N^2 \tfrac{1}{2}a_0 + \sum (N^2 - n^2)(a_n \cos nx + b_n \sin nx) - \sum \sin nx$$
$$= A_1 [\tfrac{1}{2} + \sum (-1)^n \cos nx] + A_2 \sum (-1)^n n \sin nx.$$

Equating coefficients gives

$$0 \times a_N = (-1)^N A_1, \quad 0 \times b_N = 1 + (-1)^N N A_2,$$

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and, for $n \neq N$,

$$(N^2 - n^2)\, a_n = (-1)^n A_1, \quad (N^2 - n^2)\, b_n = 1 + (-1)^n n A_2,$$

including $N^2 a_0 = A_1$. Hence $A_1 = 0$ and $a_N = A'_1$ (an arbitrary constant); $A_2 = (-1)^{N+1}/N$ and $b_N = A'_2$ (an arbitrary constant); $a_0 = A_1/N^2 = 0$; and, for $n \neq N$,

$$a_n = \frac{(-1)^n A_1}{N^2 - n^2} = 0, \quad b_n = \frac{1 - (-1)^n n A_2}{N^2 - n^2} = \frac{N - (-1)^{n+N} n}{N(N^2 - n^2)}.$$

The complete solution of 3.9 is therefore

$$y = A'_1 \cos Nx + A'_2 \sin Nx + \sum{}' \frac{N - (-1)^{n+N} n}{N(N^2 - n^2)} \sin nx,$$

where $\sum'$ denotes summation with $n = N$ omitted. It may be observed that in such a case the complementary function is obtained incidentally in the process of finding the particular integral.

The case $N = 0$ is left to the reader; the solution is

$$y = A_1 + A_2 \sum \frac{(-1)^n \sin nx}{n} - \sum \frac{\sin nx}{n^2}.$$

4. Linear equation with constant coefficients. Applied to the general linear ordinary differential equation with constant coefficients the method of §3, using the NTS $\Lambda_m(-\pi, \pi)$, yields the following theorem.

THEOREM. *Let*

(i) *$F(D)$ be a polynomial of degree m in D with constant coefficients;*

(ii) *$f(x) = \sum \gamma_n e^{inx}$, (C, k), where k is a non-negative integer, uniformly over $(-\pi, \pi)$, with the possible exclusion of the neighbourhoods of a finite number of exceptional points.*

Then the general solution of the ordinary differential equation

4.1 $$F(D)\, y = f(x)$$

is representable in $(-\pi, \pi)$ by a trigonometric series

4.2 $$y = \sum c_n e^{inx}$$

which is convergent if $k \leqslant m$ and summable (C, $k - m$) if $k \geqslant m$, uniformly as in (ii).

The coefficients c_n for which $F(in) \neq 0$ are given by

4.3 $$c_n = \left[\gamma_n + (-1)^n \sum_{r=1}^{m} A_r n^{r-1} \right] / F(in).$$

The coefficients c_N, if any, for which $F(iN) = 0$ are arbitrary constants and, for every such N,

4.4 $$\gamma_N + (-1)^N \sum_{r=1}^{m} A_r N^{r-1} = 0.$$

The constants A_r, with the c_N, form a set of m arbitrary constants.

Proof. First, by expanding $1/F(in)$ in powers of $1/n$ and appealing to the theorem on convergence factors referred to in §3 [**2**, p. 131, Theorem 76], it is seen from 4.2, 4.3 that the series for y is convergent if $k \leqslant m$ and summable $(C, k - m)$ if $k \geqslant m$, uniformly as required.

Secondly, from 4.2, 4.3, 4.4, we have formally

$$F(D)y = \sum{}' \gamma_n e^{inx} + \sum_{r=1}^{m} A_r \sum{}' (-1)^n n^{r-1} e^{inx} + \sum_N c_N F(iN) e^{iNx}$$

$$= \sum \gamma_n e^{inx} - \sum_N \left[\gamma_N + (-1)^N \sum_{r=1}^{m} A_r N^{r-1} \right] e^{iNx} + \sum_{r=1}^{m} A_r \sum (-1)^n n^{r-1} e^{inx}$$

$$= f(x),$$

where $\sum_N$ denotes summation over all integral values of N for which $F(iN) = 0$, $\sum'$ denotes summation over all other integral values of n and the summability is (C, k) or (C, m) according as $k \geqslant m$ or not.

Finally the justification of these formal equalities follows by observing that $F(D)$ is a linear combination of powers D^{r-1} with $r = 1, 2, \ldots, m + 1$ and that the conditions of the theorem on differentiation of summable series referred to in §3 [**2**, p. 349, Theorem 249) are satisfied in the closed intervals excluding the exceptional neighbourhoods. Therefore the various derived series are appropriately summable to the corresponding derivatives of y; and their combination, which is summable to $f(x)$, is summable to $F(D)y$. The theorem is established.

It may be observed that no knowledge of the factors of $F(D)$ is needed beyond that of whether or not $D - iN$ is a factor (i.e., $F(iN) = 0$) for integral N. If the coefficients of $F(D)$ are given algebraically, the existence and values of the integers N may or may not be determinable, but no difficulty can arise with numerical coefficients.

It may also be noted that the class of functions $f(x)$ which are representable as Cesàro sums of trigonometric series as required in the theorem includes in particular (a) functions which may have in $(-\pi, \pi)$ a finite number of discontinuities (or of "discontinuities in the mean" in the Cesàro sense [**3**]) and (b) functions which, regarded as analytic functions,[3] are meromorphic in a region including $(-\pi, \pi)$.

It is observed by the referee that the complementary function of 4.1 can be found independently by finding the Fourier series for the C.F. in the usual form involving the unknown linear factors of $F(D)$ and combining the resulting series.

Examples of equation of this type (besides those of §3) are

(a) $(D^3 + D^2 + 1)\, y = \frac{1}{2} \cot \frac{1}{2}x$, (b) $(D^2 + 1)(D^3 + D + 1)\, y = x^{-1}$,

whose solutions are

[3]Such functions can be reduced to the special function $\frac{1}{2} \cot \frac{1}{2}x$ and its derivatives.

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$$\begin{aligned}(a)\ A_1[\tfrac{1}{2} + \textstyle\sum(-1)^n\{(1-n^2)\cos nx - n^3 \sin nx\}/\Delta \\ + A_2\textstyle\sum(-1)^n\{n^4\cos nx + (n-n^3)\sin nx\}/\Delta \\ + A_3\textstyle\sum(-1)^n\{(n^2-n^4)\cos nx - n^5\sin nx\}/\Delta \\ - \textstyle\sum\{n^3\cos nx + (1-n^2)\sin nx\}/\Delta,\end{aligned}$$

with $\Delta = (1-n^2)^2 + n^6$;

$$\begin{aligned}&(b)\ \textstyle\sum' 2\,\mathrm{Si}(n\pi)\{(n^3-n)\cos nx + \sin nx\}/\pi R \\ &\quad + A_1\cos x + A_2 \sin x + A_3[\tfrac{1}{2} + \textstyle\sum'\theta\{\cos nx - (n^3-n)\sin nx\}] \\ &\quad + A_4\textstyle\sum'\theta\{(n^4-n^2)\cos nx + n\sin nx\} + A_5\textstyle\sum'\theta\{n^2\cos nx - (n^5-n^3)\sin nx\} \\ &\quad + A_6\textstyle\sum'\theta\{(n^6-n^4)\cos nx + n^3\sin nx\} + A_7\textstyle\sum'\theta\{n^4\cos nx - (n^7-n^5)\sin nx\},\end{aligned}$$

with $\theta^{-1} = (-1)^n R = (-1)^n(1-n^2)\{1+(n-n^3)^2\}$, $A_3 + A_5 + A_7 = 0$, $A_4 + A_6 = 2\,\mathrm{Si}\,\pi/\pi$; where $\sum$ and $\sum'$ denote summation for n from 1 to ∞ and from 2 to ∞ respectively, and $\mathrm{Si}(x)$ denotes

$$\int_0^x \frac{\sin t}{t}\,dt.$$

5. Concluding remarks. Although the principles of the method, as described in §3, are not restricted to ordinary linear equations with constant coefficients, the algebraic difficulties involved in an attempt to apply it even to linear equations with non-constant coefficients would seem forbidding. Extension to partial equations in two or more variables, using such null series as

$$\sum \exp(imx + iny)$$

might be more hopeful.

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