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# BARRELLED SPACES AND DENSE VECTOR SUBSPACES

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This note presents a structure theorem for locally convex barrelled spaces. It is shown that, corresponding to any Hamel basis, there is a natural splitting of a barrelled space into a topological sum of two vector subspaces, one with its strongest locally convex topology. This yields a simple proof that a barrelled space has a dense infinite-codimensional vector subspace, provided that it does not have its strongest locally convex topology. Some further results and examples discuss the size of the codimension of a dense vector subspace.

The notation and terminology are standard for the most part, as, for example, in [2] or [5]. We use $E$ for a locally convex Hausdorff space, $E'$ for its (continuous) dual and $E^\star$ for its algebraic dual. The strongest (finest) locally convex topology is the Mackey topology $\tau(E, E^\star)$.

Any set of the form $\{(x_i, f_i) : i \in A\} \subseteq E \times E^\star$ is a *biorthogonal system* if and only if $f_i(x_j) = 1$ for $i = j$, and $f_i(x_j) = 0$ for $i \neq j$.

THEOREM 1. *Let $E$ be a Hausdorff barrelled space, the algebraic direct sum of the vector spaces $M$ and $N$. If there is a biorthogonal system $\{(x_i, f_i) : i \in A\}$ such that $\{x_i : i \in A\}$ is a Hamel basis of $N$ and $\{f_i : i \in A\} \subseteq E' \cap M^\circ$, then $N$ has its strongest locally convex topology and $E$ is the topological direct sum of $M$ and $N$.*

PROOF: Let $V$ be an absolutely convex absorbent set in $N$. If $i \in A$, there is some $\alpha_i \neq 0$ with $\alpha_i x_i \in V$. Put $y_i = \alpha_i x_i$ and $g_i = \alpha_i^{-1} f_i$; then $\{(y_i, g_i) : i \in A\}$ is a biorthogonal system, $\{y_i : i \in A\}$ is a basis for $N$ and $\{g_i : i \in A\} \subseteq E' \cap M^\circ$. Let $G$ be the polar of $\{y_i : i \in A\}$ in $\operatorname{span}\{g_i : i \in A\}$. Then $G^\circ \subseteq V + M$. For if $x \in G^\circ$, then $x = \sum_{i \in S} \lambda_i y_i + z$ for some finite subset $S$ of $A$ and $z \in M$, and $|g(x)| \leqslant 1$ for all $g \in G$. Take $g = \sum_{i \in S} (\overline{\operatorname{sgn}} \lambda_i) g_i$. Then $g \in G$ and $|g(x)| = \sum_{i \in S} |\lambda_i| \leqslant 1$; thus $\sum_{i \in S} \lambda_i y_i \in V$ since $V$ is absolutely convex.

Also $G^\circ$ is absorbent since $M \subseteq G^\circ$ and each $y_i \in G^\circ$. Thus $G^\circ$ is a barrel since each $g_i$ is continuous, and so is a neighbourhood of the origin in $E$. Hence $V$ is a

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neighbourhood of the origin in $N$, $N$ has its strongest locally convex topology, and the projector of $E$ onto $N$ (along $M$) is continuous. ∎

We note incidentally that the existence of such a biorthogonal system is also necessary for the conclusion of the theorem.

When $E$ is a locally convex space with Hamel basis $\{x_i : i \in B\}$, the corresponding biorthogonal system $\{(x_i, f_i) : i \in B\}$ produces a natural splitting of $E$, determined by whether a linear form $f_i$ is continuous or discontinuous. Let

$$C = \{i : f_i \in E'\}, \qquad D = \{i : f_i \notin E'\},$$
$$E_C = \operatorname{span}\{x_i : i \in C\}, \qquad E_D = \operatorname{span}\{x_i : i \in D\}.$$

Then Theorem 1 has the following immediate consequence.

THEOREM 2 (SPLITTING THEOREM). *Let $E$ be a Hausdorff barrelled space and $\{x_i : i \in B\}$ a Hamel basis of $E$. Then, with the notation above, $E$ is the topological direct sum of $E_C$ and $E_D$, and $E_C$ has its strongest locally convex topology.*

From Theorem 1 we also obtain:

COROLLARY. (Saxon-Levin [8, p.92]) *Let $M$ be a closed vector subspace of countable codimension in a Hausdorff barrelled space $E$. Let $N$ be any algebraic complement of $M$ in $E$. Then $N$ is a topological complement of $M$ and has its strongest locally convex topology.*

PROOF: Given any basis $\{y_1, y_2, \ldots\}$ of $N$ we construct inductively a biorthogonal system satisfying the conditions of Theorem 1. At each stage, $x_n \in \operatorname{span}\{y_1, \ldots, y_n\}$; the span of $M$ and the finite set $\{x_i : i < n\}$ is closed, and the Hahn-Banach theorem ensures the existence of a suitable $f_n$. ∎

THEOREM 3. *Let $E$ be a Hausdorff barrelled space and suppose that every vector subspace of $E$ has finite codimension in its closure. Then $E$ has its strongest locally convex topology.*

PROOF: Choose a Hamel basis $\{x_i : i \in B\}$ for $E$. With the notation of Theorem 2, $E$ is the topological sum $E_C \oplus E_D$, where $E_C$ has its strongest locally convex topology. Suppose that $D$ contains a countably infinite set $I$. Let $M = \operatorname{span}\{x_i : i \in B \backslash I\}$. By hypothesis, $\overline{M} \subseteq M + F$ where $F = \operatorname{span}\{x_i : i \in S\}$ for some finite subset $S$ of $I$, and so $M + F = \overline{M} + F$ and is closed. Let $N = \operatorname{span}\{x_i : i \in I \backslash S\}$. Then by the Saxon-Levin Corollary, $E = (M + F) \oplus N$ and $N$ has its strongest locally convex topology. Hence for each $i \in I \backslash S$, $f_i$ is continuous, which contradicts $I \subseteq D$. Therefore $D$ is finite and so $E_D$ is finite-dimensional; since $E = E_C \oplus E_D$, $E$ has its strongest locally convex topology (and in fact $D$ is empty). ∎

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COROLLARY 1. *If $E$ is Hausdorff and barrelled with $E' \neq E^*$ then there exists a dense $\aleph_0$-codimensional vector subspace $L$ of $E$; also there exists an $\aleph_0$-dimensional vector subspace $N$ of $E^*$ with $N \cap E' = \{0\}$ such that $L = N^\circ$.*

PROOF: Since $E' \neq E^*$ the theorem shows that there exists a vector subspace $M$ of $E$ with infinite codimension in $\overline{M}$. Let $K$ be an algebraic complement of $\overline{M}$ in $E$. Then $M + K$ is dense in $E$ and of infinite codimension. Let $L$ be a vector subspace containing $M + K$ and of codimension $\aleph_0$ in $E$.

Let $\{x_i : i \in I\}$ be a basis for an algebraic complement of $L$ in $E$, and $\{(x_i, f_i) : i \in I\}$ be a biorthogonal system such that $f_i$ is zero on $L$ for all $i \in I$. Let $N = \operatorname{span}\{f_i : i \in I\}$. Then $N \cap E' = \{0\}$ since $L$ is dense, and clearly $L = N^\circ$. ∎

We note that this Corollary shows that when $E$ has the Mackey topology $\tau(E, E' + N)$ then $E$ is not barrelled. (For if $f$ is the pointwise limit of the [unconditional] series $\sum_{i \in I} f_i$, then $f$ is zero on the dense vector subspace $N^\circ$ and so is not in $E' + N$; this is also a consequence of Theorem 4 of [**7**].) This provides an alternative proof that any barrelled topology on a space $E$ with $E' \neq E^*$ has a non-barrelled countable enlargement [**6**, Theorem 3].

COROLLARY 2. *If $E$ is a locally convex Hausdorff space and its associated barrelled topology* [**3**] *is not its strongest locally convex topology, then there exists a dense infinite-codimensional vector subspace of $E$.*

(For by Corollary 1, there is an infinite-codimensional vector subspace $L$, dense when $E$ has its associated barrelled topology, and therefore dense in the original topology on $E$.)

When $E$ is any locally convex Hausdorff space, not necessarily barrelled, with $E' \neq E^*$, then of course dense vector subspaces exist; it is easy to see that a dense vector subspace of finite codimension $n$ exists if and only if the codimension of $E'$ in $E^*$ is at least $n$. Some simple considerations limit the size of the codimension of a dense vector subspace in a similar way for the infinite case.

THEOREM 4. *Let $E$ be a locally convex space with a dense $\aleph$-codimensional vector subspace, where $\aleph$ is any infinite cardinal. Then the codimension of $E'$ in $E^*$ is at least $2^{\aleph}$.*

PROOF: Let $L$ be dense in $E$ with $\aleph$-dimensional algebraic complement $M$. Then the vector subspace $L^\circ$ of $E^*$ has dimension equal to that of $M^*$, which is $2^{\aleph}$. Since $L$ is dense, $L^\circ \cap E' = \{0\}$, and so $\operatorname{codim} E' \geqslant 2^{\aleph}$. ∎

COROLLARY 1. *If $E'$ has countable codimension in $E^*$, then $E$ has no dense vector subspace of infinite codimension.*

(Therefore in this case the intersection of a sequence of dense hyperplanes, the nullspace of a linearly independent sequence of linear forms, cannot be dense.)

COROLLARY 2. *If $E'$ has codimension $2^{\aleph_0}$ in $E^*$, then $E$ has no dense vector subspace of codimension $2^{\aleph_0}$. Thus, with the continuum hypothesis, in this case $E$ can have no dense vector subspace of uncountable codimension.*

The theorem, with Corollary 2 of Theorem 3, gives the following result.

COROLLARY 3. *If the associated barrelled topology of $E$ is not $\tau(E, E^*)$, then codim $E' \geqslant 2^{\aleph_0}$.*

In [**6**], a theorem on completeness is used to deduce that the dual of a barrelled space is either $E^*$ or has uncountable codimension in $E^*$ [**6**, Theorem 2]. Here this result follows from Theorem 3, Corollary 1 and Theorem 4, Corollary 1.

**Example.** The following example, mentioned by Köthe [**4**, Section 22.5(5)] is cited by Eberhardt and Roelcke [**1**, 1.3] and also by Tsirulnikov [**9**, I, Note 2], who each demonstrate some of the points we raise here. However, we make this treatment self-contained. First, we use it to show that the condition of Corollary 2 of Theorem 3, that the associated barrelled topology is not $\tau(E, E^*)$, is not a necessary one for the existence of an infinite-codimensional dense vector subspace of $E$. In fact, there may even be a dense vector subspace of codimension equal to the dimension of $E$ when the associated barrelled topology is $\tau(E, E^*)$.

**(a).** Let $I$ be an index set of uncountable cardinality $\aleph$, and let $E = \phi(I)$, the algebraic direct sum of $\aleph$ copies of the real numbers. Then $E^* = \omega(I) = \mathbb{R}^I$. In $E^*$, consider the vector subspace consisting of those functions with at most a countable number of non-zero coordinates. This is $\sigma(E^*, E)$-dense in $E^*$; call it $E'$. For every countable subset $J$ of $I$, and for every absolutely convex absorbent subset $V$ of $\phi(J)$, let $U = V + \phi(I \backslash J)$, and give $E$ the topology with all such sets $U$ as a base of neighbourhoods of the origin.

**(b).** If $f$ is a linear form continuous for this topology, then $f$ is bounded on some $U$; hence $f$ has zero coordinates in the corresponding $I \backslash J$ and so $f \in E'$. Conversely, if $f \in E'$, there is a corresponding countable set $J$; let $U = \{x : |f(x)| \leqslant 1\}$. Then $U$ is of the form $V + \phi(I \backslash J)$ as above, and $f$ is bounded on $U$. Hence $E'$ is the dual of $E$ with this topology.

**(c).** Now in $E$, take the Hamel basis $\{e_i : i \in I\}$ of coordinate vectors and the corresponding biorthogonal linear forms $f_i$. Since each $f_i$ has only one non-zero coordinate, $f_i \in E'$ and so $f_i$ is continuous in the associated barrelled topology. Hence, by Theorem 2 applied to that topology, with the notation there, $E = E_C$ and $E$ has its strongest locally convex topology.

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**(d).** Write $I = \bigcup\{I_\gamma : \gamma \in \Gamma\}$ where the $I_\gamma$ are disjoint, each has cardinality $\aleph$ and $\Gamma$ has cardinality $\aleph$. Let $h_\gamma(x) = \sum\{f_i(x) : i \in I_\gamma\}$. Since the restriction of $h_\gamma$ to $\phi(I_\gamma)$ has an uncountable number of non-zero coordinates, it has no extension to $E$ which belongs to $E'$; hence the restriction of $h_\gamma$ to $\phi(I_\gamma)$ is not continuous. Thus $h_\gamma^{-1}(0) \cap \phi(I_\gamma)$ is a dense hyperplane in $\phi(I_\gamma)$. Let $M = \bigcap\{h_\gamma^{-1}(0) : \gamma \in \Gamma\}$. Then $h_\gamma^{-1}(0) \cap \phi(I_\gamma) \subseteq M$ and so $\phi(I_\gamma) \subseteq \overline{M}$ for all $\gamma$; hence $M$ is dense, and codim $M = \aleph$ (one dimension from each $\phi(I_\gamma)$; in fact we may choose any $i(\gamma)$ from each $I_\gamma$ and then span$\{e_i : \gamma \in \Gamma\}$ is an algebraic complement of $M$).

It is attractive to conjecture that a locally convex space $E$, with the property that every $\aleph_0$-dimensional vector subspace is isomorphic to $\phi$ (that is, has its strongest locally convex topology) and has a topological complement, must have topology $\tau(E, E^*)$. This example dispels that hope, even when the topology of $E$ is fine enough for the associated barrelled topology to be $\tau(E, E^*)$, as we now show.

**(e).** Let $J \subseteq I$ be countable. Then clearly $\phi(J)$ is isomorphic to $\phi$, from the construction in (a) of the topology of $E$, and $E = \phi(J) \oplus \phi(I \backslash J)$ (algebraically and topologically). Suppose that $N$ is any $\aleph_0$-dimensional vector subspace of $E$. Then each element in a basis for $N$ is linearly dependent on only a finite number of coordinate vectors; so there is some countable $J$ such that $N \subseteq \phi(J)$. Let $M$ be an algebraic complement of $N$ in $\phi(J)$. Since $\phi(J)$ has its strongest locally convex topology, $M$ is also a topological complement. Hence $E = (N \oplus M) \oplus \phi(I \backslash J) = N \oplus (M \oplus \phi(I \backslash J))$.

**(f).** On the other hand, we may ask whether the conjecture is correct for barrelled spaces. However, in [**1**], Eberhardt and Roelcke define the class of GM-spaces, those $E$ for which the closed graph theorem holds for any linear mapping of $E$ into any metrisable locally convex space. Such spaces are therefore barrelled. In [**1**], it is shown that every $\aleph_0$-dimensional vector subspace of a GM-space is isomorphic to $\phi$ and has a topological complement (1.5), and that GM-spaces exist which do not have the strongest locally convex topology (3.5). In fact, Theorem 7 of [**6**] shows that, for a subclass of GM-spaces, there even exist dense vector subspaces of codimension $2^{\aleph_0}$.

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