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COUNTING SYMMETRIC BRACELETS

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Abstract

An r-ary necklace (bracelet) of length n is an equivalence class of r-colourings of vertices of a regular n-gon, taking all rotations (rotations and reflections) as equivalent. A necklace (bracelet) is symmetric if a corresponding colouring is invariant under some reflection. We show that the number of symmetric r-ary necklaces (bracelets) of length n is $\frac{1}{2}(r+1)r^{n/2}$ if n is even, and $r^{(n+1)/2}$ if n is odd.

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1. Introduction

An r-ary *necklace* of length n is an equivalence class of r-colourings of vertices of a regular n-gon, taking all rotations as equivalent. It is well known that there are

$$N_r(n) = \frac{1}{n}\sum_{d|n}\varphi(d)r^{n/d}$$

r-ary necklaces of length n, where φ is the Euler function (see [2]).

In [4], a formula was obtained for the number $s_r(n)$ of symmetric r-ary necklaces of length n. Identifying the vertices of the polygon with the cyclic group $\mathbb{Z}_n$, the symmetries were taken to be the mappings $x \mapsto 2a - x$, where $a \in \mathbb{Z}_n$. This is the usual convenient notion of symmetry on an Abelian group. But in the case of the polygon it has a disadvantage. It gives us only the reflections with respect to an axis through the centre and one of the vertices of the polygon, and so, if n is even, it captures only half the reflections of the polygon.

In [3] it was shown that

$$s_r(n) = \begin{cases} \frac{1}{2}r^{n/2+1} + \frac{1}{2}r^{(m+1)/2} & \text{if } n = 2^l m,\ l \geq 1 \text{ and } m \text{ is odd,} \\ r^{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

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In this note we consider the problem of counting symmetric bracelets. An r-ary *bracelet* of length n is an equivalence class of r-colourings of vertices of a regular n-gon, taking all rotations and reflections as equivalent. It is well known that there are

$$B_r(n) = \frac{1}{2}N_r(n) + \begin{cases} \frac{1}{4}(r+1)r^{n/2} & \text{if } n \text{ is even,} \\ \frac{1}{2}r^{(n+1)/2} & \text{if } n \text{ is odd} \end{cases}$$

r-ary bracelets of length n (see [1, 5.29]). As distinguished from necklaces, bracelets leave no choice for the notion of symmetry. These have to be all the reflections. Then the symmetric bracelets turn out to be the same as the symmetric necklaces. We show that their number is

$$B_r^*(n) = N_r^*(n) = \begin{cases} \frac{1}{2}(r+1)r^{n/2} & \text{if } n \text{ is even,} \\ r^{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

In fact, we do more. We introduce the notion of symmetry on an Abelian group via the generalised dihedral group, which gives us in the case of $\mathbb{Z}_n$ all the reflections. We compute the numbers $N_r(A)$ and $B_r(A)$ of necklaces and bracelets on a finite Abelian group A and show that

$$B_r^*(A) = N_r^*(A) = 2B_r(A) - N_r(A).$$

From this we obtain the number $B_r^*(A) = N_r^*(A)$.

2. Symmetries, necklaces and bracelets

Let A be an Abelian group written multiplicatively and let G be the *generalised dihedral group* of A. That is, G is the semidirect product of A and the 2-element group $\{1, i\}$ with i representing the inversion of A. Thus, A is a subgroup of G of index 2, $G \setminus A = Ai$, $i^2 = 1$ and $ia = a^{-1}i$ for all $a \in A$. The group G naturally acts on A by

$$ax = a \cdot x \quad \text{and} \quad (ai)x = a \cdot x^{-1}.$$

Thus, the elements $a \in A$ represent the translations

$$A \ni x \mapsto ax \in A,$$

and the elements $ai \in Ai$ the mappings

$$A \ni x \mapsto ax^{-1} \in A,$$

which we call the *symmetries* on A.

Now suppose that A is finite and let $r \in \mathbb{N}$. An *r-colouring* of A is any mapping $\chi : A \to \{0, 1, \ldots, r-1\}$. Let r^A denote the set of all r-colourings of A. The action of G on A induces the action on r^A by

$$g\chi(x) = \chi(g^{-1}x).$$

That is, for every $a \in A$,

$$a\chi(x) = \chi(xa^{-1}) \quad \text{and} \quad ai\chi(x) = \chi(ax^{-1}).$$

The mappings

$$r^A \ni \chi \mapsto ai\chi \in r^A,$$

where $a \in A$, correspond to the symmetries on A. Therefore, we call them the *symmetries* on r^A. A colouring $\chi \in r^A$ is *symmetric* if it is a fixed point of some symmetry, that is, there is $a \in A$ such $\chi(x) = \chi(ax^{-1})$ for all $x \in A$.

For every $\chi \in r^A$, $A\chi$ ($G\chi$) is the A-orbit (G-orbit) of χ. We call it an r-ary *necklace* (*bracelet*) on A. A necklace $A\chi$ (bracelet $G\chi$) is *symmetric* if there is a symmetric $\psi \in A\chi$ ($\psi \in G\chi$). Let $N_r(A)$, $B_r(A)$, $N_r^*(A)$ and $B_r^*(A)$ denote the numbers of r-ary necklaces, bracelets, symmetric necklaces and symmetric bracelets on A. We conclude this section by computing the numbers $N_r(A)$ and $B_r(A)$.

Applying Burnside's lemma [1, Proposition 1.7] gives us that

$$\begin{aligned} N_r(A) &= \frac{1}{|A|}\sum_{a\in A} |\{\chi \in r^A : a\chi = \chi\}| \\ &= \frac{1}{|A|}\sum_{a\in A} |\{\chi \in r^A : \chi(xa^{-1}) = \chi(x)\}|, \\ B_r(A) &= \frac{1}{2|A|}\sum_{g\in G} |\{\chi \in r^A : g\chi = \chi\}| \\ &= \frac{1}{2|A|}\sum_{a\in A} (|\{\chi \in r^A : \chi(xa^{-1}) = \chi(x)\}| + |\{\chi \in r^A : \chi(ax^{-1}) = \chi(x)\}|). \end{aligned}$$

It is easy to see that for every $a \in A$,

$$|\{\chi \in r^A : \chi(xa^{-1}) = \chi(x)\}| = r^{|A:\langle a\rangle|}.$$

Substituting this into the expression above for $N_r(A)$, we obtain the following result.

THEOREM 2.1. *For every finite Abelian group A and $r \in \mathbb{N}$,*

$$N_r(A) = \frac{1}{|A|}\sum_{a\in A} r^{|A:\langle a\rangle|}.$$

In the case where $A = \mathbb{Z}_n$, the number of elements $a \in A$ of the same order d is $\varphi(d)$, so we obtain from Theorem 2.1 the following corollary.

COROLLARY 2.2. *For every $n, r \in \mathbb{N}$,*

$$N_r(n) = \frac{1}{n}\sum_{d|n} \varphi(d) r^{n/d}.$$

Computing $|\{\chi \in r^A : \chi(ax^{-1}) = \chi(x)\}|$ involves the subgroups A^2 and $A[2]$ of A, which are defined by

$$A^2 = \{x^2 : x \in A\} \quad \text{and} \quad A[2] = \{x \in A : x^2 = 1\}.$$

Notice that $|A[2]|$ is the 2-rank of A and $|A| = |A^2| \cdot |A[2]|$.

Lemma 2.3. *For every $a \in A$,*

$$|\{\chi \in r^A : \chi(ax^{-1}) = \chi(x)\}| = \begin{cases} r^{(|A|+|A[2]|)/2} & \text{if } a \in A^2, \\ r^{|A|/2} & \text{otherwise.} \end{cases}$$

Proof. The number on the left is equal to the number of r-colourings of the family $\{\{x, ax^{-1}\} : x \in A\}$. Since $x = ax^{-1}$ if and only if $x^2 = a$, that number is

$$r^{|K_a|+(|A|-|K_a|)/2} = r^{(|A|+|K_a|)/2},$$

where $K_a = \{x \in A : x^2 = a\}$. If $a \notin A^2$, then $K_a = \emptyset$. Let $a \in A^2$ and pick $x_0 \in K_a$. We claim that $K_a = x_0 A[2]$.

To see that $x_0 A[2] \subseteq K_a$, let $y \in A[2]$. Then $(x_0 y)^2 = x_0^2 y^2 = a$, so $x_0 y \in K_a$.

To see the converse inclusion, let $x \in K_a$. From $x_0^2 = a$ and $x^2 = a$, we obtain that $(x x_0^{-1})^2 = 1$, whence $x x_0^{-1} \in A[2]$, and so $x \in x_0 A[2]$. □

Now applying Lemma 2.3 gives us that

$$\begin{aligned} B_r(A) &= \frac{1}{2|A|} \sum_{a \in A} (|\{\chi \in r^A : \chi(xa^{-1}) = \chi(x)\}| + |\{\chi \in r^A : \chi(ax^{-1}) = \chi(x)\}|) \\ &= \frac{1}{2} N_r(A) + \frac{1}{2|A|} (|A^2| r^{(|A|+|A[2]|)/2} + (|A| - |A^2|) r^{|A|/2}). \end{aligned}$$

Taking into account that $|A|/|A^2| = |A[2]|$, we obtain the following result.

Theorem 2.4. *For every finite Abelian group A and $r \in \mathbb{N}$,*

$$B_r(A) = \frac{1}{2} N_r(A) + \frac{1}{2|A[2]|} (r^{|A[2]|/2} + |A[2]| - 1) r^{|A|/2}.$$

Since

$$\mathbb{Z}_n[2] = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$$

we obtain from Theorem 2.4 the following corollary.

Corollary 2.5. *For every $n, r \in \mathbb{N}$,*

$$B_r(n) = \tfrac{1}{2} N_r(n) + \begin{cases} \tfrac{1}{4}(r+1) r^{n/2} & \text{if } n \text{ is even,} \\ \tfrac{1}{2} r^{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

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3. Symmetric orbits

Now let G act on any finite set X. A symmetry on X is a mapping $X \ni x \mapsto aix \in X$, where $a \in A$. An element $x \in X$ is symmetric if it is a fixed point of some symmetry, that is, $aix = x$ for some $a \in A$. For every $x \in X$, Ax (Gx) is the *A-orbit (G-orbit)* of x. An A-orbit (G-orbit) is *symmetric* if it contains a symmetric element. Let $O(A)$, $O(G)$, $O^*(A)$ and $O^*(G)$ denote the numbers of A-orbits, G-orbits, symmetric A-orbits and symmetric G-orbits in X.

Theorem 3.1. *The following identity holds:*

$$O^*(G) = O^*(A) = 2O(G) - O(A).$$

Before proving Theorem 3.1 we establish two simple lemmas. The first of them gives us several characterisations of symmetric orbits and tells us that the symmetric G-orbits coincide with the symmetric A-orbits.

Lemma 3.2. *Let $x \in X$. Then the following statements are equivalent:*
(1) *x is symmetric;*
(2) *every $y \in Ax$ is symmetric;*
(3) *Ax is symmetric;*
(4) *Gx is symmetric;*
(5) *$Gx = Ax$; and*
(6) *$ix \in Ax$.*

Proof. (1) $\Rightarrow$ (2). Pick $a, b \in A$ such that $aix = x$ and $bx = y$. Then

$$ab^2iy = ab^2ibx = abix = baix = bx = y.$$

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Pick $y \in Gx$ and $a \in A$ such that $aiy = y$. Then for every $b \in A$, $biy = biaiy = ba^{-1}y$. Consequently, $Gy = Ay$. Since $Ax \subseteq Gx = Gy = Ay$, it follows that $Ax = Ay = Gx$.

(5) $\Rightarrow$ (6) is obvious.

(6) $\Rightarrow$ (1). Pick $a \in A$ such that $ix = ax$. Then $a^{-1}ix = x$. □

The second lemma tells us that if a G-orbit is not symmetric, then it is the union of two A-orbits.

Lemma 3.3. *Let $x \in X$. If Gx is not symmetric, then Gx is a disjoint union of Ax and Aix.*

Proof. That Ax and Aix are distinct A-orbits follows from Lemma 3.2. Clearly, $Ax \cup Aix \subseteq Gx$, and since $(ai)x = a(ix)$, the converse inclusion also holds. Thus, $Gx = Ax \cup Aix$. □

Proof of Theorem 3.1. By Lemma 3.2, a G-orbit is symmetric if and only if it is a symmetric A-orbit. In particular, $O^*(G) = O^*(A)$. And by Lemma 3.3, if a G-orbit is not symmetric, then it is a disjoint union of two A-orbits. It then follows that

$$O(G) = O^*(A) + \tfrac{1}{2}(O(A) - O^*(A)) = \tfrac{1}{2}(O(A) + O^*(A)),$$

whence $O^*(A) = 2O(G) - O(A)$. □

In the case where $X = r^A$, Theorem 3.1 gives us the following result.

Corollary 3.4. *For every finite Abelian group A and $r \in \mathbb{N}$,*

$$B_r^*(A) = N_r^*(A) = 2B_r(A) - N_r(A).$$

From Corollary 3.4 and Theorem 2.4 we obtain our main result.

Theorem 3.5. *For every finite Abelian group A and $r \in \mathbb{N}$,*

$$B_r^*(A) = N_r^*(A) = \frac{1}{|A[2]|}(r^{|A[2]|/2} + |A[2]| - 1)r^{|A|/2}.$$

As a special partial case we obtain the following corollary from Theorem 3.5.

Corollary 3.6. *For every $n, r \in \mathbb{N}$,*

$$B_r^*(n) = N_r^*(n) = \begin{cases} \frac{1}{2}(r+1)r^{n/2} & \textit{if } n \textit{ is even,} \\ r^{(n+1)/2} & \textit{if } n \textit{ is odd.} \end{cases}$$

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