A QUARTIC WITH 28 REAL BITANGENTS

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According to Pluecker's equations a plane quartic can have at most 28 bitangents. These can be all real as has been shown by Pluecker himself with the help of the example (see [1]):

$$(y^2 - x^2)(x - 1)(x - \tfrac{3}{2}) - 2[y^2 + x(x - 2)]^2 = k^2 .$$

Another example of a plane quartic having 28 real bitangents is:

The locus of the centres of ellipses with semi-axes a, b, which are tangent to two fixed straight lines forming an angle 2φ, provided that $\sqrt{\frac{b}{a}} < \tan\varphi < \sqrt{\frac{a}{b}}$.

A possible advantage of the above example is the fact that, besides its simple geometrical interpretation, the equations of the 28 bitangents can be given in simple algebraic terms.

Choose the bisectors of the two fixed straight lines as the axes of an orthogonal cartesian coordinate system, and let $m = \tan\varphi$. The two fixed straight lines will then have the line coordinates $[m, 1, 0]$ and $[-m, 1, 0]$. The equation in line coordinates u, v, w, for an ellipse with semi-axes a, b and centre (x, y) is

$$\text{(E)} \quad a^2(u\cos\theta - v\sin\theta)^2 + b^2(u\sin\theta + v\cos\theta)^2 - (xu + yv + w)^2 = 0 ,$$

where θ is the angle between the x-axis and an axis of the ellipse.

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The lines $[\pm m, 1, 0]$ are tangents if

$$a^2(m \cos\theta \mp \sin\theta)^2 + b^2(m \sin\theta \pm \cos\theta)^2 - (\pm mx + y)^2 = 0 .$$

These two equations imply (by addition and subtraction)

$$(m^2a^2 + b^2)\cos^2\theta + (m^2b^2 + a^2)\sin^2\theta = m^2x^2 + y^2$$

$$(a^2 - b^2)\sin\theta\cos\theta = -xy .$$

The elimination of the parameter θ between these two equations leads us finally to the equation of the locus (C) of the centres of (E):

$$\text{(C)} \quad [(x^2 + y^2) - (a^2 + b^2)][(m^4x^2 + y^2) - m^2(a^2 + b^2)] + a^2b^2(m^2 - 1)^2 = 0 .$$

The curve (C) is therefore a quartic located in the four regions of the plane lying between the circle

$$x^2 + y^2 = a^2 + b^2$$

and the ellipse

$$m^4x^2 + y^2 = m^2(a^2 + b^2)$$

(see Fig. 1).

It is now easy to determine the equations of the 28 bitangents of (C). We list them below but leave their determination as a problem to the interested reader.

Bitangents parallel to Oy:

(t_1, t_2): $x = \pm(a+b)\cos\varphi$; (t_3, t_4): $x = \pm(a-b)\cos\varphi$.

Bitangents parallel to Ox:

(t_5, t_6): $y = \pm(a+b)\sin\varphi$; (t_7, t_8): $y = \pm(a-b)\sin\varphi$.

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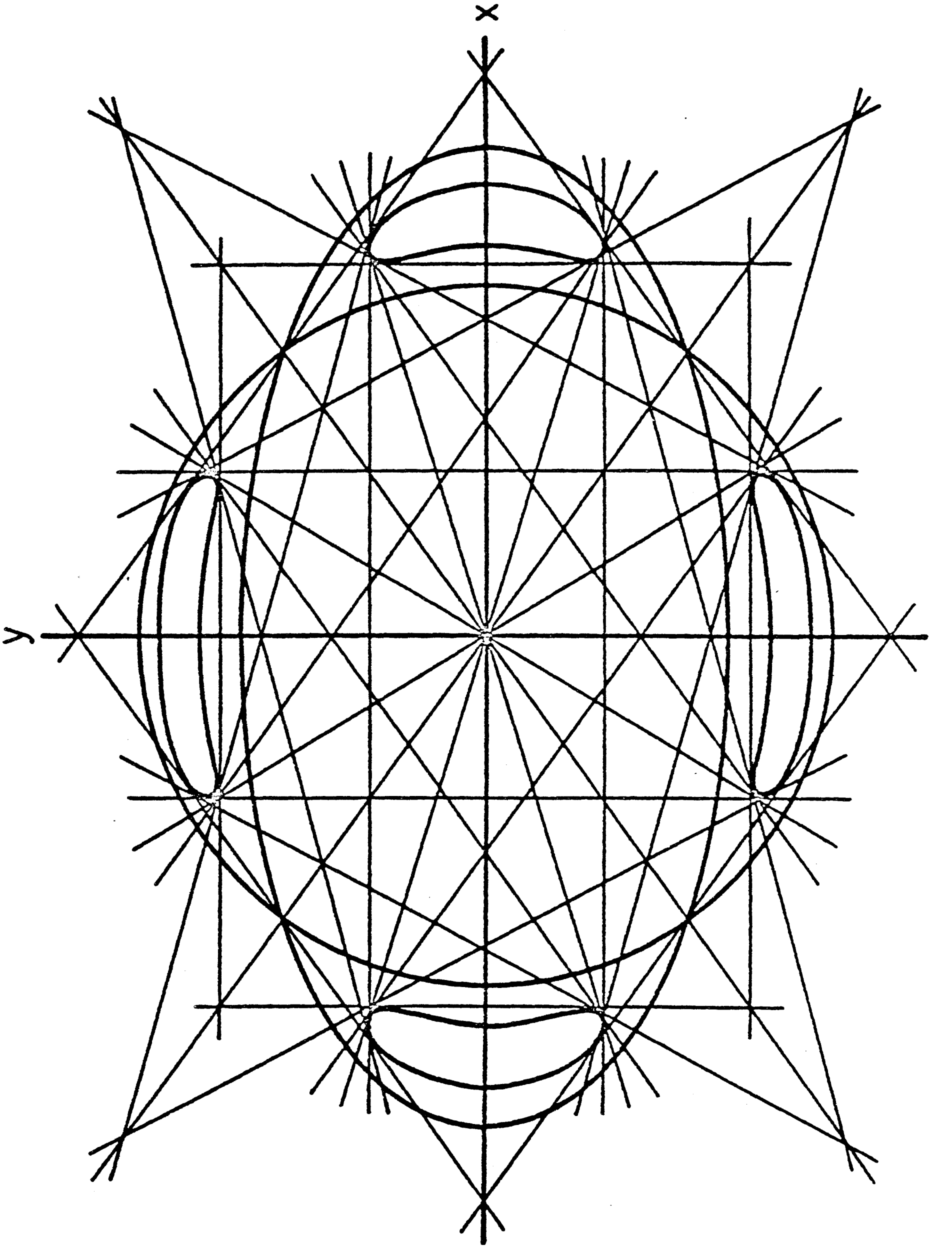

Figure 1

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Bitangents through O:

$$(t_9,\ldots,t_{12}):\ (a^2-b^2)(y^2\cos^2\varphi - x^2\sin^2\varphi) \pm 2abxy = 0 .$$

Bitangents with slope m and -m:

$$(t_{13},\ldots,t_{16}):\ y\cos\varphi \pm x\sin\varphi \pm a = 0 \ ;$$

$$(t_{17},\ldots,t_{20}):\ y\cos\varphi \pm x\sin\varphi \pm b = 0 \ .$$

The remaining bitangents are:

$$(t_{21},\ldots,t_{24}):\ by\cos\varphi \pm ax\sin\varphi \pm \sqrt{(a^2\sin^2\varphi + b^2\cos^2\varphi)(a^2\cos^2\varphi + b^2\sin^2\varphi)} = 0 \ ;$$

$$(t_{25},\ldots,t_{28}):\ ay\cos\varphi \pm bx\sin\varphi \pm \sqrt{(a^2\sin^2\varphi + b^2\cos^2\varphi)(a^2\cos^2\varphi + b^2\sin^2\varphi)} = 0 \ .$$

It is obvious that all bitangents, except possibly t_1, t_2, t_5, t_6, are real (i.e., have real points of contact with (C)) since these lie on different branches of (C). Calculating the second coordinates of the points of contact of t_1, t_2 and t_5, t_6, we get respectively:

$$y = \pm\frac{1}{\cos\varphi}\sqrt{(a\sin^2\varphi - b\cos^2\varphi)(a\cos^2\varphi - b\sin^2\varphi)}$$

and

$$x = \pm\frac{1}{\sin\varphi}\sqrt{(a\sin^2\varphi - b\cos^2\varphi)(a\cos^2\varphi - b\sin^2\varphi)} \ .$$

In order that these should be real and distinct we must have:

$$\frac{b}{a} < \tan^2\varphi < \frac{a}{b} \ ,$$

or

$$\sqrt{\frac{b}{a}} < \tan\varphi < \sqrt{\frac{a}{b}} \ .$$

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Remark 1. It can be seen that all the bitangents are easily constructible with straightedge and compass. For instance, t_{13} to t_{20} are the tangents of slope $\pm m$ to the circles with radii $\underline{a}$ and $\underline{b}$ and centre at the origin. Likewise t_{21} to t_{28} are either parallel, or intersect pairwise on the axes of coordinates and the straight lines $y = \pm mx$. In Fig. 1 we have used this for a (relatively) accurate drawing of (C).

Remark 2. The quartic

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x^2}{b^2} + \frac{y^2}{a^2} - 1\right) + k^2 = 0 \qquad (a \neq b)$$

has real branches and at least 24 real bitangents if

$$0 < k < \frac{a^2 - b^2}{2ab} .$$

The remaining 4 bitangents will also be real if

$$0 < k < \frac{ab(a^2 - b^2)}{a^4 + b^4} .$$

This curve provides, therefore, a more symmetrical example of a quartic with 28 real bitangents. However, the equations of these bitangents fail to be expressible in simple algebraic terms. According to N. Mendelsohn this example has been given by Karl Strubecker in his work on plane algebraic curves which appeared in the Goeschen collection. The authors had no opportunity to consult it.

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REFERENCE

1. George Salmon, Higher plane curves, Dublin, 1879, p. 219.