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A parameter ASIP for the quadratic family

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Abstract. Consider the quadratic family $T_a(x) = ax(1-x)$ for $x \in [0, 1]$ and mixing Collet–Eckmann (CE) parameters $a \in (2, 4)$. For bounded φ, set $\tilde{\varphi}_a := \varphi - \int \varphi \, d\mu_a$, with μ_a the unique acim of T_a, and put $(\sigma_a(\varphi))^2 := \int \tilde{\varphi}_a^2 \, d\mu_a + 2 \sum_{i>0} \int \tilde{\varphi}_a(\tilde{\varphi}_a \circ T_a^i) \, d\mu_a$. For any mixing Misiurewicz parameter a_*, we find a positive measure set Ω_* of mixing CE parameters, containing a_* as a Lebesgue density point, such that for any Hölder φ with $\sigma_{a_*}(\varphi) \neq 0$, there exists $\epsilon_\varphi > 0$ such that, for normalized Lebesgue measure on $\Omega_* \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$, the functions $\xi_i(a) = \tilde{\varphi}_a(T_a^{i+1}(1/2))/\sigma_a(\varphi)$ satisfy an almost sure invariance principle (ASIP) for any error exponent $\gamma > 2/5$. (In particular, the Birkhoff sums satisfy this ASIP.) Our argument goes along the lines of Schnellmann's proof for piecewise expanding maps. We need to introduce a variant of Benedicks–Carleson parameter exclusion and to exploit fractional response and uniform exponential decay of correlations from Baladi *et al* [Whitney–Hölder continuity of the SRB measure for transversal families of smooth unimodal maps. *Invent. Math.* **201** (2015), 773–844].

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Contents

1. *Introduction*

1.1. *Background and motivation.* Let $(\Omega_*, m_*, \mathcal{F}_*)$ be a probability space. We say that a sequence of measurable functions $\xi_i : \Omega_* \to \mathbb{R}$, $i \geq 1$ satisfies the *almost sure invariance principle (ASIP) with error exponent* $\gamma < 1/2$ if there exist a probability space $(\Omega_W, m_W, \mathcal{F}_W)$ supporting a (centred) one-dimensional Brownian motion W and a sequence of measurable functions $\eta_i : \Omega_W \to \mathbb{R}$, $i \geq 1$, such that:

(i) the random variables $\{\xi_i\}_{i\geq 1}$ and $\{\eta_i\}_{i\geq 1}$ have the same distribution. (By definition of the distribution of discrete-time real-valued stochastic processes, this means that for any $n \geq 1$ and any $\{y_i \in \mathbb{R} \mid 1 \leq i \leq n\}$, the joint probability that $\xi_i \leq y_i$ for all $1 \leq i \leq n$ coincides with the joint probability that $\eta_i \leq y_i$ for all $1 \leq i \leq n$.)

(ii) Almost surely, $|W(n) - \sum_{i=1}^n \eta_i| = O(n^\gamma)$ as $n \to \infty$.

Since a Brownian motion at integer times coincides with a sum of independent identically distributed (i.i.d.) Gaussian variables, the above definition can also be formulated as an almost sure approximation, with error $o(n^\gamma)$, by a sum of i.i.d. Gaussian variables.

It is a classical result (see [**PS**]) that if the $\{\xi_i\}$ satisfies the ASIP, then it satisfies the law of the iterated logarithm (LIL), the central limit theorem (CLT) and the functional CLT: Letting $\sigma^2 > 0$ be the variance of the Brownian motion W (the expectation is zero by assumption) and denoting Lebesgue measure by m, the LIL says that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n \log\log n}} \sum_{i=1}^n \xi_i(a) = \sigma \quad \text{for } m_*\text{-almost every (a.e.) } a \in \Omega_*,$$

and the CLT (for the functional CLT, see [**DLS**, Lemma 5.1]) says that

$$\lim_{n\to\infty} m_*\left(\left\{ a \in \Omega_* \mid \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^n \xi_i(a) \leq y \right\}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-s^2/2}\, ds \quad \text{for all } y \in \mathbb{R}.$$

We consider $I = [0, 1]$ and the quadratic family

$$T_a(x) = ax(1-x), \quad x \in I, \; a \in (2, 4].$$

Denote by $c = 1/2$ the critical point of T_a and set $c_j(a) = T_a^j(c)$ for $j \geq 1$.

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If $\liminf_{n\to\infty} n^{-1} \log \partial_x(T_a^n)(T_a(c)) > 0$, we say that a is a Collet–Eckmann (CE) parameter. If a is CE, then T_a admits a unique absolutely continuous invariant probability measure (acim) $\mu_a = h_a\, dm$. Our goal is to find a positive Lebesgue measure set Ω_* of CE parameters with a Lebesgue density point $a_* \in \Omega_*$ such that for any Hölder continuous function $\varphi\colon I \to \mathbb{R}$ with $\sigma_{a_*}(\varphi) \neq 0$ (see equation (1.2)), there exists $\epsilon_\varphi > 0$ such that the ASIP holds for m_* the normalized Lebesgue measure on $\Omega_* \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$ and

$$\xi_j(a) := \varphi_a(c_{j+1}(a)), \quad j \geq 0, \quad a \in \Omega_* \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi],$$

where φ_a is a suitable normalization of φ (see equation (1.6)). We follow the approach of Schnellmann [**Sch**], who developed this program for transversal families of piecewise expanding maps T_a, for which Ω_* can be taken to be an interval.

Our main motivation is to extend to the quadratic family the method developed by de Lima and Smania [**DLS**] in the setting of piecewise expanding maps, to study linear and fractional response. (This method requires a functional central limit theorem, see [**DLS**, Lemma 5.1].)

We say that T_a is mixing if it is topologically mixing on

$$K(a) := [c_2(a), c_1(a)].$$

It will be convenient below to restrict to mixing maps T_a. Tiozzo recently showed [**Ti**, Corollary 3.15] (his result holds in fact for more general unimodal maps) that T_a is (strongly) mixing for its unique measure of maximal entropy (MME) if its topological entropy is greater than $\log(2)/2$. If a is a CE parameter with strongly mixing MME, then T_a is topologically mixing on $K(a)$ since the measure of maximal entropy has full support there. (Indeed, since T_a has no homtervals if $a \in CE$, it is conjugated to its piecewise linear model F_a by a homeomorphism which maps the MME of F_a to the MME of T_a, and the MME of F_a is absolutely continuous with a positive density on $[F_a^2(c), F_a(c)]$.) Since the topological entropy of T_4 is equal to $\log 2$, and the topological entropy of T_a is non-decreasing and continuous (in fact Hölder continuous [**Gu**]) in a, there exists $a_{\text{mix}} < 4$ such that for all $a \in (a_{\text{mix}}, 4] \cap CE$, the map T_a is topologically mixing on $K(a)$, and μ_a is strongly mixing, with support $K(a)$.

Melbourne and Nicol [**MN**] showed the ASIP in the phase space $x \in K(a)$, setting $\xi_i = T_a^i(x)$ for a fixed CE map T_a, using an induced uniformly expanding system (then [**PS**, §7] provides an ASIP which projects to the ASIP for the original CE map). However, to the best of our knowledge, the ASIP in the parameter a is still open.

In the parameter space, typicality (the law of large numbers, LLN) and the LIL are known: Avila and Moreira [**AM2**] showed that for Lebesgue almost every CE map T_a, the critical point is typical for its unique absolutely continuous invariant measure $\mu_a = h_a dm$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(c_i(a)) = \int_0^1 \varphi\, d\mu_a \quad \text{for all } \varphi \in C^0. \tag{1.1}$$

(Benedicks and Carleson established typicality in [**BC1**] for the Cantor set of CE parameters considered there.) For Hölder continuous $\varphi\colon I \to \mathbb{R}$ and a topological mixing CE parameter a, define $\sigma_a(\varphi) \geq 0$ by

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$$(\sigma_a(\varphi))^2 := \int_0^1 \left(\varphi - \int \varphi \, d\mu_a\right)^2 d\mu_a \tag{1.2}$$

$$+ 2 \sum_{i>0} \int_0^1 \left(\varphi - \int \varphi \, d\mu_a\right)\left(\varphi - \int \varphi \, d\mu_a\right) \circ T_a^i \, d\mu_a, \tag{1.3}$$

where the sum in equation (1.3) is finite because topological mixing (that is, the fact that the map is non-renormalizable) implies [**KN**] exponential mixing for the acim and Hölder continuous observables.

In a work in progress, Gao and Shen [**GS2**] show that, for Lebesgue-a.e. a in the set of mixing CE parameters, for every Hölder observable φ, either $\sigma_a(\varphi) = 0$ or the critical point c of T_a satisfies the LIL for φ, that is,

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n \log \log n}} \sum_{i=1}^{n} \left(\varphi(T_a^i(c)) - \int \varphi \, d\mu_a\right) = \sigma_a(\varphi).$$

1.2. *Statement of the ASIP (Theorem 1.1).* To state our main result, we need more notation and definitions. For $j \geq 0$ and $a \in (a_{\mathrm{mix}}, 4]$, set

$$x_j(a) = c_{j+1}(a) = T_a^{j+1}(c), \quad T_a'(x) = \partial_x T_a(x), \quad x_j'(a) = \partial_a x_j(a).$$

The family T_a is *transversal* at a_* if (see [**Ts1**]) there exists $C \geq 1$ such that

$$\frac{1}{C} \leq \left| \frac{x_j'(a_*)}{(T_{a_*}^j)'(c_1(a_*))} \right| \leq C \quad \text{for all } j \geq 1. \tag{1.4}$$

By [**Ts2**, Theorem 3], all CE parameters are transversal. We refer to [**Ts1**, (NV_t)] for an equivalent condition expressed in terms of the postcritical orbit.

The map T_a is (H_a, κ_a)*-polynomially recurrent,* for $\kappa_a \geq 1$ and $H_a \geq 1$, if

$$|x_{j-1}(a) - c| = |T_a^j(c) - c| \geq \frac{1}{j^{\kappa_a}} \quad \text{for all } j \geq H_a. \tag{1.5}$$

If $\inf_{j\geq 1} |T_a^j(c) - c| > 0$, then a is called a *Misiurewicz parameter.* Misiurewicz parameters are CE and thus transversal. Avila and Moreira [**AM1**] showed that, for any $\kappa_0 > 1$, the set of parameters a which are (H_a, κ_0)-polynomially recurrent for some H_a has full measure in the set of CE parameters. The set of Misiurewicz parameters a is uncountable (it has full Hausdorff dimension [**Za**, Theorem 1.4] but zero Lebesgue measure).

Finally, we introduce the normalization φ_a: let φ be bounded such that $\sigma_a(\varphi) \neq 0$ for a mixing CE parameter a. Then the function

$$\varphi_a(x) := \frac{1}{\sigma_a(\varphi)}\left(\varphi(x) - \int_0^1 \varphi \, d\mu_a\right) \tag{1.6}$$

is well defined and satisfies

$$\sigma_a(\varphi_a) = 1 \quad \text{and} \quad \int \varphi_a \, d\mu_a = 0. \tag{1.7}$$

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THEOREM 1.1. (Main Theorem: ASIP) *For any Misiurewicz parameter $a_* \in (a_{\rm mix}, 4)$, there exists a positive Lebesgue measure set Ω_* of mixing polynomially recurrent parameters, containing a_* as a Lebesgue density point, such that for any Hölder continuous function φ with $\sigma_{a_*}(\varphi) \neq 0$, there exists $\epsilon_\varphi > 0$ such that the functions*

$$\xi_n(a) := \varphi_a(x_n(a)) = \varphi_a(T_a^{n+1}(c)), \quad n \geq 1, \tag{1.8}$$

satisfy the ASIP for normalized Lebesgue measure m_* on $\Omega_* \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$ and all error exponents $\gamma > 2/5$. (The choice of ϵ_φ ensures in particular that $\sigma_a(\varphi) \neq 0$ if $\sigma_{a_*}(\varphi) \neq 0$.)

The value $a_* = 4$ is not covered by our arguments for technical reasons, since c_1 and c_2 then lie on the boundary of I. (For example, the Banach space of [**BBS**] requires that the function on level zero of the tower be supported in (0, 1), so this proof cannot cover the case $a = 4$.) It is possible (but a bit cumbersome) to handle (a one-sided neighbourhood of) this value by a change of coordinates as in [**Ts1**, Lemma 2.1].

We expect that the methods of this paper can be extended to the case when the 'root' a_* is mixing, but only Collet–Eckmann and polynomially recurrent (for large enough $\kappa_0 > 1$), instead of Misiurewicz. (For example, [**A**, Lemma 8.1] would replace [**DMS**, Lemma V.6.5] in the proof of Proposition 2.2.) We restrict here to Misiurewicz parameters a_* for the sake of simplicity. What is most desirable, in view of our original motivation to extend the analysis of [**DLS**], is to obtain a 'fatter' Cantor set Ω_* (as opposed to a fatter set of root points a_*). Indeed, this extension will probably require the ASIP on a set $\widetilde\Omega$ for which there exist $\beta > 1$ and a full measure subset $\widetilde\Omega_1 \subset \widetilde\Omega$ such that

$$\lim_{\epsilon \to 0} \frac{m([a-\epsilon, a+\epsilon] \setminus \widetilde\Omega)}{\epsilon^\beta} = 0 \quad \text{for all } a \in \widetilde\Omega_1. \tag{1.9}$$

(See [**BS2**, equation (5), Proposition F], note that [**BS2**, Lemma E] even uses $\beta < 2$ close to 2.) Equation (1.9) is known for all $\beta < 2$ for the sets $\widetilde\Omega_1 \subset \widetilde\Omega$ studied by Tsujii [**Ts1**]. (Beware that Tsujii's result cannot be used immediately. In particular, the main argument in the construction of the parameter set in Theorem 1 of 'Pre-threshold fractional susceptibility function: holomorphy and response formula', arXiv:2203.07942, is flawed.) For our Cantor set $\Omega_* \subset \Omega_{\rm BC}$, we expect that for any $\kappa > 1$, taking κ_0 large enough in Proposition 2.2, the factor ϵ^β in equation (1.9) must be replaced by $\epsilon|\log\epsilon|^{-\kappa}$ (see equation (2.20)), which does not seem good enough. Attaining the goal of our original motivation may thus require establishing the ASIP on a Cantor set having larger density, and thus weakening the polynomial lower recurrence in the construction (see comments in the next paragraph). We view this as the most desirable improvement of our main theorem.

To clarify the role of Ω_*, it is useful to compare Schnellmann's proof with ours. In [**Sch**], Schnellmann studies suitable transversal one-parameter families of piecewise expanding interval maps and obtains a parameter ASIP on a set Ω_* which is just an interval $[0, \epsilon^\varphi]$ of parameters. Indeed, existence of an exponentially mixing acim enjoying fractional response (with uniform bounds) holds in an entire interval $[0, \epsilon^\varphi]$ in his setting [**Sch**, Proposition 4.3, Lemma 4.5]. So, $[0, \epsilon^\varphi]$ is the baseline parameter space for his analysis. Some parameters in this baseline cause difficulties ('exceptionally small sets'), but Schnellmann can get away with just *ignoring* them (taking advantage of the fact

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that their total measure is controlled [**Sch**, part (III), Theorem 3.2, Lemma 4.1, proof of Lemmas 6.1 and 6.2]) instead of *excluding* them from the baseline. Our situation is different, since we need to *exclude* parameters which do not have an acim or for which exponential mixing or fractional response (with uniform bounds) does not hold. Our baseline set is a Cantor set, and the best we can do is to make it as fat as possible.

The polynomial recurrence in equation (1.5) in our parameter exclusion (Proposition 2.2), which causes the 'thinness' of Ω_*, is needed to apply the results of [**BBS**] in §§2.4 and 2.5 (Propositions 2.5 and 2.6 on uniform decorrelation and fractional response, and its consequence, Lemma 2.8). (See also (4.7) and (4.22), which may cause a different error exponent.) Due to this, we already *exclude* the parameters which could have exceptionally small image and we do not need to *ignore* them (Lemma 2.3, compare also [**Sch**, proof of Lemma 6.1] with equation (4.13) below). In addition, we get an easy proof of the local distortion estimate in equation (2.31). If the required consequences of [**BBS**] could be extended to sets of parameters which enjoy only exponential recurrence bounds, then we could use the (fatter) Benedicks–Carleson Cantor set $\Omega_{\mathrm{BC}}^{\varphi}$ as a baseline instead of Ω_* (if necessary, the Benedicks–Carleson technique could be replaced by ideas from Tsujii [**Ts1**], Avila and Moreira [**AM1**] or Gao and Shen [**GS1**]). Next, one could try to *ignore* the parameters with exceptionally small images in Lemma 2.3. For equation (2.31), our proof is inspired from that of [**DMS**, Theorem V.6.2]. This is suboptimal but enough for our purposes. Adapting instead [**DMS**, Lemma V.6.4] could enhance equation (2.31).

We also note for the record here that the characteristic function $1_{\widetilde{\Omega}}$ of a fat enough Cantor set $\widetilde{\Omega}$ belongs to a Sobolev space $H_q^s(I)$ with $s > 0$ (see [**HM**, Propositions 4.9 and 4.10]). Thus, working with a Cantor set of larger density may simplify some of our arguments (in the proof of Proposition 3.2, for example).

Finally, the results of this paper probably extend to more general families of smooth unimodal maps. In the present 'proof of concept' work, we choose to restrict to the quadratic family.

1.3. *Structure of the text.* Schnellmann pointed out [**Sch**, p. 370] that the 'Markov partitions' given by the intervals in the celebrated Benedicks–Carleson [**BC1**, **BC2**] parameter exclusion construction would be the key to extend his result to non-uniformly expanding interval maps.

Our paper carries out this plan and is organized as follows. After recalling basic facts in §2.1, we adapt in §2.2 the Benedicks–Carleson procedure to construct, in a neighbourhood of a topologically mixing Misiurewicz point a_*, a sequence $\Omega_n \subset \Omega_{n-1}$ where Ω_n is a finite union of intervals in $\mathcal{P}_n$. At each step, some intervals in $\mathcal{P}_n$ are partitioned and the intervals which do not satisfy a time-n *polynomial* recurrence assumption are excluded. The remaining Cantor set $\Omega_*(a_*) = \bigcap_n \Omega_n$ is a positive Lebesgue measure set of parameters satisfying the Collet–Eckmann property, polynomial returns and distortion control, with uniform constants. (Our distortion bound in equation (2.31) is new.) In addition, the construction ensures that there are no 'exceptionally small' sets (Lemma 2.3). Applying results from [**BBS**], this ensures uniform exponential decay of correlations (Proposition 2.5) and fractional response (Proposition 2.6), from which we obtain regularity of the map $a \mapsto \sigma_a$ (Lemma 2.8).

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Sections 3 and 4 contain the proof of the ASIP along the lines of [**Sch**]: first approximate the Birkhoff sum by a sum of blocks of polynomial size (§§4.1 and 4.2), then (§4.3) approximate these blocks by a martingale difference sequence Y_j and apply Skorokhod's representation theorem linking a martingale with a Brownian motion (see [**PS**, §3]). The usual application of the approach of [**PS**, Ch. 7] in dynamics uses a strong independence condition (see [**PS**, 7.1.2]) which we do not have (the ξ_i terms are not iterations of a fixed map and there is no underlying invariant measure). (See the example (see [**Ka**, p. 646]) discussed in [**Sch**]. Also, as pointed out in [**Sch**], it is not clear how to apply the spectral techniques of [**Go**] in our setting.) We replace this strong independence condition by uniformity of constants in the exponential decay of correlations (given by [**BBS**]) which we translate into properties for the ξ_i by switching from parameter to phase space (see Proposition 3.2), giving estimates similar to those in [**PS**, §3].

For $\varpi \in (0, 1)$, we shall denote by C^{ϖ} the set of ϖ-Hölder continuous functions $\varphi\colon I \to \mathbb{R}$, putting $\|\varphi\|_{\varpi} = \sup |\varphi| + H_{\varpi}(\varphi)$, with $H_{\varpi}(\varphi)$ the smallest H_{ϖ} such that $|\varphi(x) - \varphi(y)| \le H_{\varpi}|x - y|^{\varpi}$ for all x, y in I. The letter C is used throughout to represent a (large) uniform constant, which may vary from place to place.

2. *Bounds for the quadratic family. The Cantor set $\Omega_*(a_*)$*

2.1. *Basic properties.*

Clearly, the maps

$$a \mapsto T_a'(x) = \partial_x T_a(x) = a(1-2x), \quad x \mapsto \partial_a T_a(x) = x(1-x)$$

are Lipschitz continuous uniformly in $x \in I$ and $a \in (a_{\mathrm{mix}}, 4]$, and, in addition,

$$\sup_{x \in I} |T_a'(x)| \le \Lambda := 4 \quad \text{for all } a \in (a_{\mathrm{mix}}, 4]. \tag{2.1}$$

Each T_a has two monotonicity intervals, with partition points 0, $c = 1/2$ and 1. The following easy lemma replaces [**Sch**], (30)]. (We do not need as in [**Sch**, (30)] that x has the same combinatorics under T_{a_1} and T_{a_2} up to the $(n-1)$th iteration. We thus do not need any analogue of [**Sch**, Sublemma 5.4].)

LEMMA 2.1. *There exists $C < \infty$ such that, for any $a_1, a_2 \in (2, 4]$, we have*

$$|T_{a_1}^n(x) - T_{a_2}^n(x)| \le C\Lambda^n |a_1 - a_2| \quad \textit{for all } x \in I, \textit{ for all } n \ge 1. \tag{2.2}$$

Proof. Clearly, $|T_{a_1}(x) - T_{a_2}(x)| \le |a_1 - a_2|$. For $n \ge 2$, using the definition in equation (2.1) of Λ and setting $C = \sum_{j=0}^{\infty} \Lambda^{-j}$, we get

$$\begin{aligned}
|T_{a_1}^n(x) - T_{a_2}^n(x)| &\le |T_{a_1}(T_{a_1}^{n-1}(x)) - T_{a_2}(T_{a_1}^{n-1}(x))| + |T_{a_2}(T_{a_1}^{n-1}(x)) - T_{a_2}(T_{a_2}^{n-1}(x))| \\
&\le |a_1 - a_2| + \Lambda |T_{a_1}^{n-1}(x) - T_{a_2}^{n-1}(x)| \\
&\le |a_1 - a_2|(1 + \Lambda) + \Lambda^2 |T_{a_1}^{n-2}(x) - T_{a_2}^{n-2}(x)| \le \cdots \\
&\le |a_1 - a_2| \sum_{j=0}^{n-1} \Lambda^j \le C\Lambda^n |a_1 - a_2|.
\end{aligned}$$

□

2.2. *A polynomial Benedicks–Carleson construction ($\Omega_*(a_*)$, $\mathcal{P}_n$).*

For each $j \ge 0$, the function $x_j(a) = T_a^{j+1}(c)$ is a map from the parameter space $(a_{\mathrm{mix}}, 4]$ to the phase space

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$I = [0, 1]$, with $x_j(a) \in K(a)$ for all a. The transversality condition in equation (1.4) says that the derivatives of x_j and T_a^j are comparable at a_*, so that statistical properties (such as the ASIP) can be transferred from the maps $x \mapsto T_a^j(x)$ to the maps $a \mapsto x_j(a)$. To make this precise, we next construct a sequence of partitions in the parameter space. Our starting point is the following variant of the Benedicks–Carleson Cantor set $\Omega_{\mathrm{BC}} = \Omega_{\mathrm{BC}}(a_*)$ (see [**BC1**, **BC2**]) associated to a Misiurewicz parameter a_* (which is automatically transversal). (See equation (2.17) for the construction of Ω_{BC}.)

PROPOSITION 2.2. (The Cantor set $\Omega_* = \Omega_*(a_*, \kappa_0)$) *Let $a_* \in (a_{\mathrm{mix}}, 4]$ be a Misiurewicz parameter. There exist $\lambda_{\mathrm{CE}} \in (1, \Lambda)$ and $C_0 \in (0, 1)$ such that, for any $d_1 \in (0, C_0 \log \lambda_{\mathrm{CE}}/4)$ and $d_0 > 0$, there exists $\epsilon > 0$ such that for any $\kappa_0 > 1/d_1$, for all large enough $N_0 \geq 1$, there exists a sequence $\mathcal{P}_j$ of finite sets of pairwise disjoint subintervals of*

$$\omega_0 := [a_* - \epsilon, a_* + \epsilon] \cap (a_{\mathrm{mix}}, 4]$$

such that $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_{N_0}$ and, setting

$$\Omega_* = \Omega_*(a_*, \kappa_0) := \bigcap_{j \geq N_0} \Omega_j \quad \text{with } \Omega_j := \bigcup_{\omega \in \mathcal{P}_j} \omega,$$

we have $\Omega_{j+1} \subset \Omega_j$ for $j \geq N_0$, and (the first bound of equation (2.4) implies that $a \mapsto x_j(a) = T_a^{j+1}(c)$ is monotone on $\omega \in \mathcal{P}_j$)

$$\text{for all } j \geq 1, \text{ for all } \omega \in \mathcal{P}_j, \text{ for all } 0 \leq \ell < j, \text{ there exists } \omega' \in \mathcal{P}_\ell \text{ such that } \omega \subset \omega', \tag{2.3}$$

$$|x_j'(a)| > 0, |T_a^{j+1}(c) - c| > 0 \text{ for all } a \in \omega, \text{ for all } \omega \in \mathcal{P}_j, \text{ for all } j \geq 0, \tag{2.4}$$

*and there exists (note that equation (2.6) replaces [**Sch**, Lemma 2.4]) $C < \infty$ such that for all $j \geq N_0$ and $\omega \in \mathcal{P}_j$,*

$$|(T_a^n)'(T_a(c))| \geq \lambda_{\mathrm{CE}}^n \quad \text{for all } N_0 \leq n \leq j, \text{ for all } a \in \omega, \tag{2.5}$$

$$\frac{1}{C} \leq \left| \frac{x_n'(a)}{(T_a^n)'(T_a(c))} \right| \leq C \quad \text{for all } N_0 \leq n \leq j, \text{ for all } a \in \omega, \tag{2.6}$$

$$|\tilde{\omega}| \leq C \lambda_{\mathrm{CE}}^{-n} |x_n(\tilde{\omega})| \quad \text{for all } N_0 \leq n \leq j, \text{ for all } \tilde{\omega} \subset \omega, \tag{2.7}$$

and, moreover,

$$|T_a^{n+1}(c) - c| > n^{-\kappa_0} \quad \text{for all } N_0 \leq n \leq j, \text{ for all } a \in \omega. \tag{2.8}$$

Finally, we have that $a_* \in \Omega_*$ is a Lebesgue density point of Ω_*, with

$$|\Omega_*| \geq (1 - d_0 \cdot e_j)|\Omega_{j-1}| \quad \text{for all } j \geq N_0, \text{ where } e_j := \sum_{n=j}^{\infty} n^{-d_1 \cdot \kappa_0}, \tag{2.9}$$

and we have the more precise (semi-local) bound

$$\sum_{\substack{\omega \in \mathcal{P}_\ell \\ \omega \subset \omega'}} |\omega \setminus (\omega \cap \Omega_*)| \leq d_0 \cdot e_{\ell - \ell'} |\omega'| \quad \text{for all } \omega' \in \mathcal{P}_{\ell'}, \text{ for all } \ell \geq \ell' \geq N_0. \tag{2.10}$$

See Lemma 2.3 below regarding the absence of exceptionally small sets and §2.3 for a Hölder distortion property refining equation (2.16).

Clearly, equation (2.8) means that any $a \in \Omega_*$ is (N_0, κ_0)-polynomially recurrent.

The bound in equation (2.9) implies that the Cantor set Ω_* has positive Lebesgue measure as soon as $d_1 \cdot \kappa_0 > 1$ (and N_0 is large enough). Proposition 2.2 holds for such κ_0, but we will need the stronger condition $d_1 \cdot \kappa_0 \geq 11/3$ to use equation (2.9) in the proof of Proposition 3.2 (and $d_1 \cdot \kappa_0 > 9/5$ for Lemma 4.2).

The local bound in equation (2.10) is used in the proof of Lemma 4.1.

Proof of Proposition 2.2. Let $r_0 \geq 2$ be a large integer (to be chosen later, with $\epsilon \to 0$ as r_0 increases). For $r \geq r_0$, set $I_r = I_r^- \cup I_r^+$, where

$$I_r^+ = [c + e^{-r-1}, c + e^{-r}), \quad I_r^- = (c - e^{-r}, c - e^{-r-1}], \quad U_r = (c - e^{-r}, c + e^{-r}),$$

and cover each $I_r^\pm$ by r^2 pairwise disjoint intervals $I_{r,\ell}^\pm$ of equal size, each $I_{r,\ell}^\pm$ containing its boundary point closest to c. Let $\beta_{\mathrm{BC}} > \alpha_{\mathrm{BC}} > 0$, where

$$e^{-n\alpha_{\mathrm{BC}}} \leq n^{-\kappa_0} \quad \text{for all } n \geq N_0,$$

for N_0 a large integer to be chosen later. (The constant α_{BC} is usually called α, but we shall need the letter α for another purpose in equation (2.30).)

For $a \in (a_{\mathrm{mix}}, 4]$, $\nu \geq 1$, and $r \geq r_0$ such that $T_a^\nu(c) \in I_r$, the *binding time* $p(a) = p(r, a, \nu)$ of U_r with $T_a^\nu(c)$ is the maximal $p \in \mathbb{Z}_+ \cup \{\infty\}$ such that

$$|T_a^j(x) - T_a^{j+\nu}(c)| \leq e^{-j\beta_{\mathrm{BC}}} \quad \text{for all } 1 \leq j \leq p, \text{ for all } x \in U_r.$$

The *first free return time* $\nu_1(a)$ of $a \in (a_{\mathrm{mix}}, 4]$ is the smallest integer $j \geq 1$ for which $T_a^j(c) \in U_{r_0}$. For an interval $\omega \subset (a_{\mathrm{mix}}, 4]$, the *first free return time* $\nu_1(\omega)$ is the smallest integer $j \geq 1$ for which there exists $a \in \omega$ with $T_a^j(c) \in U_{r_0}$. If there exists $r = r(\omega)$ such that $x_{\nu_1 - 1}(\omega) \subset I_r$ (recall that $T_a^{\nu_1}(c) = x_{\nu_1 - 1}(a)$), we define the *first binding time* of ω by $p_1(\omega) = \min_{a \in \omega} p(r, a, \nu_1(\omega))$. For $i \geq 2$, define inductively the ith *free return time* of (suitable) ω to be the largest integer $\nu_i(\omega) > \nu_{i-1}(\omega) + p_{i-1}(\omega) + 1$ such that

$$T_a^j(c) \cap U_{r_0} = \emptyset \quad \text{for all } \nu_{i-1}(\omega) + p_{i-1}(\omega) + 1 \leq j < \nu_i(\omega), \text{ for all } a \in \omega,$$

and, for $r(\omega)$ such that $x_{\nu_{i-1}-1}(\omega) \subset I_r$, set the ith *binding time* of ω to be

$$p_i(\omega) = \min_{a \in \omega} p(r, a, \nu_{i-1}(\omega)).$$

(Similarly, define inductively for $i \geq 2$ and a such that $T_a^{\nu_{i-1}}(c) \in I_r$, the pointwise binding times $p_i(a)$ and free returns $\nu_i(a)$.) The iterates between $\nu_i(\omega)$ and $\nu_i(\omega) + p_i(\omega)$ form the ith bound period of ω, those between $\nu_{i-1}(\omega) + p_{i-1}(\omega) + 1$ and $\nu_i(\omega) - 1$ form its ith free period. Finally, if there exist $a \in \omega$ and $j \geq \nu_1(\omega)$ such that $T_a^j(c) \in U_{r_0}$, we say that j is a *return time* of ω. (Return times either are free returns $\nu_i(\omega)$ or they occur during the bound period.)

Note that for any fixed ϵ, setting $\omega_0 = [a_* - \epsilon, a_* + \epsilon]$, there exists N_ϵ such that $x_{N_\epsilon}(\omega_0)$ contains a neighbourhood of c (indeed, by transversality, for any $a \in \omega_0 \setminus \{a_*\}$, there exists $N(a)$ such that $T_{a_*}^{N(a)+1}(c)$ and $T_a^{N(a)+1}(c)$ lie on different sides of c). In particular, $\nu_1(\omega_0) < \infty$. Similarly, all $\nu_i(\omega_0)$ and $p_i(\omega_0)$ are finite.

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Let W_{a_*} be a neighbourhood of c disjoint from $\{T^n_{a_*}(c) \mid n \geq 1\}$. From now on, we only consider r_0 large enough such that $\overline{U}_{r_0-1} \subset W_{a_*}$. Set $W^+_{a_*,r_0} = W_{a_*} \cap [c + e^{-r_0}, 1]$ and $W^-_{a_*,r_0} = W_{a_*} \cap [0, c - e^{-r_0}]$. We claim that for any fixed large r_0, we have that $x_{\nu_1(\omega_0)-1}(\omega_0)$ contains $W^+_{a_*,r_0}$ or $W^-_{a_*,r_0}$ for all small enough ϵ. (This fact is used in [**DMS**, Lemma V.6.8]. There, W_{a_*} is mistakenly mentioned instead of $W^{\pm}_{a_*,r_0}$. Our r_0 is denoted by Δ and our $x_n(a)$ is denoted $\xi_{n+1}(a)$ in [**DMS**].) Indeed, $x_{\nu_1(\omega_0)-1}(\omega_0)$ is an interval intersecting U_{r_0}, and $x_{\nu_1(\omega_0)-1}(\omega_0)$ contains $T^{\nu_1(\omega)}_{a_*}(c) \notin W_{a_*}$.

For small $\epsilon > 0$ (to be chosen depending on r_0), the sequence $\mathcal{P}_j$ can now be defined inductively: start with the single interval $\mathcal{P}_0 = \mathcal{P}_1 = \cdots = \mathcal{P}_{N_0} = \{\omega_0\}$ for ϵ small enough such that $\nu_1(\omega_0) \geq N_0$ (note that $\nu_1(\omega_0)$ increases if r_0 increases or ϵ decreases). (We refer throughout to [**DMS**, §V.6]. The original ideas and key estimates appeared previously in the work of Benedicks and Carleson [**BC1**, **BC2**]. The original construction in [**BC1**, **BC2**] is for $a_* = 4$, see [**Mo**] for a self-contained account. It extends to Misiurewicz parameters: for CE parameters, the condition in [**DMS**, Theorem 6.1] is equivalent to equation (1.4), taking large enough k in the last line of [**DMS**, p. 406, Step 2].)

For $j > N_0$, each $\omega \in \mathcal{P}_{j-1}$ is partitioned into finitely many (possibly just one) intervals, at least one of which will be included into an auxiliary partition $\mathcal{P}'_j$, as follows.

If j is not a free return time of ω, we include ω in $\mathcal{P}'_j$. (That is, either j is not a return, or it is a return within the bound period.) If j is a free return time of ω but $x_{j-1}(\omega)$ does not contain an interval $I^{\pm}_{r,\ell}$ (we call this an *inessential (free) return*), we also include ω in $\mathcal{P}'_j$.

Otherwise, j is a free return time of ω such that $x_{j-1}(\omega)$ contains at least one interval $I^{\pm}_{r,\ell}$. We call this an *essential (free) return.* In that case, we decompose $x_{j-1}(\omega)$ into the following intervals:

$$x_{j-1}(\omega) \setminus U_{r_0}, \quad \{x_{j-1}(\omega) \cap I^{\pm}_{r,\ell} \mid r \geq r_0,\ 1 \leq \ell \leq r^2\}.$$

If $x_{j-1}(\omega) \setminus U_{r_0} \neq \emptyset$, but any of the (at most two) connected components of $x_{j-1}(\omega) \setminus U_{r_0}$ has size less than $e^{-r_0}(1 - 1/e)r_0^{-2} = |I^{\pm}_{r_0,\ell}|$, we join it to its neighbour $x_{j-1}(\omega) \cap I^{\pm}_{r_0,\ell} = I^{\pm}_{r_0,\ell}$. If a connected component of $x_{j-1}(\omega) \setminus U_{r_0}$ has size larger than $S := \sqrt{|U_{r_0}|}$, we subdivide it into pairwise disjoint intervals of lengths between $S/2$ and S. If $x_{j-1}(\omega) \cap I^{\pm}_{r,\ell} \neq \emptyset$, but $I^{\pm}_{r,\ell}$ is not contained in $x_{j-1}(\omega)$ (this can happen for at most two intervals $I^{\pm}_{r,\ell}$), we join $x_{j-1}(\omega) \cap I^{\pm}_{r,\ell}$ to its neighbour $x_{j-1}(\omega) \cap I^{\pm}_{r',\ell'} = I^{\pm}_{r',\ell'}$. Denote by $\{\hat{\omega}_{r,\ell} \mid r \geq r_0 - 1\}$ the partition of $x_{j-1}(\omega)$ thus obtained, where the index (r, ℓ) refers to the 'host' interval $I_{r,\ell}$ contained in $\hat{\omega}_{r,\ell}$ if $r \geq r_0$, while $\hat{\omega}_{r_0-1,\ell} \subset I \setminus U_{r_0}$. Then we discard all intervals $\hat{\omega}_{r,\ell}$ for which

$$e^r \geq (j-1)^{\kappa_0}. \tag{2.11}$$

Mapping the remaining intervals via the inverse of the diffeomorphism (see [**DMS**, Proposition V.6.2]) x_{j-1} gives finitely many subintervals of ω which we include in $\mathcal{P}'_j$. Further intervals $\hat{\omega}_{r,\ell}$ need to be discarded from $\mathcal{P}'_j$, using a requirement denoted (FA_j) or (FA'_j) in [**DMS**, §V.6], [**Mo**], which finally defines $\mathcal{P}_j$. For further use, we denote these remaining intervals by

$$\omega_{r,\ell} = x^{-1}_{j-1}(\hat{\omega}_{r,\ell}). \tag{2.12}$$

It is well known (the original construction in [BC1, BC2] is for $a_* = 4$, see [Mo] for a self-contained account. It extends to Misiurewicz parameters: for CE parameters, the condition in [DMS, Theorem 6.1] is equivalent to equation (1.4), taking large enough k in the last line of [DMS, p. 406, Step 2]) [BC1, BC2, DMS] that, if we replace the condition in equation (2.11) (used to discard intervals) by the exponential condition

$$\omega \cap I^{\pm}_{r,\ell} \neq \emptyset \quad \text{and} \quad e^r \geq e^{\alpha_{\rm BC}(j-1)}, \tag{2.13}$$

to construct sequences $\mathcal{P}^{\prime,{\rm BC}}_j$ and $\mathcal{P}^{\rm BC}_j$, then there exists $\lambda_{\rm CE} > 1$ (called e^{γ} in [DMS, (V.6.4), Theorem V.6.2]) such that for any small enough $\beta_{\rm BC} > \alpha_{\rm BC} > 0$, there exist N_0' such that if r_0 is large enough and $\epsilon > 0$ small enough, then the $\mathcal{P}^{\rm BC}_j$ satisfy equations (2.3)–(2.7) (equation (2.5) is called (EX_j) in [DMS, §V.6]) for some $C < \infty$, and the following condition (noted (BA_j) in the literature) holds for all $j \geq N_0'$:

$$2|T_a^{n+1}(c) - c| > e^{-n\alpha_{\rm BC}} \quad \text{for all } N_0' \leq n \leq j, \text{ for all } a \in \omega \text{ for all } \omega \in \mathcal{P}^{\prime,{\rm BC}}_j. \tag{2.14}$$

(Strictly speaking, the condition (BA_j) does not involve the factor 2, and a condition (BA_j') requiring that for each $\omega \in \mathcal{P}^{\prime{\rm BC}}_j$, there exists $a \in \omega$ with $|T_a^{n+1}(c) - c| > e^{-n\alpha_{\rm BC}}$ for $N_0' \leq n \leq j$ is used in some lemmas. See [DMS, §V.6, Step 5].) Since $\lambda_{\rm CE}$ does not depend on $\alpha_{\rm BC}$, N_0 or N_0', we may assume that

$$14\alpha_{\rm BC} < \log \lambda_{\rm CE}$$

and we may replace N_0 by $\max\{N_0, N_0'\}$.

In particular [DMS, Proposition V.6.1, Lemma V.6.1(b), (c)] give $\gamma_0 > 0$, $\lambda_{\rm CE} = e^{\gamma} \in (1, e^{\gamma_0})$ and $C_0 > 0$ (independent of r_0 and ϵ) such that if $a \in \Omega_n$ and $\nu_{\ell+1}(a) \leq n$, writing p_ℓ, ν_ℓ for $p_\ell(a)$, $\nu_\ell(a)$, we have

$$\begin{cases} |(T_a^{\nu_{\ell+1}-(\nu_\ell+p_\ell+1)})'(T_a^{\nu_\ell+p_\ell+1}(c))| \geq C_0 e^{\gamma_0(\nu_{\ell+1}-(\nu_\ell+p_\ell))}, \\ |(T_a^{p_\ell+1})'(T_a^{\nu_\ell}(c))| \geq \lambda_{\rm CE}^{p_\ell/4}. \end{cases} \tag{2.15}$$

To establish equation (2.5) (the bound below will also be used for equation (2.34)), one takes r_0 such that

$$r_0^2 C_0^2 \log \lambda_{\rm CE} > |\log C_0|.$$

The key distortion bound [DMS, Proposition V.6.3] gives C such that

$$\left| \frac{x_j'(a_1)}{x_j'(a_2)} \right| \leq C \quad \text{for all } N_0 \leq j \leq n, \text{ for all } a_1, a_2 \in \omega, \tag{2.16}$$

whenever $n+1$ is a free return time of $\omega \in \mathcal{P}_n$ with $x_{n+1}(\omega) \subset U_{r_0/2}$. The bound in equation (2.6) follows from [DMS, Proposition V.6.2 and Theorem V.6.2].

Let $\Omega_j' := \bigcup_{\omega \in \mathcal{P}_j'} \omega$, recall Ω_j, and define $\Omega_j^{\rm BC}$ and $\Omega_j^{\prime,{\rm BC}}$ accordingly, setting

$$\Omega_{\rm BC} = \Omega_{\rm BC}(a_*, \alpha_{\rm BC}) = \bigcap_j \Omega_j^{\rm BC} \quad \text{so that } \Omega_*(a_*) \subset \Omega_{\rm BC}(a_*). \tag{2.17}$$

It is easy to check that equation (2.11) implies equation (2.8) (for returns during a bound period, use that $\ell^{-\kappa_0} - e^{-\ell\beta_{\rm BC}} \geq j^{-\kappa_0}$ for all $N_0 \leq \ell \leq j-1$, up to increasing N_0 again).

Our choice of N_0 implies $\Omega_j \subset \Omega_j^{\mathrm{BC}}$. Also, equation (2.6) with equation (2.4) imply that all points in Ω_* are transversal. Since equation (2.7) is an immediate consequence of equations (2.5) and (2.6), it only remains to establish that a_* is a Lebesgue density point in Ω_* (clearly, $a_* \in \Omega_*$) and that equations (2.9) and (2.10) hold.

To show that a_* is a Lebesgue density point of Ω_*, we may follow [**DMS**, Step 7 of the proof of Theorem V.6.1], replacing Ce^{-iC_0} there by $C' i^{-\kappa_0}$. (We mention a typo there: although the constant $C = C(\epsilon)$ in the unnumbered equation on [**DMS**, p. 433] tends to zero as $\epsilon = |\omega_0|/2 \to 0$, the constant C_0 is (fortunately) uniformly bounded away from zero. See the proof of [**DMS**, Lemma V.6.5].)

We next establish equations (2.9) and (2.10). For suitably small $\bar\eta > 0$, and for $J_0 \geq 1$ such that $\prod_{j=J_0}^{\infty}(1 - e^{-\bar\eta j}) > 3/4$ (since $\bar\eta$ is independent of ϵ, r_0, N_0, we may take $N_0 \geq J_0$), the parameter exclusion rule in equation (2.13) gives $d_0' > 0$ (tending to zero with ϵ) such that ([**DMS**, §V.6, Step 7], [**Mo**, §6])

$$\begin{cases} |\omega \cap \Omega_j'^{,\mathrm{BC}}| \geq (1 - d_0' e^{-j\bar\eta})|\omega| & \text{for all } \omega \in \mathcal{P}_{j-1}^{\mathrm{BC}}, \text{ for all } j \geq J_0, \\ |\Omega_j^{\mathrm{BC}}| \geq |\Omega_j'^{,\mathrm{BC}}| - e^{-j\bar\eta}|\omega_0| & \text{for all } j \geq J_0. \end{cases} \tag{2.18}$$

The above implies $|\Omega_j^{\mathrm{BC}}| \geq (1 - d_0' e^{-\bar\eta j})|\Omega_{j-1}^{\mathrm{BC}}| - e^{-\bar\eta j}|\omega_0|$ for $j \geq J_0$, and, exploiting that $|\omega_0| = |\Omega_n^{\mathrm{BC}}|$ for all $n \leq N_0$ with $N_0 \geq J_0$, and using the definition of J_0, also that

$$|\Omega_j^{\mathrm{BC}}| \geq \left(\prod_{n=J_0}^{j} (1 - d_0' e^{-\bar\eta n}) - \sum_{n=J_0}^{j} e^{-\bar\eta n} \right) |\omega_0| \geq \frac{1}{2}|\omega_0| \quad \text{for all } j \geq J_0.$$

(By taking larger J_0, that is, smaller ϵ, we could replace $1/2$ by a number close to 1.) Thus, applying inductively

$$|\Omega_j^{\mathrm{BC}}| \geq ((1 - d_0' e^{-\bar\eta j}) - 2e^{-\bar\eta j})|\Omega_{j-1}^{\mathrm{BC}}| \quad \text{for all } j \geq J_0,$$

we find $\tilde\eta > 0$ such that for any $j \geq J_0$,

$$|\Omega_{\mathrm{BC}}| \geq \prod_{n=j}^{\infty} (1 - (d_0' + 2)e^{-\bar\eta n})|\Omega_{j-1}^{\mathrm{BC}}| \geq (1 - (\tilde d_0 + 2)e^{-\tilde\eta j})|\Omega_{j-1}^{\mathrm{BC}}|. \tag{2.19}$$

Recall that we fixed $d_1 \in (0, C_0/4 \log \lambda_{\mathrm{CE}})$ (independently of κ_0). Let J_1 be such that $\prod_{j=J_1}^{\infty}(1 - e^{-\bar\eta j} - j^{-2}) > 3/4$ and return to the sets Ω_j, Ω_j' constructed using the (polynomial) exclusion rule in equation (2.11) for $\kappa_0 > 1/d_1$. We claim that for any $d_0 > 0$, if ϵ is small enough,

$$\begin{cases} |\omega \cap \Omega_j'| \geq (1 - d_0 \cdot j^{-d_1\kappa_0})|\omega| & \text{for all } \omega \in \mathcal{P}_{j-1}, \text{ for all } j \geq J_1, \\ |\Omega_j| \geq |\Omega_j'| - e^{-j\bar\eta}|\omega_0| & \text{for all } j \geq J_1. \end{cases} \tag{2.20}$$

Before establishing this claim, we note that, *mutatis mutandis*, equation (2.20) combined with the arguments leading to equation (2.19) implies equation (2.9), while the more precise claim in equation (2.10) follows from the refinement of equation (2.20) coming from the second statement of [**DMS**, Lemma V.6.9] (see the use of [**Mo**, Lemma 6.3] in [**Mo**, Lemma 6.4 and Proposition 6.5]).

To show equation (2.20), we proceed in three steps, performing the necessary changes in the proof in [**DMS**, §V.6]. Recall equation (2.12).

First, up to taking larger N_0, the conclusion of [**DMS**, Lemma V.6.5] (which deals with (BA'_j) for $\omega \in \mathcal{P}_{j-1}$ satisfying (BA'_{j-1}) and (EX_{j-1}) and having a return at time j), if we replace the exponential rate (BA'_{j-1}) there by our polynomial rate in equation (2.8), becomes

$$\frac{|\omega \setminus \bigcup_{r\geq \kappa_0 \log j} \omega_{r,\ell}|}{|\omega|} \geq 1 - Cj^{-d_1\kappa_0} \quad \text{for all } j \geq N_0. \tag{2.21}$$

To show this first claim, use that the constant $C_0 \in (0, 1)$ (introduced above) is independent of κ_0 (because λ_{CE} does not depend on κ_0) and that [**DMS**, Lemma V.6.1] gives that the bound period p of a free return $\nu < j$ with

$$I_{r',\ell'} \subset x_\nu(\omega) \quad \text{for } r_0 \leq r' \leq \kappa_0 \log \nu \leq \kappa_0 \log j, \tag{2.22}$$

satisfies $p \geq C_0 r'$. Then, up to taking larger N_0, we can replace [**DMS**, V.(6.20)] in the proof of [**DMS**, Lemma V.6.5] by

$$|x_j(\omega)| \geq \lambda_{\mathrm{CE}}^{p/4} \frac{e^{-r'}}{(r')^2} \geq \frac{e^{(-1+d_1)r'}}{(r')^2} \geq \frac{1}{j^{\kappa_0(1-d_1)}}, \quad j \geq N_0, \tag{2.23}$$

where we used $d_1 \leq (C_0/4) \log \lambda_{\mathrm{CE}}$ in the second inequality. We can thus replace the chain of inequalities after [**DMS**, V.(6.20)] (using the distortion bound in equation (2.16) for $\tilde\omega \subset \omega$ the largest interval with $x_n(\tilde\omega) \subset U_{r_0/2}$, taking ϵ small enough and N_0 large enough such that equation (2.23) also holds for $\tilde\omega$) by

$$\frac{|\bigcup_{r\geq \kappa_0 \log j} \omega_{r,\ell}|}{|\omega|} \leq \frac{|\bigcup_{r\geq \kappa_0 \log j} \omega_{r,\ell}|}{|\tilde\omega|} \leq C\frac{1}{j^{\kappa_0}} \frac{1}{|x_j(\tilde\omega)|} \leq Cj^{-d_1\cdot\kappa_0}.$$

Second, [**DMS**, Lemma V.6.6] (which deals with (FA_j)) uses equation (2.14) only via [**DMS**, Lemma V.6.3], while [**DMS**, Lemma V.6.3] still holds (with the same proof) if we replace equation (2.14) by our stronger assumption in equation (2.8). (We mention here a typo: [**DMS**, V.(6.24)] follows from [**DMS**, V.(6.22)] (and not [**DMS**, V.(6.20)] as stated there).)

Third, [**DMS**, Lemmas V.6.7–6.9] are unchanged, establishing equation (2.20). □

Lemma 2.3 is the analogue of [**Sch**, (III)′]).

LEMMA 2.3. (No exceptionally small sets) *For any $\kappa_1 > \kappa_0$, there exists $N_1 \geq N_0$ such that $|x_j(\omega)| > j^{-\kappa_1}$ for all $j \geq N_1$ and $\omega \in \mathcal{P}_j = \mathcal{P}_j(a_*, \kappa_0)$.*

Proof. We first show the lemma assuming that there exists $d_2 \in (0, 1)$ such that for any $j \geq N_0$ and any $\omega \in \mathcal{P}_j$, we have

$$|x_j(\omega)| \geq \frac{d_2 e^{-r_0}(1 - 1/e)}{(\kappa_0 \log j)^2 j^{\kappa_0}}, \tag{2.24}$$

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with r_0 as in the proof of Proposition 2.2. Indeed, equation (2.24) implies that

$$|x_j(\omega)| \geq \frac{d_2 e^{-r_0}(1-e^{-1})}{\kappa_0^2} \frac{1}{j^{\kappa_0}(\log j)^2} \quad \text{for all } \omega \in \mathcal{P}_j, \text{ for all } j \geq N_0.$$

Clearly, there exists $N_1(\kappa_1) \geq N_0$ such that the right-hand side is larger than $j^{-\kappa_1}$ for all $j \geq N_1$.

To establish equation (2.24), we shall use equation (2.8). If $j+1$ is an essential free return time of ω, then taking r minimal such that $x_j(\omega)$ contains an interval $I^{\pm}_{r,\ell}$,

$$|x_j(\omega)| \geq |I^{\pm}_{r,\ell}| = e^{-r}\frac{1-1/e}{r^2} > \frac{j^{-\kappa_0}(1-1/e)}{(\kappa_0 \log j)^2}. \tag{2.25}$$

Otherwise, letting $j'+1 = \nu_{i'}(\omega) \geq \nu_1(\omega)$ be the largest essential free return time of ω such that $j'+1 < j+1$, we have $\omega \in \mathcal{P}_{j'}$ (since if $\tilde{\omega} \supset \omega$, $\tilde{\omega} \in \mathcal{P}_{j'}$, then $\tilde{\omega}$ is never cut between time j' and j), so that equation (2.25) implies

$$|x_{j'}(\omega)| > \frac{1-1/e}{(\kappa_0 \log j')^2 (j')^{\kappa_0}} > \frac{1-1/e}{(\kappa_0 \log j)^2 j^{\kappa_0}}.$$

We shall combine the above bound with [**DMS**, Lemma V.6.3 and Propositions V.6.1 and V.6.2] to handle the three cases left, namely: the time $j+1$ is an inessential free return of ω; the time $j+1$ is a return within a bound period of ω; and the intersection of $x_j(\omega)$ and U_{r_0} is empty.

If $j+1 = \nu_i(\omega)$ is an inessential free return, then [**DMS**, V.(6.15) in Lemma V.6.3] gives, for $i' \leq i$ as defined above,

$$|x_j(\omega)| \geq 2^{i-i'}|x_{j'}(\omega)| > 2^{i-i'}\frac{1-1/e}{(\kappa_0 \log j)^2 j^{\kappa_0}}. \tag{2.26}$$

If $j+1$ is a return within the bound period of a previous free return $j''+1$ of ω, then using equation (2.25) for the bound period of an essential return, respectively equation (2.26) for the bound period of a non-essential return, and applying the first claim of [**DMS**, Lemma V.6.3], we find $d_2 \in (0, 1)$ such that

$$|x_j(\omega)| \geq d_2 \lambda_{\mathrm{CE}}^{j-j''}|x_{j''}(\omega)| > \frac{d_2(1-1/e)}{(\kappa_0 \log j)^2 j^{\kappa_0}}. \tag{2.27}$$

If $x_j(\omega) \cap U_{r_0} = \emptyset$, then [**DMS**, V.(6.2) in Propositions V.6.1 and V.6.2] and equation (2.25) give

$$|x_j(\omega)| \geq d_2 e^{-r_0}|x_{j'}(\omega)| > \frac{d_2 e^{-r_0}(1-1/e)}{(\kappa_0 \log j)^2 j^{\kappa_0}}. \tag{2.28}$$

We have shown equation (2.24) and thus Lemma 2.3. □

2.3. *A Hölder local distortion estimate.* From now on, let $a_* \in (a_{\mathrm{mix}}, 4)$ be a Misiurewicz parameter, fix $\kappa_0 \geq 11/3d_1$ and let $\Omega_* = \Omega_*(a_*, \kappa_0) \subset \Omega_{\mathrm{BC}} = \Omega_{\mathrm{BC}}(a_*)$ be the positive measure Cantor set constructed in §2.2 via families $\mathcal{P}_j = \mathcal{P}_j(a_*, \kappa_0)$. The following replaces [**Sch**, equations (33) and (31)]. (For [**Sch**, equation (30)], see equation (2.2). We do not need [**Sch**, equation (32)].) The bound in equation (2.31) is new.

LEMMA 2.4. (Hölder distortion bounds) *There exists $C < \infty$ such that for all $n \geq N_0$ (with N_0 as in Proposition 2.2) and any $\omega \in \mathcal{P}_n = \mathcal{P}_n(a_*, \kappa_0)$,*

$$\frac{1}{C} \leq \left| \frac{x_n'(a)/x_j'(a)}{(T_a^{n-j})'(x_j(a))} \right| \leq C \quad \textit{for all } 1 \leq j \leq n, \textit{ for all } a \in \omega. \tag{2.29}$$

In addition, there exist $C < \infty$ and $M_0 > \kappa_0$ such that, for all $n \geq N_0$, each $\tilde{\omega} \in \mathcal{P}_n = \mathcal{P}_n(a_*, \kappa_0)$, and every $\omega \subset \tilde{\omega}$ and $\alpha \in [0, 1)$ satisfying

$$|x_n(\omega)| \leq n^{-M_0/(1-\alpha)}, \tag{2.30}$$

we have

$$\left| \frac{x_n'(a_1)}{x_n'(a_2)} \right| \leq 1 + C|x_n([a_1, a_2])|^\alpha \quad \textit{for all } a_1, a_2 \in \omega. \tag{2.31}$$

If $\alpha = 0$, and $n + 1$ is a free return of $\omega \in \mathcal{P}_n$, the bound in equation (2.31) is just equation (2.16). We shall require equation (2.31) for some $\alpha > 0$ in Corollary 3.4.

Proof. The bound in equation (2.29) is an immediate consequence of equation (2.6).

We first claim that there exist C' and $\kappa_2 > 0$ such that for any n,

$$\sum_{i=0}^{j-1} |x_i(\omega)| \leq C' j^{\kappa_0+1+\kappa_2} |x_j(\omega)| \quad \text{for all } 1 \leq j \leq n, \quad \text{for all } \omega \subset \tilde{\omega} \in \mathcal{P}_n. \tag{2.32}$$

(Our proof is inspired from that of [**DMS**, Theorem V.6.2]. This is suboptimal but enough for our purposes. Adapting instead [**DMS**, Lemma V.6.4] could enhance equation (2.31).) To start, there is C such that for any $0 \leq i \leq j \leq n$, using equations (2.3) and (2.29), there exists $a = a(i, j, \omega) \in \omega$ such that, setting $X_{i,j} = x_j \circ x_i^{-1}$,

$$\frac{|x_i(\omega)|}{|x_j(\omega)|} = \frac{|x_i(\omega)|}{|X_{i,j}(x_i(\omega))|} = \frac{|x_i'(a)|}{|x_j'(a)|} \leq \frac{C}{|(T_a^{j-i})'(T_a^{i+1}(c))|}. \tag{2.33}$$

(We used $X_{i,j}' = x_j'/x_i'$ and the mean value theorem in the second equality.) Next, let $s_j(a)$ be the largest ℓ with $\nu_\ell(a) \leq j$, and put

$$q_\ell(a) = \nu_{\ell+1}(a) - (\nu_\ell(a) + p_\ell(a) + 1), \quad \ell = 0, \ldots, s_j(a) - 1,$$
$$q_{s_j(a)}(a) = \max\{0, j - (\nu_{s_j(a)}(a) + p_{s_j(a)}(a) + 1)\}, \quad F_j(a) = \sum_{\ell=0}^{s_j(a)} q_\ell(a).$$

(The condition $(FA)_n$ implicitly used in Proposition 2.2 says that for some fixed arbitrarily small $\tau > 0$, $F_\ell(a) \geq \ell(1 - \tau)$ for $N_0 \leq \ell \leq n$. We shall not need this here.) Set $p_\ell = p_\ell(a)$, $\nu_\ell = \nu_\ell(a)$, $q_\ell = q_\ell(a)$, and $s_j = s_j(a)$. Assume first that $i = 0$. Then, we have (see e.g. [**DMS**, V.(6.11)])

$$\begin{aligned} |(T_a^{j-i})'(T_a^{i+1}(c))| &= |(T_a^j)'(T_a(c))| \\ &= |(T_a^{\nu_1-1})'(T_a(c))| \cdot |(T_a^{j+1-\nu_{s_j}})'(T_a^{\nu_s}(c))| \\ &\quad \cdot \left(\prod_{\ell=1}^{s_j-1} |(T_a^{p_\ell+1})'(T_a^{\nu_\ell}(c))| |(T_a^{q_\ell})'(T_a^{\nu_\ell+p_\ell+1}(c))| \right). \end{aligned}$$

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Since a satisfies $(BA)_m$ and $(FA)_m$ for all $m \le n$, the bounds in equation (2.15) give λ_{CE}, $\gamma_0 > 0$ and $C_0 > 0$ such that

$$\prod_{\ell=1}^{s_j-1} |(T_a^{p_\ell+1})'(T_a^{\nu_\ell}(c))||(T_a^{q_\ell})'(T_a^{\nu_\ell+p_\ell+1}(c))| \ge \prod_{\ell=1}^{s_j-1} C_0 e^{\gamma_0 q_\ell} \lambda_{\mathrm{CE}}^{p_\ell/4}.$$

Similarly, $|(T_a^{\nu_1-1})'(T_a(c))| > C_0 e^{\gamma_0 \nu_1}$. Next, if $j \le \nu_{s_j} + p_{s_j} + 1$, we have

$$|(T_a^{j+1-\nu_{s_j}})'(T_a^{\nu_{s_j}}(c))| \ge C_0^2 \lambda_{\mathrm{BC}}^{j-\nu_{s_j}} j^{-\kappa_0} e^{-r_0},$$

where we used equation (2.8) and [**DMS**, Lemma V.6.1.b, Proposition V.6.1]. If $j > \nu_{s_j} + p_{s_j} + 1$, we have, using [**DMS**, Lemma V.6.1.c, Proposition V.6.1],

$$|(T_a^{j+1-\nu_{s_j}})'(T_a^{\nu_s}(c))| \ge C_0^2 \lambda_{\mathrm{BC}}^{p_{s_j}/4} e^{\gamma_0 (j-(\nu_{s_j}-p_{s_j}-1))} e^{-r_0}.$$

Summarizing,

$$|(T_a^j)'(T_a(c))| \ge \frac{C_0^{s_j+4}}{C} \lambda_{\mathrm{CE}}^{(j-F_j(a))/4} e^{\gamma_0 F_j(a)} j^{-\kappa_0}.$$

Since $p_\ell \ge C_0 r_0$ (see after equation (2.22)), we have $j - F_j \ge j C_0 r_0$ while $s_j \le j/(C_0 r_0)$. We took r_0 large enough (see after equation (2.15)) such that

$$C_0^{s_j+4} \lambda_{\mathrm{CE}}^{(j-F_j(a))/4} \ge 1. \tag{2.34}$$

Finally, using the trivial bound $e^{\gamma_0 F_j(a)} \ge 1$, we find

$$|(T_a^j)'(T_a(c))| \ge \frac{j^{-\kappa_0}}{C}.$$

If $i \ge 1$ and $\nu_{\ell_i}(a) + p_{\ell_i}(a) < i < \nu_{\ell_i+1}(a)$ for some $\ell_i \ge 1$, then we proceed as for $i = 0$, replacing $|(T_a^{\nu_1-1})'(T_a(c))|$ by $|(T_a^{\nu_{\ell_i+1}-i})'(T_a^{i+1}(c))|$ and setting $F_{i,j}(a) = \nu_{\ell_i+1}(a) - i + \sum_{\ell \ge \ell_i+1}^{s_j(a)} q_\ell(a)$. Then,

$$|(T_a^{j-i})'(T_a^{i+1}(c))| \ge \frac{C_0^{s_j-s_i+4}}{C} \lambda_{\mathrm{CE}}^{((j-i)-F_{i,j}(a))/4} e^{\gamma_0 F_{i,j}(a)} j^{-\kappa_0}.$$

We have $j - i - F_{i,j} \ge (j-i) C_0 r_0$, while $s_j - s_i \le (j-i)/(C_0 r_0)$, and we find, using $e^{\gamma_0 F_{i,j}(a)} \ge 1$ (we do not know or need $F_{i,j}(a) \ge (1-\tau)(j-i)$),

$$|(T_a^{j-i})'(T_a^{i+1}(c))| \ge \frac{j^{-\kappa_0}}{C}.$$

Otherwise, $\nu_{\ell_i}(a) \le i - 1 \le \nu_{\ell_i}(a) + p_{\ell_i}(a)$ for some $\ell_i \ge 1$. There may be (non-free) returns during the ℓ_ith bound period. To bypass this difficulty, we exploit that the length of the ℓth bound period is of the order r if $x_{\nu_\ell}(a) \in I_r$ [**DMS**, Lemma V.6.1a]. By equation (2.11), we have $r_{\ell_i} = O(\log(\nu_{\ell_i})) \le C\kappa_0 \log i$. Thus, the missing factor in the ℓth bound period is $\le \Lambda^{C\kappa_0 \log i} \le i^{\kappa_2}$, and

$$|(T_a^{j-i})'(T_a^{i+1}(c))| \ge \frac{j^{-\kappa_0}}{C i^{\kappa_2}}.$$

Summing over i, and recalling equation (2.33), this establishes equation (2.32).

Next, taking $a_1, a_2 \in \omega$, note that equation (2.7) (using the first bound of equation (2.4) if $i < N_0$) implies that for all $N_0 \le i \le n$, recalling $T_a'(x) = a(1-2x)$,

$$
\begin{aligned}
|T'_{a_1}(x_i(a_1)) - T'_{a_2}(x_i(a_2))| \\
&\le |T'_{a_1}(x_i(a_1)) - T'_{a_2}(x_i(a_1))| + |T'_{a_2}(x_i(a_1)) - T'_{a_2}(x_i(a_2))| \\
&\le |a_1 - a_2| + 2a_2|x_i(a_1) - x_i(a_2)| \le (C + 2a_2)|x_i(\omega)|. \qquad (2.35)
\end{aligned}
$$

(Note that equation (2.35) replaces [**Sch**, (36)].) We claim that there exists C'' with

$$
\left| \frac{(T^j_{a_1})'(x_0(a_1))}{(T^j_{a_2})'(x_0(a_2))} \right| \le 1 + C'' e^{C''} j^{2\kappa_0+1+\kappa_2}|x_j(\omega)| \quad \text{for all } 1 \le j \le n. \qquad (2.36)
$$

(The above replaces [**Sch**, (37)].) Indeed, using the classical bound

$$
\prod_{i=0}^{j-1}(1+\upsilon_i) \le \exp\left(\sum_{i=0}^{j-1}\upsilon_i\right) \le 1 + e^{\sum_{i=0}^{j-1}\upsilon_i}\sum_{i=0}^{j-1}\upsilon_i \quad \text{if all } \upsilon_i \ge 0,
$$

we have, setting $C'' = 2C'C(C + 2a_2)$,

$$
\begin{aligned}
\left| \frac{(T^j_{a_1})'(x_0(a_1))}{(T^j_{a_2})'(x_0(a_2))} \right| &= \left| \prod_{i=0}^{j-1} \frac{T'_{a_1}(x_i(a_1))}{T'_{a_2}(x_i(a_2))} \right| \\
&\le 1 + e^{\sum_i |-1+T'_{a_1}(x_i(a_1))/T'_{a_2}(x_i(a_2))|} \cdot \sum_{i=0}^{j-1} \left| \frac{T'_{a_1}(x_i(a_1))}{T'_{a_2}(x_i(a_2))} - 1 \right| \\
&\le 1 + e^{(C+2a_2)\sum_{i=0}^{j-1} C|x_i(\omega)|i^{\kappa_0}} \cdot (C+2a_2)\sum_{i=0}^{j-1} C|x_i(\omega)|i^{\kappa_0} \\
&\le 1 + e^{C'' j^{2\kappa_0+1+\kappa_2}|x_j(\omega)|} \cdot C'' j^{2\kappa_0+1+\kappa_2}|x_j(\omega)| \quad \text{for all } j \le n, \qquad (2.37)
\end{aligned}
$$

where we used equations (2.35) and (2.8) (the first bound of equation (2.4) if $i < N_0$) in the second inequality, and equation (2.32) in the last inequality. Setting $M_0 := 4\kappa_0 + 3 + 2\kappa_2$, if equation (2.30) holds for ω, then equation (2.32) gives for all $N_0 \le j \le n$,

$$
\begin{aligned}
C'' j^{2\kappa_0+1+\kappa_2}|x_j(\omega)| &\le C'' j^{2\kappa_0+1+\kappa_2} C' j^{\kappa_0+1+\kappa_2}|x_n(\omega)| \\
&\le C' \frac{j^{3\kappa_0+3+2\kappa_2}}{n^{M_0/(1-\alpha)}} \le C'''.
\end{aligned}
$$

This proves equation (2.36). Similarly, $|(T^j_{a_2})'(x_0(a_2))/(T^j_{a_1})'(x_0(a_1))| \le 1 + C'' e^{C''} j^{2\kappa_0+1+\kappa_2}|x_j(\omega)|$. Therefore,

$$
\left| \frac{1}{(T^j_{a_1})'(x_0(a_1))} - \frac{1}{(T^j_{a_2})'(x_0(a_2))} \right| \le C''' \frac{j^{2\kappa_0+1+\kappa_2}|x_j(\omega)|}{|(T^j_{a_1})'(x_0(a_1))|}.
$$

We can then adapt the end of the proof of [**Sch**, (31)]. Comparing each term on the right-hand side of

$$
\frac{x_n'(a)}{(T^n_a)'(x_0(a))} = x_0'(a) + \sum_{j=1}^{n} \frac{(\partial_a T_a)(x_{j-1}(a))}{(T^j_a)'(x_0(a))} \quad \text{for all } a \in \tilde\omega \in \mathcal{P}_n,
$$

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for $a = a_1$ and $a = a_2$, we find, since $x_0'(a) = \partial_a c_1(a) = 1/4$ and

$$|\partial_a T_a|_{a_1}(x_{j-1}(a_1)) - \partial_a T_a|_{a_2}(x_{j-1}(a_2))| \le |x_{j-1}(a_1) - x_{j-1}(a_2)| \le |x_{j-1}(\omega)|,$$

recalling equation (2.5), and applying equation (2.32) and then equation (2.30) for $M_0 = 4\kappa_0 + 3 + 2\kappa_2$,

$$\begin{aligned}\left|\frac{x_n'(a_1)}{(T_{a_1}^n)'(x_0(a_1))}\right| &\le \left|\frac{x_n'(a_2)}{(T_{a_2}^n)'(x_0(a_2))}\right| + \hat{C}|a_1 - a_2| + \hat{C}\sum_{j=1}^{n} \frac{j^{2\kappa_0+1+\kappa_2}}{\lambda_{\mathrm{CE}}^j}|x_j(\omega)| \\ &\le \left|\frac{x_n'(a_2)}{(T_{a_2}^n)'(x_0(a_2))}\right| + \bar{C}C' n^{4\kappa_0+3+2\kappa_2}|x_n(\omega)| \\ &\le \left|\frac{x_n'(a_2)}{(T_{a_2}^n)'(x_0(a_2))}\right| + \tilde{C}|x_n(\omega)|^\alpha.\end{aligned}$$

Finally, we have, using equation (2.6) (which plays the role of [**Sch**, Lemma 2.4]),

$$\begin{aligned}\left|\frac{x_n'(a_1)}{x_n'(a_2)}\right| &\le \left|\frac{(T_{a_1}^n)'(x_0(a_1))}{(T_{a_2}^n)'(x_0(a_2))}\right|\left(1 + \tilde{C}|x_n(\omega)|^\alpha \frac{|(T_{a_2}^n)'(x_0(a_2))|}{|x_n'(a_2)|}\right) \\ &\le 1 + C\tilde{C}|x_n(\omega)|^\alpha.\end{aligned}$$

□

2.4. *Uniform decorrelation and Hölder response.* The maps x_j are not the iterates of a fixed dynamical system admitting an invariant measure. To exploit statistical information on the iterates of the mixing CE map (T_{a_0}, μ_{a_0}), we will 'switch locally' from x_j to $T_{a_0}^j$ (see Lemma 3.3), using that any $a \in \Omega_*$ satisfies the following uniform decorrelation result for Hölder continuous observables. (The factor $\|\varphi\|_{L_1(dm)}$ in the right-hand side of [**Sch**, Proposition 4.3] is replaced in Proposition 2.5 by $\|\varphi\|_{L^1(d\mu_a)} \le \|\varphi\|_{L^\infty(dm)}$. This does not impact the use of [**Sch**, p. 36, Proposition 4.3].) For $q > 1$ and $s \in [0, 1/q)$, we denote by $H_q^s(I) = F_{q,2}^s(I)$ the Sobolev space of functions of differentiability s and integrability q supported in I (see [**RS**]).

PROPOSITION 2.5. (Uniform decay of correlations) *For any $s > 0$ and $q > 1$, there exist $C < \infty$ and $\rho_q^s < 1$ such that for all $\varphi \in H_q^s(I)$, $\psi \in L^\infty(dm)$, $a \in \Omega_*(a_*, \kappa_0)$,*

$$\left|\int_0^1 \varphi(\psi \circ T_a^n)\, dm - \int_0^1 \varphi\, dm \int_0^1 \psi\, d\mu_a\right| \le C\|\varphi\|_{H_q^s}\|\psi\|_{L^1(d\mu_a)}(\rho_q^s)^n \quad \textit{for all } n \ge 1.$$

For any $\varpi > 0$, there exist $C < \infty$ and $\rho_\varpi < 1$ such that for all $\varphi \in C^\varpi$, $\psi \in L^\infty(dm)$, $a \in \Omega_*(a_*, \kappa_0)$,

$$\left|\int_0^1 \varphi(\psi \circ T_a^n)\, d\mu_a - \int_0^1 \varphi\, d\mu_a \int_0^1 \psi\, d\mu_a\right| \le C\|\varphi\|_\varpi\|\psi\|_{L^1(d\mu_a)}(\rho_\varpi)^n \quad \textit{for all } n \ge 1.$$

We also use Hölder bounds on $a \mapsto \mu_a$ as a distribution (in Lemma 2.8).

PROPOSITION 2.6. (Fractional response) *For any $\Theta \in (0, 1/2)$, there exists C such that for all $\varphi \in C^{1/2}$,*

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$$\left|\int \varphi\, d\mu_a - \int \varphi\, d\mu_{a'}\right| \le C|a-a'|^{\Theta}\|\varphi\|_{1/2} \quad \textit{for all } a, a' \in \Omega_*(a_*, \kappa_0). \tag{2.38}$$

Our proof of Proposition 2.5 uses the following facts.

SUBLEMMA 2.7. *For any $a \in \Omega_{\mathrm{BC}}$, the density h_a of μ_a lies in $H^s_q(I)$ for all $s \in [0, 1/2)$ and $q \in (1, 2/(1+2s))$. In addition, for any (H_0, κ_0) polynomially recurrent a_*, there exists $C_{s,q,a_*} < \infty$ such that*

$$\sup_{a\in\Omega_* a_*,\kappa_0} \|h_a\|_{H^s_q(I)} \le C_{s,q,a_*}.$$

Proof. In the Misiurewicz case, the first claim is [Se, Theorem 10], using Ruelle's [Ru, Theorem 9, Remark 16.a] decomposition of h_a into the sum of a C^1 function and an exponentially decaying sum of 'spikes' $x \mapsto |x - c_k(a)|^{-1/2}$ and square root singularities $x \mapsto |x - c_k(a)|^{1/2}$. For a general $a \in \Omega_{\mathrm{BC}}$, set $T^{-k}_{a,\varsigma} := (T^k_a|_{U_{k,a,\varsigma}})^{-1}$ for $k \ge 1$ and $\varsigma \in \pm$, where $U_{k,a,\varsigma}$ is the monotonicity interval of T^k_a containing c, located to the right of c for $\varsigma = +$, to the left of c for $\varsigma = -$. Then, since we assumed $\lambda_{\mathrm{CE}} > e^{14\alpha_{\mathrm{BC}}}$ in the proof of Proposition 2.2, use [BS1, Proposition 2.7] that there exist a C^1 function $\psi_a : I \to \mathbb{R}_+$ and C^∞ functions $\Xi^k_{a,\pm} : [0, 1] \to [0, 1]$ supported in a neighbourhood of $c_k(a)$ in $T^k_a(U_{k,a,\pm})$, such that

$$h_a(x) = \psi_a(x) + \sum_{k=1}^{\infty} \sum_{\varsigma\in\{+,-\}} \chi_{k,a}(x) \frac{\Xi^k_{a,\varsigma}(T^{-k}_{a,\varsigma}(x))\psi_a(T^{-k}_{a,\varsigma}(x))}{|(T^k_a)'(T^{-k}_{a,\varsigma}(x))|}, \tag{2.39}$$

where $\chi_{k,a}(x) = 1_{\pm x<\pm c_k(a)}$ if $\pm T^k_a$ has a local maximum at c. Setting $\Psi := \Xi^k_{a,\varsigma} \cdot \psi_a$, we find C^1 functions $\Psi_{k,\ell}$ for $\ell = 1, 2, 3$, with

$$\frac{\Psi(T^{-k}_{a,\varsigma}(x))}{|(T^k_a)'(T^{-k}_{a,\varsigma}(x))|} = \frac{\Psi_{k,1}(x)}{|x - c_k(a)|^{1/2}} + \Psi_{k,2}(x)|x - c_k(a)|^{1/2} + \Psi_{k,3}(x) \tag{2.40}$$

for any $x \in \mathrm{supp}(\chi_{k,a})$. Finally, use [Se, Lemmas 11 and 12].

For the second claim, it is convenient to use an alternative decomposition of h_a. First, recall that [BBS, Corollary 1.6] gives a set Ω_{slow} of full measure in the set of mixing CE parameters such that for any $\tilde a \in \Omega_{\mathrm{slow}}$ and each $\kappa_0 > 1$, there exist $H_0 \ge 1$ and a set $\Delta_0(\tilde a, \kappa_0) \subset \Omega_{\mathrm{slow}}$ of (H_0, κ_0)-polynomially recurrent (and thus transversal) parameters, with $\tilde a$ as a Lebesgue density point, such that Proposition 2.5 holds for all $a \in \Delta_0$. (It is unknown whether $\tilde a \in \Delta_0$.) The proof involves constructing a tower for each parameter in Δ_0. We claim that, up to reducing the value of ϵ in the proof of Proposition 2.2, we can replace $\tilde a$ by a_* and Δ_0 by $\Omega_*(a_*, \kappa_0)$. Indeed, Δ_0 was constructed in [BBS, Proposition 2.1], and it suffices to observe that the required uniformity in constants is satisfied by equations (2.5) and (2.8), while [BBS, equations (8) and (7)] are exactly [DMS, V.(6.1), V.(6.2) in Proposition V.6.1].

Let then

$$\Pi_a(\hat\psi)(x) = \sum_{j\ge 0,\varsigma\in\pm} \frac{\lambda^j}{|(T^j_a)'(T^{-j}_{a,\varsigma}(x))|}\psi_j(T^{-j}_{a,\varsigma}(x))$$

(for a suitable $\lambda > 1$) be the projection from the tower with polynomial recurrence used in [BBS], and let $\hat{\mathcal{L}}_a$ be the lift $\mathcal{L}_a \Pi_a = \Pi_a \hat{\mathcal{L}}_a$ of the transfer operator $\mathcal{L}_a\varphi(x) = \sum_{T_a(y)=x} \varphi(y)/|T_a'(y)|$. Then ([BBS, (66)] gives uniform Lasota–Yorke estimates, [BBS, Lemmas 3.8, 4.5 and 4.6, Proposition 4.1] give the weak norm bounds needed by Keller and Liverani [KL]), there exist $C < \infty$ and $\theta < 1$ such that, letting $\|\cdot\|_a'$ be the norm of the Sobolev space $\mathcal{B}_a^{W_1^1}$ of [BBS],

$$\|\hat{\mathcal{L}}_a^n(\hat{\varphi}) - \hat{h}_a \hat{\nu}(\hat{\varphi})\|_a' \le C\|\hat{\varphi}\|_a'\theta^n \quad \text{for all } \hat{\varphi} \in \mathcal{B}_a^{W_1^1}, \text{ for all } a \in \Omega_{\tilde{a}}, \tag{2.41}$$

where $\hat{h}_a$ is the fixed point (the fixed point property determines $\hat{h}_a$ by its value on the level zero of the tower) of $\hat{\mathcal{L}}_a$ on $\mathcal{B}_a^{W_1^1}$ normalized by $\int \Pi_a \hat{h}_a \, dm = 1$, while $\hat{\nu}$, the non-negative measure whose density with respect to Lebesgue in the level j of the tower is λ^j, is the fixed point of the dual of $\hat{\mathcal{L}}_a$ (see [BS1, equation (85)], note that $\nu(\hat{h}_a) = 1$ is automatic). Since $\Pi_a \hat{h}_a = h_a$ and the W_1^1 norm dominates any H_q^s norm on I if $s \in [0, 1)$ and $1 < q < 1/s$ (by the Sobolev embedding, more precisely [RS, Ch. 2], the bounded inclusions $W_1^1 \subset W_1^\sigma = F_{1,1}^\sigma \subset F_{1,2}^\sigma \subset F_{q,2}^s = H_q^s$, if $\sigma = 1 + s - 1/q \in (0, 1)$ and $q \in (1, \infty)$), the decomposition in equation (2.40) combined with the uniform bound in equation (2.41) (for $\hat{\varphi}$ vanishing on all levels ≥ 1 and constant on level zero of the tower, with $\hat{\nu}(\varphi) = 1$) gives the second claim of the sublemma, using again [Se, Lemmas 11 and 12]. □

Proof of Proposition 2.5. Recall from the proof of Proposition 2.2 that we have $\lambda_{\mathrm{CE}} > e^{14\alpha_{\mathrm{BC}}}$. By mollification, it is enough to prove both bounds for C^1 functions φ. It is in fact enough to show the first bound for $\varphi \in C^1$ as follows. Indeed, again by mollification (see e.g. the proof of [Se, Lemma 14]), if the first bound holds for $\varphi \in C^1$, then it holds for any $\varphi \in H_q^s(I)$ with $q > 1$ and $s > 0$. Therefore, since the density h_a of μ_a lies in $H_q^s(I)$ for all $s \in (0, 1/2)$ and $q \in (1, 2/(1 + 2s))$ by Sublemma 2.7 (with norm uniformly bounded in a), the second bound follows from the first bound for $\varphi \in C^1$ (using that C^1 functions are bounded multipliers on H_q^s).

Next, we observed in the proof of Sublemma 2.7 that we can replace the set called Δ_0 in [BBS, Corollary 1.6] by $\Omega_*(a_*, \kappa_0)$. The first bound for Lipschitz continuous φ thus follows from the second assertion of [BBS, Corollary 1.6], since $\Omega_* \subset [a_{\mathrm{mix}}, 4)$. Indeed, note first that a is topologically mixing if and only if its renormalization period P_a is equal to one. Second, observe that the constant $C_{\varphi,\psi}$ in the second claim of [BBS, Corollary 1.6] can be replaced by $C\|\varphi\|_\varpi \|\psi\|_{L^1(d\mu_a)}$ for a constant C uniform in a in view of [BBS, Lemmas 4.5 and 4.6] and the principle of uniform boundedness. More precisely, using the notation from the proof of Sublemma 2.7, we have

$$\int (\psi \circ T_a^n)\varphi \, dm = \int \psi \Pi_a(\hat{\mathcal{L}}_a^n(\hat{\varphi})) \, dm \quad \text{if } \Pi_a(\hat{\varphi}) = \varphi,$$
$$\int (\psi \circ T_a^n)\varphi h_a \, dm = \int \psi \Pi_a(\hat{\mathcal{L}}_a^n(\hat{\varphi}_a)) \, dm \quad \text{if } \Pi_a(\hat{\varphi}_a) = \varphi h_a.$$

Since $\Pi_a \hat{h}_a = h_a$, any Lipschitz continuous φ can be written as $\Pi_a(\hat{\varphi})$ (take $\hat{\varphi}_0 = \varphi$ on the level zero and $\hat{\varphi}_j \equiv 0$ on levels $j \ge 1$) such that, on the one hand, $\|\hat{\varphi}\|_a' \le C\|\varphi\|_1$

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uniformly in a, and, on the other hand, $\hat{\nu}(\hat{\varphi}) = \int \varphi \, dm$. We conclude by applying equation (2.41) from the proof of Sublemma 2.7. (The Banach space of [**BBS**] requires that the function on level zero of the tower be supported in $(0, 1)$, so this proof cannot cover the case $a = 4$.) □

Proof of Proposition 2.6. If $a = a_*$, the bound is an immediate consequence of the first claim of [**BBS**, Corollary 1.6], since we can replace the set denoted Δ_0 there by $\Omega_*(a_*, \kappa_0)$, as observed in the proof of Sublemma 2.7 and used in the proof of Proposition 2.5. If $a \neq a_*$, the uniformity of the constants given by Proposition 2.2 ensures that we may construct the reference tower in [**BBS**] at a (instead of a_*), viewing a' as a perturbation of a. □

2.5. *Hölder regularity of the variance $\sigma_a(\varphi)$.* Propositions 2.5 and 2.6 will imply the following regularity of $a \mapsto \sigma_a(\varphi)$ on Ω_*.

LEMMA 2.8. (Regularity of $\sigma_a(\varphi)$) *For any $\varpi \in (0, 1]$, there exist $\theta \in (0, \min\{1/2, \varpi\})$ and $C < \infty$ such that for each $\varphi \in C^\varpi$ with $\sigma_{a_*}(\varphi) > 0$, there exists $\epsilon_\varphi > 0$ such that*

$$C_{\epsilon_\varphi}(\varphi) := \inf_{a \in \Omega_{a_*} \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]} \sigma_a(\varphi) > 0,$$

and such that for all $a, a' \in \Omega_*(a_*, \kappa_0) \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$, we have

$$|\sigma_a(\varphi) - \sigma_{a'}(\varphi)| \leq \frac{C}{2C_\epsilon(\varphi)} \|\varphi\|_\varpi |a - a'|^\theta. \tag{2.42}$$

Proof. Let $k_0 > 1$ be a large integer to be chosen at the end of the proof. By the second claim of Proposition 2.5, there exist $\rho = \rho_\varpi < 1$ and C_0 such that

$$\sum_{k > k_0} \left| \int \left(\varphi - \int \varphi \, d\mu_a \right) \cdot \left(\left(\varphi - \int \varphi \, d\mu_a \right) \circ T_a^k \right) d\mu_a \right|$$
$$\leq C_0 \|\varphi\|_\varpi^2 \cdot \frac{\rho^{k_0}}{1 - \rho} \quad \text{for all } k_0 \geq 1, \text{ for all } a \in \Omega_{a_*}, \text{ for all } \varphi \in C^\varpi.$$

Set $A_a = \int \varphi \, d\mu_a$. Since $\int ((\varphi - A_a) \circ T_a^k)(\varphi - A_a) \, d\mu_a = \int (\varphi \circ T_a^k) \varphi \, d\mu_a - A_a^2$, we have

$$\begin{aligned} |\sigma_a(\varphi)^2 - \sigma_{a'}(\varphi)^2| \leq {} & 2 \sum_{k=0}^{k_0 - 1} \left| \int \varphi(\varphi \circ T_a^k) \, d\mu_a - \int \varphi(\varphi \circ T_{a'}^k) \, d\mu_a \right| \\ & + 2 \sum_{k=0}^{k_0 - 1} \left| \int \varphi(\varphi \circ T_{a'}^k) \, d\mu_a - \int \varphi(\varphi \circ T_{a'}^k) \, d\mu_{a'} \right| \\ & + 2 \sum_{k=0}^{k_0 - 1} \left| \left(\int \varphi \, d\mu_a \right)^2 - \left(\int \varphi \, d\mu_{a'} \right)^2 \right| \\ & + 4C_0 \|\varphi\|_\varpi^2 \frac{\rho^{k_0}}{1 - \rho} \quad \text{for all } k_0 \geq 1. \end{aligned}$$

Assume for a moment that $\varpi \geq 1/2$. The ϖ-Hölder constant of $\varphi(\varphi \circ T_{\bar{a}}^k)$ (for $\bar{a} = a$ or a') is bounded by $\Lambda^k \|\varphi\|_\varpi^2$. Thus, Proposition 2.6 gives for any $\Theta < 1/2$ a constant

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$C_1 = C_1(\Theta)$ such that for $a, a' \in \Omega_{a_*}$ and $\varphi \in C^{\varpi}$,

$$|\sigma_a(\varphi)^2 - \sigma_{a'}(\varphi)^2| \le k_0 C_1 \|\varphi\|_{\varpi}^2 \Lambda^{k_0} |a-a'|^{\Theta} + C_0 \|\varphi\|_{\varpi}^2 \frac{\rho^{k_0}}{1-\rho}$$
$$+ 2 \sum_{k=0}^{k_0-1} \left| \int \varphi(\varphi \circ T_a^k)\, d\mu_a - \int \varphi(\varphi \circ T_{a'}^k)\, d\mu_a \right| \quad \text{for all } k_0 \ge 1.$$

Next, equation (2.2) gives that

$$\int |\varphi \circ T_a^k - \varphi \circ T_{a'}^k|\, d\mu_a \le \|\varphi\|_{\varpi} (C\Lambda^k |a-a'|)^{\varpi}.$$

Therefore, we find

$$|\sigma_a(\varphi)^2 - \sigma_{a'}(\varphi)^2| \le k_0 C_1 \|\varphi\|_{\varpi}^2 \Lambda^{k_0} |a-a'|^{\Theta} + 4 C_0 \|\varphi\|_{\varpi}^2 \frac{\rho^{k_0}}{1-\rho}$$
$$+ k_0 \|\varphi\|_{\varpi} (C\Lambda^{k_0} |a-a'|)^{\varpi}. \tag{2.43}$$

We conclude the proof for $\varpi \ge 1/2$ by dividing equation (2.43) by $|a-a'|^{\theta}$ for small enough $\theta > 0$ and optimizing in k_0, using also $(\sigma_a - \sigma_{a'})(\sigma_a + \sigma_{a'}) = \sigma_a^2 - \sigma_{a'}^2$.

If $\varpi \in (0, 1/2)$, mollification gives $\varphi_{\upsilon} \in C^{1/2}$ and C_4 such that

$$\|\varphi_\upsilon\|_{1/2} \le C_4 \upsilon^{\varpi - 1/2} \|\varphi\|_{\varpi}, \quad \sup |(\varphi \circ T_{\bar a}^k)\varphi - (\varphi_\upsilon \circ T_{\bar a}^k)\varphi_\upsilon| \le C_4 \upsilon^{\varpi} \Lambda^k \|\varphi\|_{\varpi}$$

for all small $\upsilon > 0$, all $0 \le k \le k_0$ and all $\bar a \in \Omega_{a_*}$. To conclude, optimize in $\upsilon = |a-a'|^{\theta_0}$ for small $\theta_0 > 0$, taking θ smaller (in particular $\theta < \varpi\theta_0$). □

3. *Switching locally from the parameter to the phase space*

Let a_*, $\mathcal{P}_j(a_*, \kappa_0)$ and $\Omega_* = \Omega_*(a_*, \kappa_0)$ be as in Proposition 2.2 for $\kappa_0 \ge 11/(3d_1)$, and fix $\varpi \in (0, 1)$. This section is devoted to Proposition 3.2, the main estimate (analogous to [**Sch**, Proposition 5.1]) towards a law of large numbers for the squares of the blocks which will be defined in §4 (see Lemma 4.2).

From now on, fix $\varpi \in (0, 1)$ and a ϖ-Hölder continuous function $\varphi\colon I \to \mathbb{R}$, recalling $\varphi_a, \sigma_a(\varphi)$ from equations (1.6) and (1.3), and assume $\sigma_{a_*}(\varphi) > 0$. Lemma 2.8 gives $\epsilon_\varphi > 0$ such that

$$\sigma_a(\varphi) > 0 \quad \text{for all } a \in \Omega_*^{\varphi} := \Omega_* \cap [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]. \tag{3.1}$$

If $\epsilon_\varphi < \epsilon$, we replace Ω_* by Ω_*^{φ} by replacing ϵ in the proof of Proposition 2.2 with ϵ_φ. (This is harmless as it can only improve the constants.)

Remark 3.1. (θ-Hölder–Whitney extensions of φ_a and $\xi_n(a)$) By Proposition 2.6, the function $a \mapsto \int \varphi\, d\mu_a$ is Θ-Hölder continuous on Ω_* for any $\Theta < 1/2$. By Lemma 2.8, the function $a \mapsto \sigma_a(\varphi) \ge 0$ is θ-Hölder continuous on Ω_* for some $\theta < \min\{1/2, \varpi\}$, and uniformly bounded away from zero on Ω_*^{φ}. Taking $\Theta \ge \theta$, the map $a \mapsto \varphi_a(u) = (\varphi(u) - \int \varphi d\mu_a)/\sigma_a$ is θ-Hölder continuous on Ω_*^{φ} uniformly in $u \in I$. By the Whitney extension theorem, we extend each map $a \mapsto \varphi_a(u)$ to a θ-Hölder continuous map on

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$[a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$, uniformly in $u \in I$. In addition, there exists $\widetilde{C} < \infty$ such that

$$\|\varphi_a\|_\infty \le \|\varphi_a\|_\varpi \le \widetilde{C}\|\varphi\|_\varpi \quad \text{for all } a \in [a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]. \tag{3.2}$$

Then, using equation (2.2), we may extend each map $a \mapsto \xi_n(a) = \varphi_a(T_a^{n+1}(c))$ to a θ-Hölder continuous map on $[a_* - \epsilon_\varphi, a_* + \epsilon_\varphi]$, with θ-Hölder constant bounded by $C\Lambda^{\theta(n+1)}$. Indeed, recalling $x_n(a) = T_a^{n+1}(c)$, just decompose

$$\xi_n(a) - \xi_n(a') = \varphi_a(x_n(a)) - \varphi_{a'}(x_n(a)) + \varphi_{a'}(T_a^{n+1}(c)) - \varphi_{a'}(T_{a'}^{n+1}(c)). \tag{3.3}$$

Fix $\alpha \in (0, 1)$ such that (in view of the use of equation (2.30) in Corollary 3.4)

$$\frac{M_0}{1-\alpha} \le \frac{3}{\alpha}. \tag{3.4}$$

Fix $q > 1$ and $0 < s < \min\{\varpi, 1/q\}$, and let

$$\lambda_0 = \min(\lambda_{\mathrm{CE}}^\theta, \rho^{-1/2}) > 1, \tag{3.5}$$

where $\lambda_{\mathrm{CE}} > 1$ is given by equation (2.5), while $\rho = \max\{\rho_q^s, \rho_\varpi\} < 1$ is given by Proposition 2.5, and $\theta \in (0, \min\{1/2, \varpi\})$ is given by Lemma 2.8. Finally, recalling Λ from equation (2.1), let $\eta \in (0, 1/2)$ be so small that

$$\left(\frac{2\Lambda}{\lambda_{\mathrm{CE}}}\right)^\eta \le \lambda_0 < \frac{\lambda_{\mathrm{CE}}^\varpi}{\Lambda^{\eta\varpi}}. \tag{3.6}$$

Define the expectation $E(\psi)$ of $\psi \in L^\infty(\Omega_*^\varphi)$ by

$$E(\psi) := \frac{1}{m(\Omega_*^\varphi)} \int_{\Omega_*^\varphi} \psi \, dm. \tag{3.7}$$

(We restrict to the Cantor set Ω_*^φ here and thus in equation (3.8). The bound in equation (2.9) is used in the proof of equation (3.8) (but not for equation (3.9), Lemma 3.3 or Corollary 3.4).) The following result is the key estimate on $\xi_j(a) = \varphi_a(T_a^{j+1}(c))$.

PROPOSITION 3.2. *There exist $C_\varphi < \infty$ and $K < \infty$ such that*

$$\left| E\left(\sum_{j=k}^{k+n-1} \xi_j\right)^2 - n \right| \le C_\varphi \quad \textit{for all } k \ge \max\{2K, [2/\eta]\}, \textit{ for all } 1 \le n \le \eta k/2, \tag{3.8}$$

and, setting (the stretched exponent $1/4$ for $v(k)$ and the lower bound can be replaced by any number in $(0, 1)$, without changing the statements, up to adjusting intermediate constants) $v(k) = [k - k^{1/4}]$ for every non-trivial interval $\omega \subset \tilde{\omega} \in \mathcal{P}_{v(k)}$ with $\omega \cap \Omega_*^\varphi \ne \emptyset$ and $\lambda_0^{-k^{1/4}} \le |x_{v(k)}(\omega)| \le v(k)^{-3/\alpha}$, we have

$$\left| \frac{1}{|\omega|} \int_\omega \left(\sum_{j=k}^{k+n-1} \xi_j\right)^2 dm - n \right| \le C_\varphi \quad \textit{for all } k \ge [2/\eta], \textit{ for all } 1 \le n \le \eta k/2, \tag{3.9}$$

and, for any sequence Ψ_k with $C_\Psi := \sup_k k^{-8/3} \sup |\Psi_k| < \infty$ and for any refinement $\mathcal{Q}_{v(k)}$ of $\mathcal{P}_{v(k)}$ such that $\lambda_0^{-k^{1/4}} \le |x_v(\omega)| \le v^{-3/\alpha}$ for all $\omega \in \mathcal{Q}_{v(k)}$ (we have $\lambda_0^{-k^{1/4}} \le |x_v(\omega)|$ for all $\omega \in \mathcal{P}_{v(k)}$ by equation (3.21)), setting

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$$\mathcal{Q}_{*,v(k)} := \{\omega \in \mathcal{Q}_{v(k)} \mid \omega \cap \Omega_*^{\varphi} \neq \emptyset\}, \quad \Omega^{\mathcal{Q}}_{*,v(k)} = \bigcup_{\omega \in \mathcal{Q}_{*,v(k)}} \omega, \tag{3.10}$$

we have

$$\left| E(\Psi_k) - \frac{1}{|\Omega^{\mathcal{Q}}_{*,v(k)}|} \int_{\Omega^{\mathcal{Q}}_{*,v(k)}} \Psi_k \, dm \right| \le C_\Psi C_\varphi \quad \text{for all } k \ge [2/\eta]. \tag{3.11}$$

Proposition 3.2 is proved in §3.1. Like for its analogue [**Sch**, Proposition 5.1], the first step will be to show the local estimate in equation (3.9) using Lemma 3.3 through its Corollary 3.4 (the analogues of [**Sch**, Lemma 5.3, Corollary 5.5]).

LEMMA 3.3. (Switching locally from parameter to phase space) *Fix* $\ell_0 \in \{1, 2, 3, 4\}$. *There exists* $C < \infty$ *such that we have, for any integers*

$$n \le n_i \le n + \eta n, \quad 1 \le i \le \ell_0,$$

for every $\tilde{\omega} \in \mathcal{P}_n$ *and each non-trivial interval* $\omega \subset \tilde{\omega}$ *with* $\omega \cap \Omega_*^{\varphi} \neq \emptyset$,

$$\int_{x_n(\omega)} \left| \prod_{\ell=1}^{\ell_0} \xi_{n_\ell}(x_n|_\omega^{-1}(y)) - \prod_{\ell=1}^{\ell_0} \varphi_{a_0}(T_{a_0}^{n_\ell - n}(y)) \right| dy \tag{3.12}$$
$$\le C\lambda_0^{-n} |x_n(\omega)| \quad \text{for all } a_0 \in \omega \cap \Omega_*^{\varphi}.$$

COROLLARY 3.4. *There exists* $C_3 > 1$ *such that for* ℓ_0, n, $n_1, \ldots, n_{\ell_0}$, *and* ω *as in Lemma 3.3, if, in addition,* $|x_n(\omega)| \le n^{-3/\alpha}$, *then for any* $a_0 \in \omega \cap \Omega_*^{\varphi}$,

$$\left| \frac{1}{|\omega|} \int_\omega \prod_{\ell=1}^{\ell_0} \xi_{n_\ell}(a) \, da - \frac{1}{|x_n(\omega)|} \int_{x_n(\omega)} \prod_{\ell=1}^{\ell_0} \varphi_{a_0}(T_{a_0}^{n_\ell - n}(y)) \, dy \right|$$
$$\le C_3(|x_n(\omega)|^\alpha + \lambda_0^{-n}).$$

Proof. Since equation (3.4) implies equation (2.30) for ω, the change of variables $y = x_n(a)$ on ω, combined with the distortion estimate in equation (2.31), gives

$$\left| \frac{1}{|\omega|} \int_\omega \prod_{\ell=1}^{\ell_0} \xi_{n_\ell}(a) \, da - \frac{1}{|x_n(\omega)|} \int_{x_n(\omega)} \prod_{\ell=1}^{\ell_0} \xi_{n_\ell}(x_n|_\omega^{-1}(y)) \, dy \right|$$
$$= \frac{1}{|x_n(\omega)|} \left| \int_{x_n(\omega)} \prod_{\ell=1}^{\ell_0} \xi_{n_\ell}(x_n|_\omega^{-1}(y)) \left(\frac{1}{|x_n'(x_n|_\omega^{-1} y)|} \frac{|x_n(\omega)|}{|\omega|} - 1 \right) dy \right| \tag{3.13}$$
$$\le C \frac{|x_n(\omega)|^\alpha}{|x_n(\omega)|} \int_{x_n(\omega)} \prod_{\ell=1}^{\ell_0} |\xi_{n_\ell}(x_n|_\omega^{-1}(y))| \, dy.$$

Since $\sup_k \|\xi_k\|_{L^\infty} < \infty$, the claim then follows from Lemma 3.3. □

Proof of Lemma 3.3. For $a_0 \in \omega$ as in the statement, the functions

$$\tilde{\varphi}_\ell(y) = \tilde{\varphi}_{\ell,a_0}(y) = \varphi_{a_0}(T_{a_0}^{n_\ell - n}(y)), \quad \tilde{\xi}_\ell(y) = \tilde{\xi}_{\ell,\omega}(y) = \xi_{n_\ell}(x_n|_\omega^{-1}(y))$$

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with

$$\xi_{n_\ell}(x_n|_\omega^{-1}(y)) = \varphi_{x_n|_\omega^{-1}(y)}(x_{n_\ell}(x_n|_\omega^{-1}(y)) = \varphi_{x_n|_\omega^{-1}(y)}(T^{n_\ell+1}_{x_n|_\omega^{-1}(y)}(c))$$

are bounded on $x_n(\omega \cap \Omega_*^\varphi)$. Decomposing

$$|\tilde{\xi}_1\tilde{\xi}_2\tilde{\xi}_3\tilde{\xi}_4 - \tilde{\varphi}_1\tilde{\varphi}_2\tilde{\varphi}_3\tilde{\varphi}_4| \le |(\tilde{\xi}_1 - \tilde{\varphi}_1)\tilde{\xi}_2\tilde{\xi}_3\tilde{\xi}_4| + |\tilde{\varphi}_1(\tilde{\xi}_2 - \tilde{\varphi}_2)\tilde{\xi}_3\tilde{\xi}_4| \\ + |\tilde{\varphi}_1\tilde{\varphi}_2(\tilde{\xi}_3 - \tilde{\varphi}_3)\tilde{\xi}_4| + |\tilde{\varphi}_1\tilde{\varphi}_2\tilde{\varphi}_3(\tilde{\xi}_4 - \tilde{\varphi}_4)|, \tag{3.14}$$

it is enough to find a uniform constant $\bar{C} > 1$ such that

$$\frac{1}{|x_n(\omega)|}\int_{x_n(\omega)} |\tilde{\xi}_{\ell,\omega} - \tilde{\varphi}_{\ell,a_0}|\, dy \le \bar{C}\lambda_0^{-n} \quad \text{for all } a_0 \in \omega \cap \Omega_*^\varphi,\ 1 \le \ell \le \ell_0.$$

We will do so by showing the pointwise estimate

$$|\tilde{\xi}_{\ell,\omega}(y) - \tilde{\varphi}_{\ell,a_0}(y)| \le \bar{C}\lambda_0^{-n} \quad \text{for all } y \in x_n(\omega),\ \text{for all } a_0 \in \omega \cap \Omega_*^\varphi,\ 1 \le \ell \le \ell_0.$$

For $a = x_n|_\omega^{-1}(y)$, we decompose

$$\begin{aligned}\tilde{\xi}_{\ell,\omega}(y) - \tilde{\varphi}_{\ell,a_0}(y) &= \xi_{n_\ell}(a) - \varphi_{a_0}(T_{a_0}^{n_\ell-n}(x_n(a))) \\ &= \varphi_a(x_{n_\ell}(a)) - \varphi_{a_0}(T_{a_0}^{n_\ell-n}(x_n(a))) \\ &= \varphi_a(x_{n_\ell}(a)) - \varphi_{a_0}(x_{n_\ell}(a)) + \varphi_{a_0}(x_{n_\ell}(a)) - \varphi_{a_0}(T_{a_0}^{n_\ell-n}(x_n(a))).\end{aligned} \tag{3.15}$$

Using Remark 3.1, there exists C, independent of n_ℓ, such that

$$|\varphi_a(x_{n_\ell}(a)) - \varphi_{a_0}(x_{n_\ell}(a))| \le C|\omega|^\theta \quad \text{for all } \{a, a_0\} \subset \omega. \tag{3.16}$$

Hence, using our choice in equation (3.5) of λ_0, and since $|\omega| \le C\lambda_{\mathrm{CE}}^{-n}$ by equation (2.7), we get

$$|\varphi_a(x_{n_\ell}(a)) - \varphi_{a_0}(x_{n_\ell}(a))| \le C|\omega|^\theta \le C\lambda_0^{-n}. \tag{3.17}$$

For the last two terms in the right-hand side of equation (3.15), note that since $a = x_n|_\omega^{-1}(y)$ implies $x_{n_\ell}(a) = T_a^{n_\ell+1}(c) = T_a^{n_\ell-n}(T_a^{n+1}(c)) = T_a^{n_\ell-n}(y)$, we have, using equation (2.2),

$$\begin{aligned}|x_{n_\ell}(x_n|_\omega^{-1}(y)) - T_{a_0}^{n_\ell-n}(y)| &= |T_a^{n_\ell-n}(y) - T_{a_0}^{n_\ell-n}(y)| \\ &\le C\Lambda^{n_\ell-n}|a-a_0| \le C\Lambda^{n_\ell-n}|\omega| \quad \text{for all } y \in x_n(\omega).\end{aligned} \tag{3.18}$$

Then, since $n_\ell - n \le \eta n$, our choice of λ_0, η, with equation (3.2) at $a = a_0$ give (we do not need the analogue of [**Sch**, Sublemma 5.4] here)

$$\begin{aligned}&|\varphi_{a_0}(x_{n_\ell}(a)) - \varphi_{a_0}(T_{a_0}^{n_\ell-n}(x_n(a)))| \\ &\quad = |\varphi_{a_0}(x_{n_\ell}(x_n|_\omega^{-1}(y))) - \varphi_{a_0}(T_{a_0}^{n_\ell-n}(y))| \\ &\quad \le C\tilde{C}\Lambda^{(n_\ell-n)\varpi}|\omega|^\varpi \le C\tilde{C}\Lambda^{\varpi\eta n}|\omega|^\varpi \le C\tilde{C}\lambda_0^{-n},\end{aligned} \tag{3.19}$$

using again in the last inequality that $|\omega| \le C\lambda_{\mathrm{CE}}^{-n}$ from equation (2.7). We conclude by combining equations (3.17) and (3.19) into equation (3.15). □

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3.1. *Proof of Proposition 3.2.* We first show equation (3.9). Let $\omega \subset \tilde{\omega} \in \mathcal{P}_{[k-k^{1/4}]}$, with $k \geq 2n/\eta$, be as in the assertion. Writing

$$\int_\omega \bigg(\sum_{j=k}^{k+n-1} \xi_j \bigg)^2 \, dm = \sum_{j=k}^{k+n-1} \bigg(\int_\omega \xi_j^2 \, dm + 2 \sum_{\ell=j+1}^{k+n-1} \int_\omega \xi_j \xi_\ell \, dm \bigg),$$

it is sufficient to show that

$$\sum_{j=k}^{k+n-1} \bigg| 1 - \frac{1}{|\omega|} \int_\omega \bigg(\xi_j^2 + 2 \sum_{\ell=j+1}^{k+n-1} \xi_j \xi_\ell \bigg) \, dm \bigg| = O(1). \tag{3.20}$$

Fix $a_0 \in \omega \cap \Omega_*^\varphi$. By Corollary 3.4 for $\ell_0 = 2$, we have, for $k \leq j \leq k+n-1$,

$$\begin{aligned}
&\frac{1}{|\omega|} \int_\omega \bigg(\xi_j^2 + 2 \sum_{\ell=j+1}^{k+n-1} \xi_j \xi_\ell \bigg) \, dm \\
&\quad = \frac{1}{|x_v(\omega)|} \int_{x_v(\omega)} \bigg(\varphi_{a_0}^2 \circ T_{a_0}^{j-v} + 2 \sum_{\ell=j+1}^{k+n-1} \varphi_{a_0} \circ T_{a_0}^{j-v} \varphi_{a_0} \circ T_{a_0}^{\ell-v} \bigg) \, dm \\
&\qquad + O((k+n-j)(\lambda_0^{-(k-k^{1/4})} + |x_v(\omega)|^\alpha))
\end{aligned}$$

(recall $v = [k - k^{1/4}]$). Since $0 < s < 1/q < 1$, we have that $1_{x_v(\omega)} \in H_q^s$, uniformly in v and ω (see [**St**]), so the first claim of Proposition 2.5 gives

$$\begin{aligned}
&\int_{x_v(\omega)} (\varphi_{a_0} \circ T_{a_0}^{j-v})(\varphi_{a_0} \circ T_{a_0}^{\ell-v}) \, dm \\
&\quad = |x_v(\omega)| \int \varphi_{a_0} \cdot (\varphi_{a_0} \circ T_{a_0}^{\ell-j}) \, d\mu_{a_0} + O(\rho^{j-v}) \quad \text{for all } \ell \geq j.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{|\omega|} \int_\omega \bigg(\xi_j^2 + 2 \sum_{\ell=j+1}^{k+n-1} \xi_j \xi_\ell \bigg) \, dm &= \int \bigg(\varphi_{a_0}^2 + 2 \sum_{\ell=k+1}^{k+n-1} \varphi_{a_0} \cdot (\varphi_{a_0} \circ T_{a_0}^{\ell-j}) \bigg) \, d\mu_{a_0} \\
&\quad + O((k+n-j)(\lambda_0^{-(k-k^{1/4})} + |x_v(\omega)|^\alpha + \rho^{j-v} |x_v(\omega)|^{-1})).
\end{aligned}$$

By equations (1.3) and (1.7), we have

$$1 = \int \varphi_{a_0}^2 \, d\mu_{a_0} + 2 \sum_{i=1}^\infty \int \varphi_{a_0} \cdot \varphi_{a_0} \circ T_{a_0}^i \, d\mu_{a_0}.$$

Therefore, the second claim of Proposition 2.5 gives

$$\int \bigg(\varphi_{a_0}^2 + 2 \sum_{\ell=j+1}^{k+n-1} \varphi_{a_0} \cdot (\varphi_{a_0} \circ T_{a_0}^{\ell-j}) \bigg) \, d\mu_{a_0} = 1 + O(\rho^{k+n-j}).$$

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Hence, we find, for $k \le j \le k+n-1$ and $v = [k - k^{1/4}]$,

$$\bigg|1 - \frac{1}{|\omega|}\int_\omega \bigg(\xi_j^2 + 2\sum_{\ell=j+1}^{k+n-1} \xi_j \xi_\ell\bigg)\bigg|$$
$$\le C(k+n-j)(\lambda_0^{-(k-k^{1/4})} + |x_v(\omega)|^\alpha + \rho^{j-v}|x_v(\omega)|^{-1}) + C\rho^{k+n-j}.$$

To proceed, we shall use several times that

$$\sup_n \sup_k \sum_{j=k}^{k+n-1} \frac{1}{(k+n-j)^2} \le \sup_n \sum_{\ell=1}^{n} \frac{1}{\ell^2} < \infty.$$

Clearly, $\rho^{k+n-j} \le C/(k+n-j)^2$. For the term $(k+n-j)\rho^{j-v}|x_v(\omega)|^{-1}$, we use $|x_v(\omega)| \ge \lambda_0^{-k^{1/4}}$ and the definition in equation (3.5) of λ_0 to get, since $k \ge 2n/\eta$,

$$\frac{\rho^{j-v}}{|x_v(\omega)|} \le \rho^{k^{1/4}} \lambda_0^{k^{1/4}} \le \lambda_0^{-k^{1/4}} \le \frac{C}{n^3} \le \frac{C}{(k+n-j)^3}, \quad k \le j \le k+n-1.$$

The term $(k+n-j)\lambda_0^{-(k-k^{1/4})}$ is similar. Finally, $|x_v(\omega)| \le v^{-3/\alpha}$ gives

$$\sum_{j=k}^{k+n-1} (k+n-j)|x_v(\omega)|^\alpha \le \sum_{j=k}^{k+n-1} \frac{k+n-j}{n^3} \le n\frac{k+n-k}{n^3} = \frac{1}{n}.$$

This proves equation (3.20), and hence equation (3.9).

We will next deduce equations (3.8) and (3.11) from equation (3.9). Fix $\kappa_1 > \kappa_0$, let $N_1(\kappa_1) \ge N_0$ be given by Lemma 2.3 and let $K \ge N_1$ be such that $k^{\kappa_1} \le \lambda_0^{k^{1/4}}$ for all $k \ge K$. Then, if $v = v(k) \ge K$ (so that $k \ge K$), we have

$$|x_v(\tilde\omega)| > v^{-\kappa_1} = [k - k^{1/4}]^{-\kappa_1} > \lambda_0^{-k^{1/4}} \quad \text{for all } \tilde\omega \in \mathcal{P}_v. \tag{3.21}$$

Refining $\mathcal{P}_v$ to a partition $\mathcal{Q}_v$ such that

$$\lambda_0^{-k^{1/4}} \le |x_v(\omega)| \le v^{-3/\alpha} \quad \text{for all } \omega \in \mathcal{Q}_v,$$

we set $\Omega_{*,v}^{\mathcal{Q}}$ as in equation (3.10) and we decompose

$$|\Omega_*^\varphi| \cdot E\bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 = \int_{\Omega_{*,v}^{\mathcal{Q}}} \bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 dm - \int_{\Omega_{*,v}^{\mathcal{Q}} \setminus \Omega_*^\varphi} \bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 dm$$
$$= \int_{\Omega_v} \bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 dm - \int_{\Omega_v \setminus \Omega_*^\varphi} \bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 dm.$$

Then, using equation (2.9), $\sup_k \sup |\xi_k| < \infty$, $\kappa_0 \ge 3/d_1$ and $v(k) \ge k/2 \ge n/\eta$,

$$0 \le \int_{\Omega_{*,v}^{\mathcal{Q}} \setminus \Omega_*^\varphi} \bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 dm \le \int_{\Omega_v \setminus \Omega_*^\varphi} \bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 dm$$
$$\le Cn^2 e_v \le Cn^2 n^{1-d_1\kappa_0} \le C, \tag{3.22}$$

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which shows that

$$\int_{\Omega_*^{\varphi}} \bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 \, dm = \int_{\Omega_{*,v}^{\mathcal{Q}}} \bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 \, dm + O(1)$$
$$= \int_{\Omega_v} \bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 \, dm + O(1).$$

By equation (3.9),

$$\int_{\omega} \bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 \, dm = |\omega|(n + O(1)) \quad \text{for all } \omega \in \mathcal{Q}_v^*. \tag{3.23}$$

Summing equation (3.23) over $\omega \in \mathcal{Q}_v^*$, we get that

$$\int_{\Omega_{*,v}^{\mathcal{Q}}} \bigg(\sum_{j=k}^{k+n-1} \xi_j\bigg)^2 \, dm = n + O(1).$$

Finally, using again equation (2.9) to see

$$0 \le \frac{|\Omega_{*,v}^{\mathcal{Q}}|}{|\Omega_*^{\varphi}|} - 1 \le \frac{|\Omega_v|}{|\Omega_*^{\varphi}|} - 1 = O(e_v) = O(e_n),$$

we have established equation (3.8), and also equation (3.11) in the case $\Psi_k = (\sum_{j=k}^{k+n-1} \xi_j)^2$ (note that $|\Psi_k| \le Cn^2 \le Ck^2$). For more general Ψ_k, the same argument, using $d_1\kappa_0 \ge 11/3$ in equation (3.22), gives equation (3.11). This ends the proof of Proposition 3.2.

4. *Proof of Theorem 1.1 via Skorokhod's representation theorem*

We will rearrange the Birkhoff sum as a sum of blocks of polynomial size, approximate the blocks by a martingale and finally apply Skorokhod's representation theorem to this martingale. The size for the jth block $\mathbb{I}_j$ is $j^{2/3}$, which will give the error exponent $\gamma > 2/5$ in our ASIP. (A block size $\#\mathbb{I}_j = j^b$ replaces $3/5$ in equation (4.4) by $1/(1+b)$, so that the first constraint becomes $N^{\gamma} > N^{b/(1+b)}$, see equation (4.5). Our bounds in equations (4.25) and (4.26) (with Gál–Koksma's strong law of large numbers, Theorem 4.3, and $M(N) \sim N^{1/(1+b)}$) give $N^{\gamma} > N^{(b+2)/(4(b+1))}$. Hence, $b = 2/3$ is the optimum. In the independent and identically distributed case, a block size $j^{1/2}$ gives $\gamma > 1/3$ [**PS**, p. 25], see also the beginning of [**Sch**, §6].)

4.1. *Blocks $\mathbb{I}_M$. Approximations χ_i and y_j.*

Fix $a_*, \varpi \in (0, 1), q, s \in (0, \min\{\varpi, 1/q\})$, $\rho, \theta, \lambda_0, \eta, \alpha, \varphi \in C^{\varpi}$, $\Omega_*^{\varphi} = \Omega_* \cap [a_* - \epsilon_{\varphi}, a_* + \epsilon_{\varphi}]$ as in the beginning of §3. Set

$$\mathcal{P}_{*,k} := \{\omega \in \mathcal{P}_k \mid |\omega \cap \Omega_*^{\varphi}| > 0\}, \quad \Omega_{*,k} := \bigcup_{\omega \in \mathcal{P}_{*,k}} \omega, \quad k \ge 1.$$

Fix $\gamma \in (2/5, 1/2)$ and $\delta \in (0, \min\{1/5, 2(\gamma - 2/5)\})$. (See Lemma 4.4 for the condition $\delta < 2(\gamma - 2/5)$.) For $i \ge 1$, we shall approach $\xi_i : [a_* - \epsilon_{\varphi}, a_* + \epsilon_{\varphi}] \to \mathbb{C}$ (see Remark 3.1) by the stepfunction

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$$\chi_i : \Omega_{*,r_i} \to \mathbb{C}, \quad \chi_i = E(\xi_i|\mathcal{F}_{r_i}) \quad \text{where } r_i = i + [i^\delta],$$

with $\mathcal{F}_k$ the σ-algebra generated by the intervals in $\mathcal{P}_{*,k}$. Conditional expectations are only defined almost everywhere, but we may set (see equation (3.7))

$$\chi_i|_\omega \equiv \frac{\int_{\omega\cap\Omega_*^\varphi} \xi_i \, dm}{|\omega \cap \Omega_*^\varphi|} \quad \text{for all } \omega \in \mathcal{P}_{*,r_i}, \text{ for all } i \geq 1. \tag{4.1}$$

Thus, χ_i is defined everywhere on Ω_{*,r_i}, allowing pointwise claims about it.

Recalling e_ℓ from equation (2.9) and our assumption $\lambda_{\mathrm{CE}} > e^{14\alpha_{\mathrm{BC}}}$ in the proof of Proposition 2.2, we have the following basic lemma.

LEMMA 4.1. *For any $\tilde{\lambda}_{\mathrm{CE}} \in (e^{\alpha_{\mathrm{BC}}}, \sqrt{\lambda_{\mathrm{CE}}} \cdot e^{-\alpha_{\mathrm{BC}}})$, there exists C such that*

$$|\xi_i(a) - \chi_i(a)| \leq C\tilde{\lambda}_{\mathrm{CE}}^{-\theta i^\delta} \quad \textit{for all } i \geq 1, \textit{ for all } a \in \Omega_{*,r_i}, \tag{4.2}$$

and for all $i \geq 1$, $j \geq 0$ and all $a \in \Omega_{*,r_i}$,

$$|E(\xi_{i+j}|\mathcal{F}_{r_i})(a)| = |E(\chi_{i+j}|\mathcal{F}_{r_i})(a)| \leq C \min(1, e_{[\eta(j-2i^\delta)]}). \tag{4.3}$$

(The constant C in equation (4.3) goes to infinity as $\delta \to 0$, that is, if $\gamma \to 2/5$.)

Following [**PS**, §3.3], [**Sch**, §6.1], we define inductively consecutive blocks $\mathbb{I}_j$ of integers and associated functions y_j as follows. Let $\mathbb{I}_1 = \{1\}$ and let $\mathbb{I}_j$ for $j \geq 2$ contain $[j^{2/3}]$ consecutive integers. The first blocks are

$$\underbrace{1,}_{\mathbb{I}_1} \ \underbrace{2,}_{\mathbb{I}_2} \ \underbrace{3, 4,}_{\mathbb{I}_3} \ \underbrace{5, 6,}_{\mathbb{I}_4} \ \underbrace{7, 8,}_{\mathbb{I}_5} \ \underbrace{9, 10, 11,}_{\mathbb{I}_6} \ \underbrace{12, 13, 14,}_{\mathbb{I}_7} \ \underbrace{15, 16, 17, 18,}_{\mathbb{I}_8} \ \underbrace{19, 20, 21, 22,}_{\mathbb{I}_9} \ldots$$

Let $M = M(N)$ be uniquely defined by $N \in \mathbb{I}_M$. There exists C such that

$$C^{-1}N^{3/5} \leq M(N) \leq CN^{3/5} \quad \text{for all } N \geq 1. \tag{4.4}$$

By equation (4.2) in Lemma 4.1, there is C such that, for all $i \geq 1$ and all $a \in \Omega_{*,r_i}$,

$$\left| \sum_{i=1}^{N} \xi_i(a) - \sum_{j=1}^{M(N)} \sum_{i\in\mathbb{I}_j} \chi_i(a) \right| \leq \sum_{i=1}^{N} |\xi_i(a) - \chi_i(a)| + C\#\mathbb{I}_M \leq CN^{2/5} \tag{4.5}$$

for all $N \geq 1$. Hence, to prove Theorem 1.1, it is sufficient to consider

$$y_j : \Omega_{*,[Cr_j^{5/3}]} \to \mathbb{C}, \quad y_j := \sum_{i\in\mathbb{I}_j} \chi_i, \quad j \geq 1.$$

Proof of Lemma 4.1. By equation (4.1), since ξ_i is continuous (see Remark 3.1), for any $\omega \in \mathcal{P}_{*,r_i}$, there exists $a' \in \omega$ such that $\chi_i|_\omega = \xi_i(a')$. Revisiting the decomposition in equation (3.3), and using equation (3.2) and the θ-Hölder continuity of $a \mapsto \varphi_a(u)$ (as for equation (3.16)), we find C such that for all $i \geq 1$ and $\omega \in \mathcal{P}_{*,r_i}$,

$$|\xi_i(a) - \chi_i(a)| = |\xi_i(a) - \xi_i(a')| \leq C(|\omega|^\theta + |x_i(\omega)|^\varpi) \leq C|x_i(\omega)|^\theta \quad \text{for all } a \in \omega,$$

where we used $\theta \leq \varpi$ and equation (2.7) in the second inequality. This establishes equation (4.2), since for any $\bar{\lambda}_{\mathrm{CE}} \in (e^{\alpha_{\mathrm{BC}}}, \sqrt{\lambda_{\mathrm{CE}}} \cdot e^{-\alpha_{\mathrm{BC}}})$, there exists $\bar{C}$ such that

$$|x_i(\omega)| \leq \bar{C} \cdot \bar{\lambda}_{\mathrm{CE}}^{-i^\delta} \cdot i^{\kappa_0} \quad \text{for all } \omega \in \mathcal{P}_{*,r_i}, \text{ for all } i. \tag{4.6}$$

To show equation (4.6), first note, using equation (2.29), that there exists $a \in \omega$ such that

$$|x_i(\omega)| \leq C \frac{|x_{r_i}(\omega)|}{|(T_a^{i^\delta})'(x_i(a))|}.$$

Then, if $a \in \Omega_*$, the polynomial recurrence in equation (2.8) and standard arguments give

$$|(T_a^{i^\delta})'(x_i(a))| \geq C i^{-\kappa_0} \bar{\lambda}_{\mathrm{CE}}^{-i^\delta} \tag{4.7}$$

(see e.g. [**BS1**, Proposition 3.7] in the exponentially recurrent case). If $a \notin \Omega_*$, we may use bounded distortion in equation (2.31) ($\alpha = 0$ suffices here) since $|\omega \cap \Omega_*| > 0$.

The equality in equation (4.3) follows from the definition since $\mathcal{F}_{r_i} \subset \mathcal{F}_{r_{i+j}}$. Indeed, for $a \in \omega \in \mathcal{P}_{*,r_i}$,

$$\begin{aligned} |\omega \cap \Omega_*^\varphi| \cdot |E(\xi_{i+j}|\mathcal{F}_{r_i})(a)| &= \int_{\omega \cap \Omega_*^\varphi} \xi_{i+j}\, dm \\ &= \sum_{\substack{\omega' \in \mathcal{P}_{*,r_{i+j}} \\ \omega' \subset \omega}} |\omega' \cap \Omega_*^\varphi| \cdot \frac{\int_{\omega' \cap \Omega_*^\varphi} \xi_{i+j}\, dm}{|\omega' \cap \Omega_*^\varphi|} \\ &= \sum_{\substack{\omega' \in \mathcal{P}_{*,r_{i+j}} \\ \omega' \subset \omega}} |\omega' \cap \Omega_*^\varphi| \cdot \chi_{i+j}|_{\omega'} = \sum_{\substack{\omega' \in \mathcal{P}_{*,r_{i+j}} \\ \omega' \subset \omega}} \int_{\omega' \cap \Omega_*^\varphi} \chi_{i+j}\, dm. \end{aligned} \tag{4.8}$$

Since $\sup_k \|\xi_k\|_{L^\infty} < \infty$, we may and shall assume that $j \geq 2i^\delta$ to prove the upper bound in equation (4.3). For such j, recalling $\eta \in (0, 1/2)$ from equation (3.6), define

$$k = k(i, j) = \max \left\{ i + [i^\delta] + \eta(j - i^\delta), \left\lceil \frac{i+j}{1+\eta} \right\rceil \right\} \tag{4.9}$$

so that $k \leq i + j - \eta/(1+\eta)(j - i^\delta) \leq i + j$ and $i + j \leq k(1+\eta)$.

Since δ is fixed, we may and shall assume that i is large enough such that $k(i, j) \geq N_1$ (with N_1 from Lemma 2.3) and

$$\max\{\lambda_0^{-(j+i)/(1+\eta)}, \rho^{\eta j/3} \cdot (2j)^{(\kappa_0+1)/\delta}\} \leq e_{[\eta(j-i^\delta)]}. \tag{4.10}$$

Since $k(i, j) \geq r_i$, we have, similarly as for equation (4.8),

$$|E(\xi_{i+j}|\mathcal{F}_{r_i})(a)| = |E(E(\xi_{i+j}|\mathcal{F}_{k(i,j)})|\mathcal{F}_{r_i})(a)| \quad \text{for all } a \in \tilde{\omega} \in \mathcal{P}_{*,r_i}.$$

We must analyse the above decomposition more closely than in the proof of [**Sch**, Lemma 6.1] as follows. Let $a \in \tilde{\omega} \in \mathcal{P}_{*,r_i}$, then,

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$$
\begin{aligned}
|\tilde{\omega} \cap \Omega_*^{\varphi}| \cdot |E(E(\xi_{i+j}|\mathcal{F}_{k(i,j)})|\mathcal{F}_{r_i})(a)| &= \bigg| \sum_{\substack{\omega \in \mathcal{P}_{*,k(i,j)} \\ \omega \subset \tilde{\omega}}} \frac{|\omega \cap \Omega_*^{\varphi}|}{|\omega \cap \Omega_*^{\varphi}|} \int_{\omega \cap \Omega_*^{\varphi}} \xi_{i+j}\, dm \bigg| \\
\leq \bigg| \sum_{\substack{\omega \in \mathcal{P}_{*,k(i,j)} \\ \omega \subset \tilde{\omega}}} \frac{|\omega|}{|\omega|} \int_{\omega} \xi_{i+j}\, dm \bigg| &+ \sup_{\tilde{a}} \|\varphi_{\tilde{a}}\|_{L^\infty} \cdot \sum_{\substack{\omega \in \mathcal{P}_{*,k(i,j)} \\ \omega \subset \tilde{\omega}}} |\omega \setminus (\omega \cap \Omega_*^{\varphi})|. \qquad (4.11)
\end{aligned}
$$

Since $\tilde{\omega} \in \mathcal{P}_{*,r_i}$, the bound in equation (2.10) implies

$$
\begin{cases}
\dfrac{\sum_{\substack{\omega \in \mathcal{P}_{*,k(i,j)} \\ \omega \subset \tilde{\omega}}} |\omega \setminus (\omega \cap \Omega_*)|}{|\tilde{\omega} \cap \Omega_*^{\varphi}|} \leq \dfrac{d_0 e_{k(i,j)-r_i} |\tilde{\omega}|}{(1-d_0 e_{r_i})|\tilde{\omega}|} \leq C d_0 e_{[\eta(j-i^\delta)]}, \\
|\tilde{\omega}|/|\tilde{\omega} \cap \Omega_*^{\varphi}| \leq \dfrac{|\tilde{\omega}|}{(1-d_0 e_{r_i})|\tilde{\omega}|} \leq C.
\end{cases} \qquad (4.12)
$$

In view of equations (2.10), (4.12) and (4.11), it suffices to show

$$
\frac{1}{|\omega|} \bigg| \int_{\omega} \xi_{i+j}\, dm \bigg| \leq C \min(1, e_{[\eta(j-2i^\delta)]}) \quad \text{for all } \omega \in \mathcal{P}_{*,k(i,j)}.
$$

Fix $\omega \in \mathcal{P}_{*,k(i,j)}$. First note that, by equation (2.31) for $\alpha = 0$,

$$
\frac{1}{|\omega|} \bigg| \int_{\omega} \xi_{i+j}(a)\, da \bigg| \leq \frac{C}{|x_k(\omega)|} \bigg| \int_{x_k(\omega)} \xi_{i+j}(x_k|_{\omega}^{-1}(y))\, dy \bigg|. \qquad (4.13)
$$

Then, on the one hand, Lemma 3.3 for $\ell_0 = 1$ gives $a_0 \in \omega \cap \Omega_*^{\varphi}$ such that

$$
\begin{aligned}
&\frac{1}{|x_k(\omega)|} \bigg| \int_{x_k(\omega)} (\xi_{i+j}(x_k|_{\omega}^{-1}(y)) - \varphi_{a_0}(T_{a_0}^{i+j-k}(y)))\, dy \bigg| \\
&\quad \leq C \lambda_0^{-k(i,j)} \leq C \lambda_0^{-(i+j)/(1+\eta)}. \qquad (4.14)
\end{aligned}
$$

On the other hand, recalling $0 < s < 1/q$, since $1_{x_k(\omega)} \in H_q^s$ (uniformly in k and ω), the first claim of Proposition 2.5, with $\int \varphi_{a_0} d\mu_{a_0} = 0$, gives (a factor $|x_k(\omega)|^{-1}$ was omitted when applying [**Sch**], Proposition 4.3] on [**Sch**, p. 400]. We fix this by using our polynomial lower bound on $|x_k(\omega)|$ (considering two different values of δ should work for [**Sch**]))

$$
\begin{aligned}
&\frac{1}{|x_k(\omega)|} \bigg| \int_{x_k(\omega)} \varphi_{a_0}(T_{a_0}^{i+j-k}(y))\, dy \bigg| \leq C \cdot k(i,j)^{(\kappa_0+1)} \rho^{i+j-k(i,j)} \\
&\quad \leq C \cdot (i+j)^{\kappa_0+1} \rho^{\eta(j-i^\delta)/(1+\eta)} \leq C \cdot (2j)^{(\kappa_0+1)/\delta} \rho^{\eta j/(2+2\eta)}. \qquad (4.15)
\end{aligned}
$$

(We used $|x_k(\omega)| > Ck^{-\kappa_0+1}$ from Lemma 2.3.) Putting together equations (4.13), (4.14), (4.15) and (4.10), we conclude the proof of equation (4.3). □

4.2. *Law of large numbers for* y_j^2. Recall that $\gamma \in (2/5, 1/2)$ is fixed. The main ingredient in the proof of Theorem 1.1 is the following analogue of [**Sch**, Lemma 6.2], itself inspired by [**PS**, Lemma 3.3.1].

LEMMA 4.2. *For m_*-a.e. $a \in \Omega_*^{\varphi}$, there exists $C(a)$ such that*

$$\left|N - \sum_{j=1}^{M(N)} y_j^2(a)\right| \le C(a)N^{2\gamma} \quad \text{for all } N \ge 1. \tag{4.16}$$

The proof of Lemma 4.2 (which uses Proposition 3.2 and equation (4.2), but not equation (4.3)) is based on the following theorem ([**GK**], see also [**PS**, Theorem A.1]).

THEOREM 4.3. (Gál–Koksma's strong law of large numbers) *Let z_j, $j \ge 1$, be zero-mean random variables. Assume there exist $p \ge 1$ and $C < \infty$ with*

$$E\bigg(\sum_{j=m+1}^{m+n} z_j\bigg)^2 \le C((m+n)^p - m^p) \quad \text{for all } m \ge 0 \text{ and } n \ge 1.$$

Then, for all $\iota > 0$, we have $(1/n^{p/2+\iota})\sum_{j=1}^{n} z_j \to 0$ almost surely.

Proof of Lemma 4.2. Set $w_j = \sum_{i\in\mathbb{I}_j} \xi_i$. Since $y_j^2 - w_j^2 = (y_j + w_j)(y_j - w_j)$ and $|y_j + w_j| \le Cj^{2/3}$, the bound in equation (4.2) gives C such that $|y_j^2 - w_j^2| \le Cj^{2/3}\tilde{\lambda}_{\mathrm{CE}}^{-\theta j^{\delta}}$ for all $j \ge 1$ and $a \in \Omega_{*,r_{Cj^{5/3}}}$. Hence, $\sup_{a\in\Omega_*^{\varphi}} \sum_{j\ge1} |y_j^2 - w_j^2|$ is finite, and it suffices to show equation (4.16) with y_j replaced by w_j.

By equation (3.8), we have $|E(w_j^2) - \#\mathbb{I}_j| \le C$, and, since $\sum_{j=1}^{M(N)} \#\mathbb{I}_j = N$, we get $|\sum_{j=1}^{M(N)} E(w_j^2) - N| \le CM(N)$. Therefore,

$$\left|N - \sum_{j=1}^{M(N)} w_j^2\right| \le CM(N) + \left|\sum_{j=1}^{M(N)} w_j^2 - E(w_j^2)\right|. \tag{4.17}$$

Assume there exists C such that

$$E\bigg(\sum_{j=m+1}^{m+n} w_j^2 - E(w_j^2)\bigg)^2 \le C((m+n)^{8/3} - m^{8/3}) \quad \text{for all } m \ge 0,\ n \ge 1. \tag{4.18}$$

Then, Theorem 4.3 (Gál–Koksma) applied to $\iota \in (0, 10(\gamma - 2/5)/3]$, $p = 8/3$ and the zero-mean random variables $z_j = w_j^2 - E(w_j^2)$, implies that

$$\sum_{j=1}^{M(N)} w_j^2 - E(w_j^2) = o(M^{4/3+\iota}) \quad \text{almost surely.}$$

Hence, equation (4.17) gives $|N - \sum_{j=1}^{M(N)} w_j^2(a)| \le C(a)N^{4/5+3\iota/5} \le C(a)N^{2\gamma}$, almost surely (recall $M(N) \sim N^{3/5}$ by equation (4.4)). It remains to prove equation (4.18).

By Jensen's inequality, we have $(E(w_j^2))^2 \le E(w_j^4)$ and, therefore,

$$E\bigg(\sum_{j=m+1}^{m+n} w_j^2 - E(w_j^2)\bigg)^2 \le 2\sum_{j=m+1}^{m+n}\bigg(E(w_j^4) + \sum_{k=j+1}^{m+n} |E(w_j^2w_k^2) - E(w_j^2)E(w_k^2)|\bigg). \tag{4.19}$$

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We consider first $E(w_j^4)$. Fix $\upsilon \in (0, 1/6)$ and, for $j \geq 1$, let

$$S_j = \{ \vec{v} \in \mathbb{I}_j^4 \mid v_1 \leq v_2 \leq v_3 \leq v_4 \text{ and } \max\{v_2 - v_1, v_4 - v_3\} \geq j^\upsilon \}.$$

Then, since $\#(\{ \vec{v} \in \mathbb{I}_j^4 \mid v_1 \leq v_2 \leq v_3 \leq v_4\} \setminus S_j) \leq (j^{2/3+\upsilon})^2 = j^{4/3+2\upsilon}$, we find

$$\begin{aligned}\int_{\Omega_*^\varphi} w_j(a)^4 \, da &= \sum_{\vec{v} \in \mathbb{I}_j} \bigg| \int_{\Omega_*^\varphi} \prod_{\ell=1}^4 \xi_{v_\ell}(a) \, da \bigg| \leq C \sum_{\substack{\vec{v} \in \mathbb{I}_j^4 \\ v_1 \leq \cdots \leq v_4}} \bigg| \int_{\Omega_*^\varphi} \prod_{\ell=1}^4 \xi_{v_\ell}(a) \, da \bigg| \\ &\leq C \sum_{\vec{v} \in S_j} \bigg| \int_{\Omega_*^\varphi} \prod_{\ell=1}^4 \xi_{v_\ell}(a) \, da \bigg| + C j^{4/3+2\upsilon}. \end{aligned} \tag{4.20}$$

Let $\vec{v} \in S_j$ be such that $v_4 - v_3 \geq j^\upsilon$. For $\omega \in \mathcal{P}_{v_3}$ such that $\omega \cap \Omega_*^\varphi \neq \emptyset$, the change of variable in equation (3.13), together with an easy variant of Lemma 3.3 deduced from equation (3.14), give $a_0 \in \omega \cap \Omega_*^\varphi$ such that

$$\begin{aligned}&\frac{1}{|\omega|} \bigg| \int_\omega \prod_{\ell=1}^4 \xi_{v_\ell}(a) \, da \bigg| \\ &\quad \leq \frac{C}{|x_{v_3}(\omega)|} \bigg| \int_{x_{v_3}(\omega)} \bigg(\prod_{\ell=1}^3 \xi_{v_\ell}(x_{v_3}|_\omega^{-1}(y)) \bigg) \varphi_{a_0}(T_{a_0}^{v_4 - v_3}(y)) \, dy \bigg| + C\lambda_0^{-v_3}.\end{aligned}$$

For $y \in x_{v_3}(\omega)$, setting $a = x_{v_3}|_\omega^{-1}(y)$, and recalling Remark 3.1, we find

$$\begin{aligned}&|\xi_{v_\ell}(x_{v_3}|_\omega^{-1}(y)) - \varphi_{a_0}(x_{v_\ell} \circ x_{v_3}|_\omega^{-1}(y))| \\ &\quad = |\varphi_a(x_{v_\ell} \circ x_{v_3}|_\omega^{-1}(y)) - \varphi_{a_0}(x_{v_\ell} \circ x_{v_3}|_\omega^{-1}(y))| \leq C|\omega|^\theta\end{aligned}$$

for $\ell = 1, 2, 3$. Thus, equations (2.9) and (2.7) imply (using $\sup_k \|\xi_k\|_{L^\infty} < \infty$)

$$\begin{aligned}\bigg| \int_{\Omega_*^\varphi} \prod_{\ell=1}^4 \xi_{v_\ell}(a) \, da \bigg| &\leq C e_{v_3} + \sum_{\omega \in \mathcal{P}_{*,v_3}} |\omega| \bigg[\frac{C\lambda_{\mathrm{CE}}^{-v_3\theta}}{|x_{v_3}(\omega)|} + C\lambda_0^{-v_3} \bigg] \\ &\quad + \sum_{\omega \in \mathcal{P}_{*,v_3}} |\omega| \frac{C}{|x_{v_3}(\omega)|} \bigg| \int_{x_{v_3}(\omega)} \bigg(\prod_{\ell=1}^3 \varphi_{a_0}(x_{v_\ell} \circ (x_{v_3}|_\omega^{-1})(y)) \bigg) \varphi_{a_0}(T_{a_0}^{v_4 - v_3}(y)) \, dy \bigg|. \end{aligned} \tag{4.21}$$

We claim that for $\ell = 1, 2, 3$, and for each $\omega \in \mathcal{P}_{*,v_3}$,

$$|\partial_y(x_{v_\ell} \circ (x_{v_3}|_\omega^{-1}))(y)| \leq C v_\ell^{\kappa_0} \quad \text{for all } y \in x_{v_3}(\omega). \tag{4.22}$$

Indeed, by equation (2.29), there exists $a \in \omega$ such that

$$|\partial_y(x_{v_\ell} \circ (x_{v_3}|_\omega^{-1}))(y)| \leq C|(T_a^{v_3 - v_\ell})'(T_a^{v_\ell + 1}(c))|^{-1}.$$

Thus, if $a \in \Omega_*$, standard arguments (see e.g. [**BS1**, Proposition 3.7], using our polynomial recurrence in equation (2.8)) give the claim. Otherwise, since $|\omega \cap \Omega_*| > 0$, we may use equation (2.31) as for equation (4.6).

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Therefore, we find C such that for each v_3 and $\omega \in \mathcal{P}_{*,v_3}$,

$$\|1_{x_{v_3}(\omega)} \cdot \prod_{\ell=1}^{3}(\varphi_{a_0} \circ x_{v_\ell} \circ x_{v_3}|_\omega^{-1})\|_{H_q^s} \le C(v_1 v_2)^{\varpi\kappa_0}\|\varphi_{a_0}\|_{C^\varpi}^2 \|\varphi_{a_0}\|_{H_q^s}.$$

Indeed, on the one hand, there exists C such that, for any C^2 map $\mathcal{T}$, we have

$$\|\varphi_{a_0} \circ \mathcal{T}\|_{C^\varpi} \le C \sup |\mathcal{T}'|^\varpi \|\varphi_{a_0}\|_{C^\varpi}.$$

On the other hand, since $0 < s < 1/q < 1$, the characteristic function of an interval is a bounded multiplier on $H_q^s(I)$ (uniformly in the size of the interval), and since $s < \varpi$, a function in C^ϖ is a bounded multiplier on $H_q^s(I)$ [**St**, **Th**].

Hence, by the first claim of Proposition 2.5 (with equation (3.2) and $\int \varphi_{a_0} d\mu_{a_0} = 0$), we have

$$\begin{aligned}\frac{|\int_{x_{v_3}(\omega)}(\prod_{\ell=1}^{3}\varphi_{a_0}(x_{v_\ell} \circ x_{v_3}|_\omega^{-1}))\varphi_{a_0}(T_{a_0}^{v_4-v_3})\, dy|}{|x_{v_3}(\omega)|} &\le C(v_1v_2)^{\varpi\kappa_0}\frac{\rho^{v_4-v_3}}{|x_{v_3}(\omega)|}\\ &\le Cj^{10\varpi\kappa_0/3}v_3^{\kappa_1}\rho^{j^\upsilon}.\end{aligned}$$

(We used Lemma 2.3 and that $v_\ell \in \mathbb{I}_j$ implies $v_\ell \le Cj^{5/3}$.) Next,

$$\left|\int_{\Omega_*^\varphi}\prod_{\ell=1}^{4}\xi_{v_\ell}\, da\right| \le C(e_{[Cj^{5/3}]} + j^{10\varpi\kappa_0/3}j^{5\kappa_1/3}\rho^{j^\upsilon} + j^{5\kappa_1/3}\lambda_0^{j^{5/3}}) \le Ce_{[Cj^{5/3}]}$$

for all $\vec{v} \in S_j$ with $v_4 - v_3 \ge j^\upsilon$ (if j is large enough).

Let now $\vec{v} \in S_j$ with $v_2 - v_1 \ge j^\upsilon$. Then applying directly Lemma 3.3 with $\ell_0 = 4$, a similar reasoning gives $|\int_{\Omega_*^\varphi}\prod_{\ell=1}^{4}\xi_{v_\ell}\, da| \le Ce_{[Cj^{5/3}]}$.

Finally, since $\#S_j \le \#\mathbb{I}_j^4 \le j^{8/3}$ and $e_j \le j^{-d_1\kappa_0+1}$ with $d_1\kappa_0 \ge 3 > 9/5$, the bound in equation (4.20) gives C such that

$$E(w_j^4) \le C(j^{8/3}e_{[Cj^{5/3}]} + j^{4/3+2\upsilon}) \le Cj^{4/3+2\upsilon} \quad \text{for all } j \ge 1. \tag{4.23}$$

(For the purposes of the present lemma, a version of equation (4.23) with $Cj^{5/3}$ in the right-hand side would suffice. The stronger statement is needed for equation (4.31).)

We next bound $|E(w_j^2w_k^2) - E(w_j^2)E(w_k^2)|$ for $k \ge j+1$. If $k = j+1$, by Cauchy's inequality and equation (4.23),

$$E(w_j^2w_{j+1}^2) \le \sqrt{E(w_j^4)E(w_{j+1}^4)} \le Cj^{5/3}.$$

By equation (3.8), we have $E(w_j^2)E(w_{j+1}^2) \le Cj^{4/3}$. Hence,

$$|E(w_j^2w_{j+1}^2) - E(w_j^2)E(w_{j+1}^2)| \le Cj^{5/3}. \tag{4.24}$$

Assume now that $k \ge j+2$. By construction, y_j is constant on elements of $\mathcal{P}_v$ if $v \ge r_{j_1} = j_1 + [j_1^\delta]$, where j_1 is the largest number in $\mathbb{I}_j$. Let

$$k_0 = k_0(k) := \min \mathbb{I}_k \ge \frac{k^{5/3}}{C}.$$

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Then, for large enough j, using that $x \mapsto x - x^{1/4}$ is increasing for large x, we find

$$k_0 - k_0^{1/4} \geq j_1 + \#\mathbb{I}_{j+1} - (j_1 + \#\mathbb{I}_{j+1})^{1/4} \geq j_1 + 2j_1^{2/3} - 2j_1^{1/4} \geq j_1 + j_1^{1/4}.$$

Since $k \geq j + 2$ and $\delta < \frac{1}{4}$, we have that y_j is constant on elements of $\mathcal{P}_v$ for

$$v = v(k_0) = [k_0 - k_0^{1/4}].$$

Lemma 2.3 gives $|x_{v(k_0)}(\omega)| \geq \lambda_0^{-k_0{}^{1/4}}$ if $\omega \in \mathcal{P}_{v(k_0)}$. Thus, there exists a refinement $\mathcal{Q}_{v(k_0)}$ of $\mathcal{P}_{v(k_0)}$ such that

$$\lambda_0^{-k_0{}^{1/4}} \leq |x_{v(k_0)}(\omega)| \leq v(k_0)^{-3/\alpha} = [k_0 - k_0^{1/4}]^{-3/\alpha} \quad \text{for all } \omega \in \mathcal{Q}_{v(k_0)}.$$

Therefore, for large enough k, the local bound in equation (3.9) in Proposition 3.2 gives for all $\omega \in \mathcal{Q}_v$ with non-empty intersection with Ω_*^{φ} that

$$\left| \frac{1}{|\omega|} \int_{\omega} w_k^2 \, dm - \#\mathbb{I}_k \right| \leq C,$$

since $n = \#\mathbb{I}_k = [k^{2/3}] \leq \eta k_0/2$. As in equation (3.10), we write $\mathcal{Q}_{*,v}$ for the set of $\omega \in \mathcal{Q}_v$ with non-empty intersection with Ω_*^{φ}, and $\Omega_{*,v}^{\mathcal{Q}} = \cup \mathcal{Q}_{*,v}$. Thus, using that y_j is constant on each $\omega \in \mathcal{Q}_v$ (since $\mathcal{Q}_v$ refines $\mathcal{P}_v$),

$$\begin{aligned}\int_{\Omega_{*,v}^{\mathcal{Q}}} (y_j^2 w_k^2) \, dm &= \sum_{\omega \in \mathcal{Q}_v^*} |\omega| \cdot y_j^2|_{\omega} \cdot \frac{1}{|\omega|} \int_{\omega} w_k^2 \, dm \\ &\in \bigg[\int_{\Omega_{*,v}^{\mathcal{Q}}} y_j^2 \, dm(\#\mathbb{I}_k - C), \int_{\Omega_{*,v}^{\mathcal{Q}}} y_j^2 \, dm(\#\mathbb{I}_k + C) \bigg].\end{aligned}$$

Recall that $j \leq k - 2$. Since $|y_j^2| \leq Cj^{4/3} \leq Ck^{4/3}$, we get

$$\frac{1}{m(\Omega_{*,v}^{\mathcal{Q}})} \int_{\Omega_{*,v}^{\mathcal{Q}}} y_j^2 \, dm = E(y_j^2) + O(1),$$

by equation (3.11) applied to $\Psi_k = y_j^2$, and since $|y_j^2 w_k^2| \leq Ck^{8/3}$, we have

$$\frac{1}{m(\Omega_{*,v}^{\mathcal{Q}})} \int_{\Omega_{*,v}^{\mathcal{Q}}} (y_j^2 w_k^2) \, dm = E(y_j^2 w_k^2) + O(1),$$

by equation (3.11) applied to $\Psi_k = (y_j^2 w_k^2)$. That is,

$$|E(y_j^2 w_k^2) - \#\mathbb{I}_k E(y_j^2)| \leq C(E(y_j^2) + 1).$$

Next, the global estimate in equation (3.8) in Proposition 3.2 gives $|E(y_j^2)E(w_k^2) - \#\mathbb{I}_k E(y_j^2)| \leq CE(y_j^2)$. Therefore, (the expression $\#\mathbb{I}_j = j^b = j^{2/3}$ in the right-hand side already leads to $\gamma > 2/5$. A block size $\#\mathbb{I}_j = j^b$ replaces $3/5$ in equation (4.4) by $1/(1+b)$, so that the first constraint becomes $N^{\gamma} > N^{b/(1+b)}$, see equation (4.5). Our bounds in equations (4.25) and (4.26) (with Gál–Koksma's strong law of large numbers, Theorem 4.3, and $M(N) \sim N^{1/(1+b)}$) give $N^{\gamma} > N^{(b+2)/(4(b+1))}$. Hence, $b = 2/3$ is the optimum. In the independent and identically distributed case, a block size $j^{1/2}$ gives

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$\gamma > 1/3$ [**PS**, p. 25], see also the beginning of [**Sch**, §6])

$$|E(y_j^2 w_k^2) - E(y_j^2)E(w_k^2)| \le C(2E(y_j^2) + 1)) \le C\#\mathbb{I}_j.$$

Hence, for large enough j and all $k \ge j+2$, since $\sup |w_j + y_j| \le C\#\mathbb{I}_j$,

$$\begin{aligned}
|E(w_j^2 w_k^2) - E(w_j^2)E(w_k^2)| &\le |E(y_j^2 w_k^2) - E(y_j^2)E(w_k^2)| \\
&\quad + |E(y_j^2 w_k^2) - E(w_j^2 w_k^2)| + |E(w_j^2)E(w_k^2) - E(y_j^2)E(w_k^2)| \\
&\le C\#\mathbb{I}_j + CE(w_k^2) \sup |w_j - y_j| \cdot \sup |w_j + y_j| \\
&\le Cj^{2/3} + Ck^{2/3} j^{2/3} \tilde{\lambda}_{\mathrm{CE}}^{-\theta j^{5\delta/3}}. \qquad (4.25)
\end{aligned}$$

(We used equation (4.2) to get $\sup |w_j - y_j| \le C\#\mathbb{I}_j \tilde{\lambda}_{\mathrm{CE}}^{-\theta j^{5\delta/3}}$.)

Finally, we plug equations (4.25), (4.24) and (4.23) into equation (4.19), and get, since $2\upsilon < 1/3$,

$$\begin{aligned}
&E\bigg(\sum_{j=m+1}^{m+n} w_j^2 - E(w_j^2)\bigg)^2 \\
&\quad \le C \sum_{k=m+3}^{m+n} k^{2/3} \sum_{j=m+1}^{\infty} j^{2/3} \tilde{\lambda}_{\mathrm{CE}}^{-\theta j^{5\delta/3}} + C \sum_{j=m+1}^{m+n} \bigg(j^{5/3} + \sum_{k=j+2}^{m+n} j^{2/3}\bigg) \\
&\quad \le C\bigg((m+n)^{5/3} - m^{5/3} + \sum_{j=m+1}^{m+n} (j^{5/3} + (m+n-j)j^{2/3})\bigg). \qquad (4.26)
\end{aligned}$$

This proves equation (4.18). □

4.3. *Martingale differences Y_j. Skorokhod's representation theorem.* As in Schnellmann's adaptation of [**PS**, §§3.4 and 3.5] in [**Sch**, §6.3], let $\mathcal{L}_j$ be the σ-algebra generated by $\{y_\ell\}_{1 \le \ell \le j}$, and set

$$u_j = \sum_{k \ge 0} E(y_{j+k} \mid \mathcal{L}_{j-1}), \quad Y_j = y_j + u_{j+1} - u_j, \quad j \ge 2. \qquad (4.27)$$

Then, $\{Y_j, \mathcal{L}_j\}$ is a martingale difference sequence. Using equation (4.3), we show that $\{Y_j\}$ inherits the law of large numbers established for $\{y_j\}$ in Lemma 4.2:

LEMMA 4.4. *For m_*-a.e. $a \in \Omega_*^{\varphi}$, there exists $C(a)$ such that*

$$\bigg|N - \sum_{j=1}^{M(N)} Y_j^2(a)\bigg| \le C(a) N^{2\gamma} \quad \textit{for all } N \ge 1, \qquad (4.28)$$

and

$$\bigg|\sum_{j=1}^{M(N)} E(Y_j^2 \mid \mathcal{L}_{j-1}) - Y_j^2(a)\bigg| \le C(a) N^{2\gamma} \quad \textit{for all } N \ge 1. \qquad (4.29)$$

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Proof. Recalling the σ-algebra $\mathcal{F}_{r_i}$ generated by the intervals in $\mathcal{P}_{r_i}$, we have $\mathcal{L}_{\ell-1} \subset \mathcal{F}_{r_{i(\ell)}}$, where $i(\ell) = \max\{i \in \mathbb{I}_{\ell-1}\} \le C\ell^{5/3}$ by equation (4.4). Then,

$$u_\ell = \sum_{j\ge 1} E(E(\xi_{i(\ell)+j} \mid \mathcal{F}_{r_{i(\ell)}}) \mid \mathcal{L}_{\ell-1}).$$

Since $\sum_{j=1}^{\infty} e_j < \infty$, the bound in equation (4.3) in Lemma 4.1 gives

$$|u_\ell(a)| \le \sum_{j\ge 1} C \min\{1, e_{[\eta(j-2i(\ell)^\delta))]}\} \le \frac{2C}{\eta} i(\ell)^\delta \le C\ell^{5\delta/3}. \tag{4.30}$$

Put $v_j = u_j - u_{j+1}$, so that $Y_j^2 = y_j^2 - 2y_jv_j + v_j^2$.

We claim that equation (4.28) follows if for a.e. $a \in \Omega_*^\varphi$, there exists C such that $\sum_{j=1}^{M(N)} v_j^2 \le CN^{4\gamma-1}$. Indeed, since $\gamma < 1/2$, Lemma 4.2 and Cauchy's inequality then give (using $\sum_{j=1}^{M(N)} y_j^2 \le CN$)

$$\begin{aligned}\left|N - \sum_{j=1}^{M(N)} Y_j^2\right| &= \left|N - \sum_{j=1}^{M(N)} (y_j^2 - 2y_jv_j + v_j^2)\right| \\ &\le \left|N - \sum_{j=1}^{M(N)} y_j^2\right| + \sum_{j=1}^{M(N)} v_j^2 + 2\sqrt{\sum_{j=1}^{M(N)} y_j^2 \sum_{j=1}^{M(N)} v_j^2} \\ &\le C(a)N^{2\gamma} + CN^{2\gamma} + C\sqrt{NN^{4\gamma-1}} \le C(a)N^{2\gamma}.\end{aligned}$$

However, since we have $v_j^2 \le Cj^{10\delta/3}$ (by equation (4.30)), we find, using $\delta < 2(\gamma - 2/5)$,

$$\sum_{j=1}^{M(N)} v_j^2 \le CM^{1+10\delta/3} \le N^{3/5+2\delta} \le CN^{4\gamma-1}.$$

It remains to prove equation (4.29). Set $R_j = Y_j^2 - E(Y_j^2 \mid \mathcal{L}_{j-1})$ and observe that $\{R_j, \mathcal{L}_j\}$ is a martingale difference sequence. By Minkowski's inequality

$$E(R_j^2) \le \left(\sqrt{E(Y_j^4)} + \sqrt{E(E(Y_j^2 \mid \mathcal{L}_{j-1})^2)}\right)^2 \le \left(2\sqrt{E(Y_j^4)}\right)^2 = 4E(Y_j^4).$$

Since $Y_j = y_j - v_j$, we have, again by Minkowski's inequality,

$$\begin{aligned}E(R_j^2) &\le 4E(Y_j^4) \le 4((E(y_j^4))^{\frac{1}{4}} + (E(v_j^4))^{\frac{1}{4}})^4 \le C(E(y_j^4) + E(v_j^4)) \\ &\le C(E(w_j^4) + E(|w_j^4 - y_j^4|) + E(v_j^4)).\end{aligned}$$

Since $w_j^4 - y_j^4 = (w_j^2 + y_j^2)(w_j + y_j)(w_j - y_j)$, we get from equation (4.2) that $E(|w_j^4 - y_j^4|)$ is uniformly bounded. By equation (4.30), we have $|u_j| \le Cj^{5\delta/3}$.

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Hence, $|v_j| \le |u_j| + |u_{j-1}| \le Cj^{5\delta/3}$ and $E(v_j^4) \le Cj^{20\delta/3} \le Cj^{4/3}$, since $\delta < 1/5$. For arbitrary $\iota > 0$, the bound in equation (4.23) gives C such that $E(w_j^4) \le Cj^{4/3+\iota}$. Thus,

$$\sum_{j\ge1} \frac{E(R_j^2)}{j^{7/3+\iota}} < \infty, \tag{4.31}$$

and a martingale result (see [**Ch**]) implies that $\sum_{j\ge1} R_j/j^{7/6+\iota}$ converges almost surely. For m_*-a.e. $a \in \Omega_*^\varphi$, Kronecker's lemma gives $C(a)$ with

$$\sum_{j=1}^{M(N)} R_j \le C(a)M^{7/6+\iota} \le C(a)N^{21/30+\iota},$$

using equation (4.4) in the last inequality. Since $21/30 < 2\gamma$, this establishes equation (4.29). $\square$

We shall apply the following embedding result. (See [**HH**, Theorem A.1].)

THEOREM 4.5. (Skorokhod's representation theorem) *For any zero-mean square-integrable martingale* $\{\sum_{k=1}^{j} Y_k,\ \mathcal{L}_j \mid j \ge 1\}$, *there exist a probability space supporting a (standard) Brownian motion W, and non-negative variables* $\{T_k,\ k \ge 1\}$, *such that* $\{\sum_{k=1}^{j} Y_k\}_{j\ge1}$ *and* $\{W(\sum_{k=1}^{j} T_k)\}_{j\ge1}$ *have the same distribution, and, in addition, letting* $\mathcal{G}_0$ *be the trivial* σ*-algebra (the empty set and the entire space), and* $\mathcal{G}_j$, *for* $j \ge 1$, *be the* σ*-algebra generated by*

$$\{W(t) \mid 0 \le t \le \tau_j\} \quad \textit{where } \tau_j := \sum_{k=1}^{j} T_k,$$

then τ_j *is* $\mathcal{G}_j$*-measurable, while* $E(T_1 \mid \mathcal{G}_0) = E(W(T_1)^2 \mid \mathcal{G}_0)$, *and*

$$E(T_j \mid \mathcal{G}_{j-1}) = E((W(\tau_j) - W(\tau_{j-1}))^2 \mid \mathcal{G}_{j-1}) \quad \textit{for all } j \ge 2, \ \textit{almost surely.}$$

By the last claim of Theorem 4.5 and properties of Brownian motion,

$$E(T_j \mid \mathcal{G}_{j-1}) = E(W(T_j)^2 \mid \mathcal{G}_{j-1}) \quad \text{for all } j \ge 1, \tag{4.32}$$

almost surely. (Indeed, letting W_1 be an independent copy of W, we have $W(\tau_j) = W_1(\tau_{j-1} + T_j) = W_1(\tau_{j-1}) + W(T_j)$ in distribution, so that $W(\tau_j) - W(\tau_{j-1}) = W(T_j)$ in distribution.)

We need one last lemma. Recall that $\gamma \in (2/5, 1/2)$ is fixed.

LEMMA 4.6. (Strong law of large numbers for the sequence T_j) *For* m_**-a.e.* $a \in \Omega_*^\varphi$, *there exists* $C(a)$ *such that*

$$\left| N - \sum_{j=1}^{M(N)} T_j \right| \le C(a)N^{2\gamma} \quad \textit{for all } N \ge 1. \tag{4.33}$$

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Proof. To start, apply Theorem 4.5 to the martingale difference sequence Y_j from equation (4.27), with $\mathcal{L}_j$ generated by $\{y_\ell\}_{1\le\ell\le j}$. Let $\tilde{Y}_j = W(\tau_j) - W(\tau_{j-1})$, so that $W(\tau_j) = \sum_{k=1}^{j} \tilde{Y}_k$ and $\tilde{Y}_j = W(T_j)$. By equation (4.32), we have, almost surely,

$$N - \sum_{j=1}^{M(N)} T_j = \bigg[N - \sum_{j=1}^{M} \tilde{Y}_j^2\bigg] + \sum_{j=1}^{M}[\tilde{Y}_j{}^2 - E(\tilde{Y}_j^2 \mid \mathcal{G}_{j-1})]$$
$$+ \sum_{j=1}^{M}[E(T_j \mid \mathcal{G}_{j-1}) - T_j] \quad \text{for all } N \geq 1.$$

Then, since Y_j and $\tilde{Y}_j$ have the same distribution, the bound in equation (4.28) in Lemma 4.4 gives $C(a)$ such that, for all $N \geq 1$, the first sum in the right-hand side above is not larger than $C(a)N^{2\gamma}$.

For the second sum in the right-hand side above, we use equation (4.29). Since conditional expectations can be expressed in terms of distributions, equation (4.29) is also valid with Y_j replaced by $\tilde{Y}_j$. Thus, the second sum in the right-hand side is also bounded by $C(a)N^{2\gamma}$ for all $N \geq 1$.

Finally, let $R_j = E(T_j \mid \mathcal{G}_{j-1}) - T_j$. Then, $\{R_j, \mathcal{G}_j\}$ is a martingale difference sequence by equation (4.32). As in the proof of equation (4.29), we can estimate $E(R_j^2) \leq 4E(W(T_j)^4)$, and thus there exists $C(a)$ such that, for all $N \geq 1$, we have $\sum_{j=1}^{M(N)} R_j \leq CN^{21/30+\iota} \leq C(a)N^{2\gamma}$ almost surely. □

Proof of Theorem 1.1. Just like Schnellmann, we follow the proof of [**PS**, Lemma 3.5.3], replacing their $1/2 - \alpha/2 + \gamma$ by γ, and replacing Lemma 3.5.1 there by our Lemma 4.6. We then obtain that, almost surely,

$$\bigg|\sum_{j=1}^{M(N)} Y_j - W(N)\bigg| = O(N^\gamma).$$

Then, using equations (4.30) and (4.4), we find

$$\bigg|\sum_{j=1}^{M(N)} y_j - Y_j\bigg| = \bigg|\sum_{j=1}^{M(N)} (u_{j+1} - u_j)\bigg| = |u_{M(N)+1} - u_1| \leq CN^\delta. \tag{4.34}$$

Since $\delta < 2/5$, and recalling equation (4.5), this establishes Theorem 1.1. □

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