Can. J. Math., Vol. XXXI, No. 1, 1979, pp. 93–111

RINGS CHARACTERIZED BY THEIR CYCLIC MODULES

P. F. SMITH

1. Introduction. A ring R (with identity element) is called a *right PCI-ring* if and only if every proper cyclic right R-module is injective; that is, if C is a cyclic right R-module then either $C \cong R$ or C is injective. Faith [**3**, Theorems 14 and 17] (or see [**2**, Proposition 6.12 and Theorem 6.17]) proved that if a ring R is a right PCI-ring then R is semiprime Artinian or R is a simple right semihereditary right Ore domain. These latter rings we shall call *simple right PCI-domains*. Examples of non-Artinian simple right PCI-domains were produced by Cozzens [**1**]. The object of this paper is to examine rings with similar properties and thus extend Faith's results.

Let S be a semiprime Artinian ring and T a simple right PCI-domain. Then the ring $A = S \oplus T$ is not a right PCI-ring by Faith's theorems. However, A has the property that every cyclic right A-module is projective or injective and we call rings with this property *right CPOI-rings*. Any right CPOI-ring is the direct sum of a semiprime Artinian ring and a simple right PCI-domain (Theorem 2.12). If $B = T \oplus T$ then B is neither a right PCI-ring nor a right CPOI-ring. Any right ideal E of B has the form $F \oplus G$ where F and G are right ideals of T and $B/E \cong (T/F) \oplus (T/G)$. It follows that every cyclic right B-module is the direct sum of a projective right B-module and an injective right B-module and rings with this property we call *right CDPI-rings*.

These classes of rings are related to a class of rings studied by Goodearl [**5**]. He called a ring R a *right SI-ring* in case every singular right R-module is injective. Recall that if R is a ring and X a right R-module then the *singular submodule* $Z(X)$ of X is the set of elements x of X such that $xE = 0$ for some essential right ideal E of R and X is called *singular* provided $X = Z(X)$. A ring R with right socle V is a right SI-ring if and only if R is a right RIC-ring and the ring R/V is right Noetherian (Theorem 3.1). By a *right RIC-ring* ("RIC" for restricted injective condition) we mean a ring R such that R/E is an injective right R-module for every essential right ideal E of R. It is not difficult to see that right RIC-rings are precisely the rings R for which every finitely generated singular right R-module is injective. Recall that Osofsky [**7**] proved that a ring R is semiprime Artinian if and only if every cyclic right R-module is injective.

A ring R is a *right V-ring* (after Villamayor) if and only if every simple right R-module is injective. If R is a right RIC-ring with right socle S then the ring R/S is a right V-ring (Lemma 2.5). It follows immediately by [**6**, Theorem 2.1] that if X is a singular right module over a right RIC-ring then X has zero

Received August 10, 1977.

Jacobson radical. By contrast, a ring R is a right SI-ring if and only if $Z(R) = 0$ and every singular right R-module is semisimple (see [**5**, Proposition 3.1]).

There is one final class of rings we wish to mention. This is the class of rings R such that every cyclic right R-module is the extension of a projective right R-module by an injective right R-module and we call such rings *right CEPI-rings*. There are the following implications:

semiprime Artinian ring $\Rightarrow$ right PCI-ring $\Rightarrow$ right CPOI-ring $\Rightarrow$
right CDPI-ring $\Rightarrow$ right CEPI-ring $\Rightarrow$ right RIC-ring

and

semiprime Artinian ring $\Rightarrow$ right SI-ring $\Rightarrow$ right CEPI-ring.

We do not know whether right PCI-rings are right SI-rings. This is true if and only if right PCI-rings are right Noetherian (see [**5**, Proposition 3.6]) and this is a question of Cozzens and Faith [**2**, p. 109]. If it is true then right CPOI-rings are right SI-rings. If K is a field then the ring of 2×2 upper triangular matrices with entries in K is a right Artinian right SI-ring but is not a right CPOI-ring [**5**, Theorem 3.11]. We do not know if

right RIC-ring $\Rightarrow$ right CEPI-ring and
right CEPI-ring $\Rightarrow$ right SI-ring

are true.

With additional assumptions about the rings these questions can be answered. For example for right Noetherian rings the following implications hold:

right CDPI-ring $\Rightarrow$ right SI-ring $\Leftrightarrow$ right RIC-ring

and an example is given in § 4 of a right and left Artinian right and left SI-ring which is not a right CDPI-ring. On the other hand, for commutative rings we have

semiprime Artinian ring $\Leftrightarrow$ CDPI-ring $\Rightarrow$ SI-ring $\Leftrightarrow$ RIC-ring.

Perhaps we ought to note that all of the classes of rings discussed above are closed under taking homomorphic images and the classes of right SI-rings, right CDPI-rings, right CEPI-rings and right RIC-rings are closed under taking finite direct sums.

Before proceeding with the proofs of these results we mention two items of notation. If R is a ring and A is a non-empty subset of R then the *right annihilator* of A is $\{r \in R : Ar = 0\}$ and is denoted by $r(A)$. The *left annihilator* of A is $l(A) = \{r \in R : rA = 0\}$. If $a \in R$ and $A = \{a\}$ then we shall write $r(a)$ and $l(a)$ for $r(A)$ and $l(A)$, respectively. Also if S and T are rings and M is a left S-, right T-bimodule then

$$\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$$

will denote the ring of "matrices"

$$\begin{bmatrix} s & m \\ 0 & t \end{bmatrix}$$

with s in S, m in M and t in T, and with the usual matrix addition and multiplication. Recall that if R is a ring and e is an idempotent element of R such that $(1 - e)Re = 0$ then

$$R \cong \begin{bmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{bmatrix}.$$

2. Cyclic modules are projective or injective. Recall the following result of Osofsky [**7**].

LEMMA 2.1. *A ring R is semiprime Artinian if and only if every cyclic right R-module is injective.*

Let R be a right CPOI-ring such that R is right self-injective. Let I be a right ideal of R. If the right R-module R/I is projective then $R \cong I \oplus (R/I)$ and it follows that R/I is injective. Thus every cyclic right R-module is injective and R is semiprime Artinian by Lemma 2.1. Thus right self-injective right CPOI-rings are semiprime Artinian.

LEMMA 2.2. *Let R be a right* CPOI*-ring and A be a proper ideal of R. Then either $A = eR$ for some idempotent element e of R or the ring R/A is semiprime Artinian.*

Proof. If the right R-module R/A is projective then $R = A \oplus I$ for some right ideal I of R and hence $A = eR$ for some idempotent element e. Otherwise, R/A is an injective right R-module and hence an injective right (R/A)-module. The above remark shows that in this case the ring R/A is semiprime Artinian.

COROLLARY 2.3. *Let R be a right* CPOI*-ring with Jacobson radical J. Then either $J = 0$ or the ring R/J is semiprime Artinian.*

LEMMA 2.4. *Right* CEPI*-rings are right semihereditary right* RIC*-rings.*

Proof. Let R be a right CEPI-ring. Let n be a positive integer and I be a right ideal of R generated by elements $a_1, a_2, \ldots, a_n$. If $n = 1$ let $J = 0$ and if $n > 1$ let $J = a_1R + a_2R + \ldots + a_{n-1}R$. Suppose that J is projective. Because I/J is cyclic there exists a right ideal K of R such that $J \subseteq K \subseteq I$, K/J is projective and I/K is injective. Since K/J is projective it follows that $K \cong J \oplus (K/J)$ and hence K is projective. Since I/K is injective it follows that $R/K = (I/K) \oplus (L/K)$ for some right ideal L containing K. Then $R = I + L$ and $K = I \cap L$, and we can form the exact sequence

$$0 \rightarrow K \xrightarrow{\lambda} I \oplus L \xrightarrow{\mu} R \rightarrow 0$$

https://doi.org/10.4153/CJM-1979-011-2 Published online by Cambridge University Press

where $\lambda(k) = (k, k)$ for each k in K and $\mu(i, b) = i - b$ for each i in I, b in L. Thus $I \oplus L \cong R \oplus K$ and, because K is projective, it follows that I is projective. By induction on n it follows that R is right semihereditary.

Now let E be an essential right ideal of R. There exists a right ideal F containing E such that F/E is a projective R-module and R/F is an injective R-module. But F/E projective implies that $F = E \oplus G$ for some right ideal G of R and hence $F = E$ because E is essential. Hence R/E is an injective R-module. Thus R is a right RIC-ring.

LEMMA 2.5. *Let R be a right* RIC*-ring with Jacobson radical J and right socle S. Then the ring R/S is a right* V*-ring and $J \subseteq S$.*

Proof. Let M be a maximal right ideal of R with $S \subseteq M$. Then M is an essential right ideal of R and the right R-module R/M is injective. Thus R/M is an injective right (R/S)-module and it follows that R/S is a right V-ring. By [**6**, Theorem 2.1] $J \subseteq S$.

LEMMA 2.6. *Let R be a right* CPOI*-ring with Jacobson radical J. Then $J = 0$.*

Proof. Suppose on the contrary that $J \neq 0$. By Corollary 2.3 the ring R/J is semiprime Artinian. If S is the right socle of R then by Lemma 2.5 $J \subseteq S$ and hence $J^2 = 0$. Thus R is semiprimary and does not contain an infinite collection of orthogonal idempotents. Hence without loss of generality we can suppose that R is indecomposable.

Let A be an ideal of R. By Lemma 2.2 either $A \cap J = eR$ for some idempotent element e of R and so $A \cap J = 0$ or the ring $R/(A \cap J)$ is semiprime Artinian in which case $J \subseteq A$. If $A \cap J = 0$ then R/A is not a semiprime Artinian ring and by Lemma 2.2 there exists an idempotent element f of R such that $A = fR$. Moreover, $A \cap l(A)$ and $A \cap r(A)$ are both nilpotent ideals of R and hence $A \cap l(A) \subseteq J$ and $A \cap r(A) \subseteq J$. Thus $A \cap l(A) = 0$, $A \cap r(A) = 0$ and it follows that $l(A) = r(A)$. Now $l(A) = R(1 - f)$ and so $fR(1 - f) = 0$ and $fR \subseteq Rf$. That is, $A = fR = Rf$ and $A = 0$ since R is indecomposable. Thus $J \subseteq A$ for every non-zero ideal A of R.

Since R/J is semiprime Artinian it follows that the right annihilator $r(J)$ of J is a semisimple left R-module and hence $r(J) = T$, where T is the left socle of R. Let $M_1, M_2, \ldots, M_k$ be the maximal ideals of R. If $J \subseteq JM_i$ for each $1 \leqq i \leqq k$ then $J \subseteq J(M_1M_2 \ldots M_k) \subseteq J^2 = 0$. Thus there exists $1 \leqq i \leqq k$ such that $JM_i = 0$ and hence T is a maximal ideal of R. Moreover, since R is right semihereditary by Lemma 2.4 and R does not contain an infinite collection of orthogonal idempotents it follows by [**9**, Theorem 1] that $T = tR$ for some non-trivial idempotent element t of R. Then $Rt \subseteq tR$ and $(1 - t)Rt = 0$. Thus without loss of generality we can identify R with the ring

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where $A = tRt$, $B = tR(1 - t)$ and $C = (1 - t)R(1 - t) = (1 - t)R \cong R/T$.

Note that C is a simple Artinian ring and B is a semisimple right C-module. Let I be the right ideal

$$\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$$

of R. Then we claim that the right R-module R/I is neither projective nor injective. Consider the right C-module B. Let V be a minimal right ideal of C. Then there exist an index set Λ, submodules B_λ of B and isomorphisms $f_\lambda : B \rightarrow V$ for each λ in Λ, such that $B = \bigoplus_{\lambda \in \Lambda} B_\lambda$. Define $h : I \rightarrow R/I$ by

$$\begin{bmatrix} 0 & \sum b_\lambda \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & \sum f_\lambda(b_\lambda) \end{bmatrix} + I$$

where $b_\lambda \in B_\lambda (\lambda \in \Lambda)$ and $b_\lambda \neq 0$ for only a finite number of λ in Λ. Then h is a homomorphism of the right R-module I to the right R-module R/I. If R/I is injective then there exists an element

$$r = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

of R such that $h(i) = ri + I$ for each i in I. Let $x \in B_\lambda$. Then

$$h\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} + I = \begin{bmatrix} 0 & ax \\ 0 & 0 \end{bmatrix} + I = 0,$$

and it follows that $x = 0$. Thus R/I injective implies that $B = 0$ and hence $R = A \oplus C$. This contradicts the fact that R is indecomposable. On the other hand, if the right R-module R/I is projective then $I = jR$ for some idempotent element j of R and $I = 0$ because $I \subseteq J$. Again the fact that R is indecomposable is contradicted. This proves that $J = 0$, as required.

LEMMA 2.7. *Let e be an idempotent element of a right* CPOI-*ring R. Then eR is a semisimple right R-module or $(1 - e)R$ is an injective right R-module.*

Proof. Suppose that eR is not semisimple. Then there exists a right ideal G of R such that $G \subseteq eR$ and G is a proper essential submodule of eR. If R/G is projective then $R = G \oplus H$ for some right ideal H and hence $eR = G \oplus (eR \cap H)$, a contradiction. Thus R/G is an injective right R-module. Consider the exact sequence

$$0 \rightarrow eR/G \rightarrow R/G \rightarrow R/eR \rightarrow 0.$$

Since $R/eR \cong (1 - e)R$ is projective it follows that $R/G \cong (R/eR) \oplus (eR/G)$ and hence $(1 - e)R$ is an injective module.

An ideal A of a ring R is called *von Neumann regular* if and only if every element a of A is von Neumann regular; that is, to each a in A there corresponds b in R such that $a = aba$. If a is an element of R such that R/aR is projective then $R = aR \oplus B$ for some right ideal B and it is an easy matter to show that this implies that the element a is von Neumann regular.

LEMMA 2.8. *Let R be a right* CPOI-*ring with right socle S. Then S is a von Neumann regular ideal of R.*

Proof. Suppose the result is false. Let $a \in S$, $a \notin aRa$. Then the above remark shows that R/aR is injective. Since aR is semisimple it follows that $aR = U_1 \oplus U_2 \oplus \ldots \oplus U_k$ where k is a positive integer and U_i is a minimal right ideal of R for each $1 \leqq i \leqq k$. Since R is semiprime (Lemma 2.6) it is well known that $U_i = e_iR$ where e_i is an idempotent element for each $1 \leqq i \leqq k$. Choose any $1 \leqq i \leqq k$ and let $e = e_i \in S$. If $(1 - e)R \subseteq S$ then $R = eR \oplus (1 - e)R$ is semiprime Artinian and S is von Neumann regular, a contradiction. Thus $(1 - e)R \nsubseteq S$ and eR is injective by Lemma 2.7. Hence e_iR is injective for each $1 \leqq i \leqq k$ and this implies that aR is injective. By Lemma 2.4 aR is projective and hence $R \cong (aR) \oplus (R/aR)$. This implies that R is right self-injective. But we have already remarked that right self-injective right CPOI-rings are semiprime Artinian and again S is von Neumann regular, a contradiction. Thus S is von Neumann regular as required.

LEMMA 2.9. *Let R be a right* CPOI-*ring with zero right socle. Then R is a simple right* PCI-*domain.*

Proof. Let I be a proper right ideal of R. If R/I is a projective right R-module then, as before, $I = eR$ for some idempotent element e of R. If $e = 0$ then $I = 0$. Otherwise, eR and $(1 - e)R$ are both injective by Lemma 2.7 and hence $R = eR \oplus (1 - e)R$ is right self-injective. This implies that R is semiprime Artinian, a contradiction. It follows that if $I \neq 0$ then R/I is injective. Thus R is a right PCI-ring and by [**3**, Theorem 14] a simple right PCI-domain.

Note that the above argument gives immediately the fact that R is an integral domain. For if R is a right CPOI-ring with zero right socle and if $a \in R$ then $r(a) = eR$ for some idempotent element e of R because aR is projective (Lemma 2.4). Since $e = 0$ or 1 by the proof of Lemma 2.9 it follows that R is an integral domain.

LEMMA 2.10. *If R is a right* RIC-*ring then the right singular ideal $Z(R)$ is zero.*

Proof. Let $x \in Z(R)$ and $E = r(x)$. Then $xR \cong R/E$ is an injective right R-module and hence $R = xR \oplus I$ for some right ideal I. Thus $xR = eR$ for some idempotent element e. But this implies that $e \in Z(R)$ and hence $(1 - e)R$ is an essential right ideal of R. Thus $eR = 0$ and hence $x = 0$. It follows that $Z(R) = 0$.

LEMMA 2.11. *Let R be a von Neumann regular right* CPOI-*ring. Then R is semiprime Artinian.*

Proof. Let S be the right socle of R. If $S = 0$ then R is an integral domain by Lemma 2.9 and since R is von Neumann regular it follows easily that R is a division ring, a contradiction. Thus $S \neq 0$. Let

$$K = \{r \in R : rE \subseteq S \text{ for some essential right ideal } E \text{ of } R\}.$$

Then K is an ideal of R and $S \subseteq K$. By Lemma 2.2 $K = eR$ for some idempotent element e or the ring R/K is semiprime Artinian. Suppose firstly that $K \neq R$ and the ring R/K is semiprime Artinian. Let $V = U/K$ be a minimal right ideal of R/K where U is a right ideal containing K. Then $V \cong R/M$ for some maximal right ideal M containing K. Since $S \subseteq K \subseteq M$ it follows that M is an essential right ideal. Then $uM \subseteq K$ for some element u in U but not K and it can easily be checked that since $Z(R) = 0$ (Lemma 2.10) this implies that $u \in K$, a contradiction. Thus $K = eR$ for some ideompotent element e. Since R is semiprime by Lemma 2.6 it follows that $K \cap r(K) = 0$, $K \cap l(K) = 0$, $l(K) = r(K)$ and hence $K = Re$ (see the proof of Lemma 2.6). Thus $R = K \oplus L$ where L is the ideal $(1 - e)R$. If $L \neq 0$ then L is a von Neumann regular right CPOI-ring with zero right socle and as before we get a contradiction. Thus $L = 0$ and $R = K$. Since $Z(R) = 0$ it follows that S is an essential right ideal of R.

Next we claim that R does not contain an infinite direct sum of non-zero ideals. For suppose otherwise and let $I_1 \oplus I_2 \oplus I_3 \oplus \ldots$ be a direct sum of non-zero ideals I_j of R. Let $G = I_1 \oplus I_3 \oplus I_5 \oplus \ldots$ and $H = I_2 \oplus I_4 \oplus I_6 \oplus \ldots$. If the right R-module R/G is projective then $G = gR$ for some idempotent element g and thus there exists a positive integer n_0 such that $I_{2n+1} = 0$ for all $n \geqq n_0$, a contradiction. On the other hand by Lemma 2.2 the ring R/G is semiprime Artinian and hence H is a finitely generated right ideal so that there exists a positive integer m_0 such that $I_{2m} = 0$ for all $m \geqq m_0$, another contradiction. Thus R does not contain an infinite direct sum of non-zero ideals and without loss of generality we can suppose that R is indecomposable.

If I is an ideal of R such that $I = fR$ for some idempotent element f then by the argument used earlier in this proof $R = I \oplus J$ where J is the ideal $(1 - f)R$. Since R is indecomposable it follows that $I = 0$ or $I = R$.

Recall that $S = \cap \{E : E \text{ is an essential right ideal of } R\}$. Let C be a non-zero ideal of R. If C has non-zero left annihilator D then by Lemma 2.2 and the last remark it follows that the rings R/C and R/D are both semiprime Artinian. Since $D \cap C = 0$ we deduce that R is right Artinian and because R is von Neumann regular as well, R is semiprime Artinian as required. So suppose that every non-zero ideal C has zero left annihilator. In this case C is an essential right ideal and $S \subseteq C$. Thus we can suppose that S is contained in every non-zero ideal of R. By Lemma 2.6 $S = S^2$. Thus R is a prime ring.

We have seen already that S is an essential right ideal of R. By Lemma 2.2 it follows that $S = R$ and the result is proved or the ring R/S is semiprime Artinian. Suppose that the latter is the case. If S is a finitely generated right ideal of R then R is right Artinian and the result follows. Therefore suppose that S is not finitely generated. There exist an infinite index set Λ and minimal

https://doi.org/10.4153/CJM-1979-011-2 Published online by Cambridge University Press

right ideals $S_\lambda (\lambda \in \Lambda)$ such that $S = \oplus_{\lambda \in \Lambda} S_\lambda$ and, because R is prime, $S_\lambda \cong S_\mu$ for all λ, μ in Λ. Thus there exist submodules X and Y of the right R-module S such that $S = X \oplus Y$, X is not finitely generated and $Y \cong S$. If $E(A)$ denotes the injective hull of an arbitrary right R-module A then $E(R) \cong E(S) \cong E(Y)$.

Now consider the right R-module R/X. Since X is not finitely generated it follows that R/X is not projective. Thus R/X is injective and Faith's argument (see [**3**, p. 106, proof of Proposition 12]) goes through to show that R is right self-injective and hence semiprime Artinian. This completes the proof of Lemma 2.11.

THEOREM 2.12. *A ring R is a right* CPOI-*ring if and only if R is a direct sum $V \oplus T$ of a semiprime Artinian ring V and a simple right* PCI-*domain T.*

Proof. Let R be a right CPOI-ring. Let S be the right socle of R. By Lemma 2.8 S is a von Neumann regular ideal of R and by Zorn's Lemma S is contained in a maximal von Neumann regular ideal V. It can easily be checked that the maximality of V implies that the ring R/V contains no non-zero von Neumann regular ideals. By Lemma 2.2, $V = vR$ for some idempotent element v of R. Since R is semiprime (Lemma 2.6) it follows that $T = (1 - v)R$ is an ideal and $R = V \oplus T$ (see the first paragraph of the proof of Lemma 2.11). By Lemma 2.11 V is semiprime Artinian and by Lemma 2.9 T is a simple right PCI-domain.

Conversely, let V be a semiprime Artinian ring, T a simple right PCI-domain and $R = V \oplus T$. Let E be a right ideal of R. Then there exist right ideals F and G of V and T, respectively, such that $E = F \oplus G$. Moreover, $R/E \cong (V/F) \oplus (T/G)$ and it easily follows that R/E is projective or injective. Thus R is a right CPOI-ring.

3. Right SI-rings. Let R be a ring and X a right R-module. Then X has *finite Goldie dimension* if and only if X does not contain an infinite direct sum of non-zero R-submodules. In this case there exists a positive integer n, called the *Goldie dimension* of X, such that every maximal direct sum of submodules of X contains n non-zero members (see [**4**, Theorem 1.1]). Moreover if $X = R$ we say that R has *finite right Goldie dimension* and call n the *right Goldie dimension* of R. A ring R is called a *right Goldie ring* in case R has finite right Goldie dimension and satisfies the ascending chain condition on right annihilators.

Recall that a ring R is a right SI-ring if and only if every singular right R-module is injective. Goodearl [**5**, Proposition 3.6] proved that if R is a right SI-ring with right socle S then the ring R/S is right Noetherian. In addition, it is clear that right SI-rings are right RIC-rings. In fact together these properties characterize right SI-rings.

THEOREM 3.1. *Let R be a ring with right socle S. Then R is a right* SI-*ring if and only if R is a right* RIC-*ring and the ring R/S is right Noetherian.*

Proof. By the above remarks we need prove only the sufficiency. So suppose that R is a right RIC-ring and R/S is right Noetherian. If X is a singular right R-module then $XS = 0$, because S is the intersection of all essential right ideals of R. Thus without loss of generality we may suppose that R is right Noetherian. By Zorn's Lemma X contains a maximal injective submodule A, because every right ideal of R is finitely generated. Then $X = A \oplus B$ for some submodule B of X and we claim that $B = 0$. If $B \neq 0$ let b be a non-zero element of B. Then $bR \cong R/E$ where $E = \{r \in R : br = 0\}$ is an essential right ideal of R. Since R is a right RIC-ring it follows that bR is injective and hence so is $A \oplus bR$, contradicting the choice of A. Thus $B = 0$ and $X = A$ is injective. It follows that R is a right SI-ring.

Let R be a right RIC-ring with right socle S. If the ring R/S is right Noetherian then R is a right SI-ring by Theorem 3.1. Now suppose that R/S is a right Goldie ring. Note that if right Goldie right RIC-rings are right Noetherian then right PCI-rings are right Noetherian by [**3**, Theorem 17]. By adapting the proof of [**5**, Theorem 3.11] we obtain the following information about the structure of R.

LEMMA 3.2. *Let R be a right* RIC-*ring with right socle S such that the ring R/S is right Goldie. Then R is a finite direct sum $A \oplus B_1 \oplus B_2 \oplus \ldots \oplus B_n$ where $S \subseteq A$, A/S is a semiprime Artinian ring and B_i is a simple right Goldie right* RIC-*ring for each $1 \leqq i \leqq n$.*

Proof. Let $A = \{r \in R : rE \subseteq S$ for some essential right ideal E of $R\}$. Then A is an ideal of R and S is an essential right ideal of A by Lemma 2.10. As in the proof of [**5**, Theorem 3.11] there exists an ideal B of R such that $A \cap B = 0$, $(A \oplus B)/A$ is an essential right ideal of the ring R/A and R/A has zero right socle. Since the right R-module $B \cong (B \oplus S)/S$ has finite Goldie dimension it follows that R/A has finite right Goldie dimension. It can easily be checked that $Z(R/A) = 0$ and hence by [**4**, Theorem 2.3(iii)] R/A is a right Goldie ring. Moreover, by Lemma 2.5 R/A is a right V-ring and is semiprime by [**6**, Theorem 2.1]. Thus $(A \oplus B)/A$ contains a regular element $c + A$ of the ring R/A where $c \in R$. Let $\bar{R} = R/A$, $\bar{c} = c + A$. Clearly $\bar{R}/\bar{c}\bar{R} \cong \bar{c}\bar{R}/\bar{c}^2\bar{R}$. Since $\bar{R}$ is a right RIC-ring it follows that $\bar{c}\bar{R}/\bar{c}^2\bar{R}$ is an injective right $\bar{R}$-module and hence there exists a right ideal I of $\bar{R}$ containing $\bar{c}^2$ such that $\bar{R}/\bar{c}^2\bar{R} = (\bar{c}\bar{R}/\bar{c}^2\bar{R}) \oplus (I/\bar{c}^2\bar{R})$. As in the proof of [**5**, Theorem 3.11] this implies that $\bar{R} = \bar{R}\bar{c} + \bar{c}\bar{R}$ and hence $R = A \oplus B$.

By [**6**, Corollary 2.2 and Lemma 3.1] the right Goldie right V-ring B is a finite direct sum $B_1 \oplus B_2 \oplus \ldots \oplus B_n$ of simple right Goldie rings B_i $(1 \leqq i \leqq n)$. Clearly each ring B_i is a right RIC-ring. Note that A is also a right RIC-ring. If C is a right ideal of the ring $\bar{A} = A/S$ then $C = D/S$ for some right ideal D of A containing S. Then $E = D \oplus B$ is an essential right ideal of R and $\bar{A}/C \cong A/D \cong R/E$ is injective. By Lemma 2.1 it follows that $\bar{A}$ is semiprime Artinian.

COROLLARY 3.3. *Let R be a right Goldie right* RIC*-ring with right socle S. Then the ring R/S is a right Goldie ring.*

Proof. In the notation of Lemma 3.2, the right R-module B has finite Goldie dimension and hence so does the essential right ideal $(B \oplus A)/A$ of the ring R/A. It follows that R/A is a right Goldie ring by [**4**, Theorem 2.3(iii)] and Lemma 2.10. By the proof of Lemma 3.2 the ring $R/S \cong (A/S) \oplus (R/A)$ is a right Goldie ring.

LEMMA 3.4. *Let R be a right* RIC*-ring with right socle S such that the ring R/S is right Goldie. Then R is right semihereditary.*

Proof. In the notation of Lemma 3.2, $R = A \oplus B_1 \oplus B_2 \oplus \ldots \oplus B_n$. Since A is a right SI-ring by [**5**, Proposition 3.1] it follows that A is right hereditary by [**5**, Proposition 3.3(d)]. Thus it is sufficient to prove the result when R is a simple right Goldie ring. Let E be a finitely generated right ideal of R. Since R is a right Goldie ring there exists a finitely generated right ideal F of R such that $E \cap F = 0$ and $G = E \oplus F$ is an essential right ideal of R. By [**4**, Theorem 3.9] there exists a regular element c of R such that $c \in G$. Then $X = G/cR$ is a finitely generated singular right R-module by [**4**, Theorems 3.9 and 4.1]. Thus X is injective and there exists a right ideal H containing cR such that $R/cR = (G/cR) \oplus (H/cR)$. By the proof of Lemma 2.4 $R \oplus (cR) \cong G \oplus H$. Thus G, and hence E, is projective. It follows that R is right semihereditary.

Note that in the above proof G, and hence E, is a homomorphic image of $R \oplus (cR)$. Therefore E can be generated by two elements. This gives the next result.

COROLLARY 3.5. *Let R be a semiprime right Goldie right* RIC*-ring. Then every finitely generated right ideal of R can be generated by two elements.*

In Lemma 2.4 we saw that right CEPI-rings are right semihereditary right RIC-rings. We have the following partial converse.

LEMMA 3.6. *Right hereditary right* RIC*-rings are right* CEPI*-rings.*

Proof. Let R be a right hereditary right RIC-ring. Let E be a right ideal of R. There exists a right ideal F of R such that $E \oplus F$ is an essential right ideal of R. If $G = E \oplus F$ then R/G is an injective right R-module because R is a right RIC-ring. Since R is right hereditary it follows that the right R-module $G/E \cong F$ is projective. Thus R is a right CEPI-ring.

Combining Lemma 3.6 with [**5**, Proposition 3.3 (d)] we have immediately:

COROLLARY 3.7. *Right* SI*-rings are right* CEPI*-rings.*

LEMMA 3.8. *Let R be a right* RIC*-ring and I an ideal of R with zero left annihilator. Then the ring R/I is semiprime Artinian.*

Proof. Let E be a right ideal of R containing I. Since I is an essential right ideal of R then so is E and it follows that R/E is an injective (R/I)-module. Thus every cyclic right (R/I)-module is injective. By Lemma 2.1 R/I is semiprime Artinian.

Note that if R is a right RIC-ring such that the right socle S of R is an essential right ideal then $l(S) = 0$ by Lemma 2.10 and R/S is a semiprime Artinian ring by Lemma 3.8. In this case R is a right SI-ring by Theorem 3.1.

THEOREM 3.9. *The following statements are equivalent for a commutative ring R with socle S.*

(i) *R is an* SI-*ring.*
(ii) *R is a* CEPI-*ring.*
(iii) *R is an* RIC-*ring.*
(iv) *R is a von Neumann regular ring and R/S is semiprime Artinian.*

Proof. By Corollary 3.7, (i) implies (ii). By Lemma 2.4, (ii) implies (iii). Suppose (iii) holds. If I is an essential ideal of R then R/I is a semisimple R-module by Lemma 3.8. By [**5**, Proposition 3.1] and Lemma 2.10 it follows that R is an SI-ring. Finally, the equivalence of (i) and (iv) is proved in [**5**, Theorem 3.9].

4. Right CDPI-rings.

We begin this section with a characterization of right CDPI-rings.

LEMMA 4.1. *A ring R is a right* CDPI-*ring if and only if for every right ideal E of R there exists an idempotent element e of R such that E is contained in eR and the right R-module eR/E is injective.*

Proof. Let R be a right CDPI-ring and let E be a right ideal of R. Then there exist right ideals F, G of R containing E such that F/E is projective, G/E is injective and $R/E = (F/E) \oplus (G/E)$. Then $F/E \cong R/G$ and R/G is a projective right R-module. It follows that $G = eR$ for some idempotent element e. Thus $E \subseteq G = eR$ and eR/E is injective. Conversely, suppose that R has the stated property. Let A be a right ideal of R. By hypothesis there exists an idempotent element f of R such that $A \subseteq fR$ and fR/A is injective. Let $B = (1 - f)R \oplus A$. Then $B/A \cong (1 - f)R$ is projective and $R/A = (B/A) \oplus (fR/A)$. It follows that R is a right CDPI-ring.

COROLLARY 4.2. *Let R be an integral domain. Then R is a simple right* PCI-*domain if and only if R is a right* CDPI-*ring.*

Proof. We can use the lemma since the only idempotent elements of R are the trivial ones 0, 1.

LEMMA 4.3. *Let R be a right* CDPI-*ring and E be a right ideal of R such that R/E is an injective right R-module. Then R/F is an injective right R-module for every right ideal F containing E.*

Proof. Let F be a right ideal of R containing E. There exist a projective right R-module P and an injective right R-module Q such that $R/F = P \oplus Q$. Then P is a homomorphic image of R/E and hence $R/E \cong P \oplus P_1$ for some right R-module P_1. Thus P is injective and hence so is R/F.

COROLLARY 4.4. *Let R be a right* CDPI*-ring and I be an ideal of R such that the right R-module R/I is injective. Then the ring R/I is semiprime Artinian.*

Proof. By Lemmas 4.3 and 2.1.

COROLLARY 4.5. *Let R be a commutative* CDPI*-ring and I be an ideal of R. Then there exists an idempotent element e of R such that I is contained in eR and eR/I is semisimple.*

Proof. By Lemma 4.1 there exists an idempotent element e of R such that $I \subseteq eR$ and eR/I is injective. Let $A = \{r \in R : er \in I\}$. Then A is an ideal of R and the R-modules R/A and eR/I are isomorphic. By Corollary 4.4 R/A, and hence eR/I, is a semisimple R-module.

THEOREM 4.6. *A commutative ring R is a* CDPI*-ring if and only if R is a semiprime Artinian ring.*

Proof. If R is semiprime Artinian then R is a CDPI-ring by Lemma 2.1. Conversely, suppose that R is a CDPI-ring. By Theorem 3.9 it follows that R is von Neumann regular and R/S is semiprime Artinian, where S is the socle of R. Suppose that S is not finitely generated. Since R is semiprime, every minimal ideal of R is generated by an idempotent element. Thus there exist an infinite index set A and non-zero idempotent elements e_α $(\alpha \in A)$ of R such that $S = \oplus_A e_\alpha R$. Let B be a non-empty subset of A. If B is finite then, because R is von Neumann regular, there exists an idempotent element f of R such that $fR = \oplus_B e_\alpha R$. It follows that $fe_\alpha = e_\alpha (\alpha \in B)$ and $fe_\alpha = 0 (\alpha \in A, \alpha \notin B)$. Now suppose that B is infinite. Let $I = \oplus_B e_\alpha R$. By Corollary 4.5 there exists an idempotent element g of R such that $I \subseteq gR$ and gR/I is semisimple. Let $C = A \backslash B$. Because $\oplus_C ge_\alpha R$ is isomorphic to a submodule of gR/I there exist at most a finite number of elements α of C such that $ge_\alpha \neq 0$. Let $D = \{\alpha \in C : ge_\alpha \neq 0\}$. Then there exists an idempotent element h of R such that $\oplus_D e_\alpha R = hR$. Let $a = g(1 - h)$. Then $ae_\alpha = e_\alpha (\alpha \in B)$ and $ae_\alpha = 0$ $(\alpha \in A, \alpha \notin B)$. Let $r \in R$ satisfy $e_\alpha r = 0 (\alpha \in A)$. Then $rS = 0$ and since R/S is semiprime Artinian it follows that rR is semisimple. Thus $rR \subseteq S$ and hence $r^2 = 0$. Since R is von Neumann regular it follows that $r = 0$. By [**7**, Theorem] the R-module R/S is not injective. But every simple R-module is injective (see [**8**, Theorem 6]) and R/S is semisimple, and hence R/S is injective. This contradiction shows that S is finitely generated. Hence R is Artinian and, because R is von Neumann regular, R is semiprime.

Next we give a new characterization of right CDPI-rings R when R does not contain an infinite collection of orthogonal idempotents. A ring R is called a *right-PP ring* if and only if every principal right ideal of R is projective.

https://doi.org/10.4153/CJM-1979-011-2 Published online by Cambridge University Press

LEMMA 4.7. *Let R be a ring which does not contain an infinite set of orthogonal idempotents. Then R is a right* CDPI*-ring if and only if R is a right* PP*-ring such that $rl(E)/E$ is an injective right R-module for each right ideal E of R.*

Proof. Suppose that R is a right PP-ring and $rl(E)/E$ is injective for each right ideal E. (By $rl(E)$ we mean of course the right annihilator of the left annihilator of E!) For any right ideal E there exists an idempotent element e of R such that $rl(E) = eR$ by [**9**, Theorem 1]. Thus $E \subseteq eR$ and eR/E is injective. By Lemma 4.1 R is a right CDPI-ring. Conversely, suppose that R is a right CDPI-ring. Then R is a right PP-ring by Lemma 2.4. Let A be a right ideal of R. By Lemma 4.1 there exists an idempotent element b of R such that if $B = bR$ then $A \subseteq B$ and B/A is an injective module. Let $C = rl(A)$. Then $A \subseteq C \subseteq B$ and by [**9**, Theorem 1] $C = cR$ for some idempotent element c of R. It follows that $B = C \oplus D$ where D is the right ideal $(1 - c)R \cap B$. Then $B/A = (C/A) \oplus ((D + A)/A)$ and hence C/A is injective. This completes the proof.

THEOREM 4.8. *Let R be a semiprime right and left Goldie ring. Then R is a right* CDPI*-ring if and only if R is a right* RIC*-ring.*

Proof. If R is a right CDPI-ring then R is a right RIC-ring by Lemma 2.4. Conversely, suppose that R is a right RIC-ring. By Corollary 3.3 and Lemma 3.4 R is right semihereditary. By [**4**, Theorems 4.1 and 4.4] R has a classical right and left quotient ring Q which is semiprime Artinian. If E is a right ideal of R then the left annihilator of E in Q is

$$l(EQ) = \{c^{-1}r : r \in l(E) \text{ and } c \text{ is a regular element of } R\}.$$

Thus $rl(E)$ is contained in $rl(EQ)$. But because Q is semiprime Artinian $rl(EQ) = EQ$. It follows that $rl(E) \subseteq (EQ) \cap R$. It can now easily be checked that E is an essential submodule of the right R-module $rl(E)$. Then $rl(E)/E$ is an injective right R-module. The result follows by Lemma 4.7.

Recall that in Lemma 2.5 we saw that if R is a right RIC-ring with Jacobson radical J then J is a semisimple right R-module. If in addition R is a right Goldie ring then J is a finite direct sum of minimal right ideals. We can extend Theorem 4.8 to the case when J is a minimal right ideal.

THEOREM 4.9. *Let R be a right and left Goldie ring with Jacobson radical J such that $J = 0$ or J is a minimal right ideal. Then R is a right* CDPI*-ring if and only if R is a right* RIC*-ring.*

Proof. The necessity is a consequence of Lemma 2.4. Conversely, suppose that R is a right RIC-ring. By Corollary 3.3 and Lemma 3.4 R is right semihereditary. By Lemma 3.2 R is a direct sum $A \oplus B_1 \oplus B_2 \oplus \ldots \oplus B_n$ where A is a right Artinian ring and B_i is a simple right and left Goldie ring for each $1 \leqq i \leqq n$. By Theorem 4.8 B_i is a right CDPI-ring for each $1 \leqq i \leqq n$. Thus $B_1 \oplus B_2 \oplus \ldots \oplus B_n$ is a right CDPI-ring and the result will follow once we have proved that A is a right CDPI-ring. Thus without loss of generality we can suppose that $R = A$ is right Artinian and hence right hereditary. Also without loss of generality we can suppose that R is indecomposable.

If $J = 0$ then R is semiprime Artinian and hence a right CDPI-ring by Lemma 2.1. Therefore suppose that J is a minimal right ideal of R. Since by [**9**, Theorem 1] R has zero left singular ideal it follows that J is not an essential left ideal of R. Thus there exists a maximal ideal P of R such that P is not an essential left ideal. Let E be a non-zero left ideal of R such that $E \cap P = 0$. Then $PE = 0$ and it follows that $P = lr(P)$. By [**9**, Theorem 1] there exists an idempotent element e of R such that $P = Re$. Then $eR(1 - e) = 0$ and R is isomorphic to the ring

$$\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$$

where $S = (1 - e)R(1 - e) = R(1 - e) \cong R/Re = R/P$ is a simple Artinian ring, $T = eRe$ is a right Artinian ring and $M = (1 - e)Re$ is a left S-, right T-bimodule. Without loss of generality we can identify R with this ring of "matrices". Then

$$J = \begin{bmatrix} 0 & M \\ 0 & J_1 \end{bmatrix}$$

where J_1 is the Jacobson radical of T. Because R is indecomposable it follows that $J_1 = 0$ and M is a simple right T-module.

Let F be a right ideal of R. Suppose that $F \cap J \neq 0$. Then $J \subseteq F$ and it can easily be checked that there exist idempotent elements e of S and f of T such that

$$F = \begin{bmatrix} eS & M \\ 0 & fT \end{bmatrix}.$$

If $e \neq 0$ then eM is a T-submodule of M and hence $eM = M$ since S is a simple ring and M is a simple right T-module. It follows that in this case $F = hR$, where h is the idempotent element

$$h = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix},$$

and hR/F is an injective right R-module. If $e = 0$ then

$$F = \begin{bmatrix} 0 & M \\ 0 & fT \end{bmatrix}$$

and it can easily be checked that

$$rl(F) = \begin{bmatrix} S & M \\ 0 & fT \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & f \end{bmatrix} R.$$

https://doi.org/10.4153/CJM-1979-011-2 Published online by Cambridge University Press

Let G be the set of elements

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

of R such that

$$\begin{bmatrix} 1 & 0 \\ 0 & f \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in F.$$

Then

$$G = \begin{bmatrix} 0 & M \\ 0 & T \end{bmatrix}$$

and $rl(F)/F \cong R/G$. Since G is an essential right ideal of R it follows that R/G, and hence $rl(F)/F$, is an injective right R-module.

Now suppose that $F \cap J = 0$. Then F is contained in the ideal

$$\begin{bmatrix} 0 & M \\ 0 & T \end{bmatrix}$$

and $F = aR$ for some idempotent element a of R since T is semiprime Artinian. Thus aR/F is an injective right R-module. By Lemma 4.1 R is a right CDPI-ring and the theorem is proved.

COROLLARY 4.10. *Let R be a right and left Goldie ring with right socle S such that the right R-module S has Goldie dimension $\leqq 2$. Then R is a right* CDPI-ring *if and only if R is a right* RIC-*ring.*

Proof. In the notation of the proof of Theorem 4.9, $J \subseteq S \subseteq A$ where A is a right Artinian ring. Since J is not an essential right ideal of A it follows that $J = 0$ or J is a minimal right ideal. The result follows by the theorem.

COROLLARY 4.11. *Let R be a right and left Noetherian ring with Jacobson radical J such that $J = 0$ or J is a minimal right ideal. Then R is a right* SI-*ring if and only if R is a right* CDPI-*ring.*

To highlight these last few results we now give an example of a right and left Artinian right and left SI-ring of right Goldie dimension 3 (that is, the right socle of R has Goldie dimension 3 as a right R-module) such that the Jacobson radical of R is the direct sum of two minimal right ideals but R is not a right CDPI-ring.

Example 4.12. Let K be a field and V be a two-dimensional vector space over K. Let

$$R = \begin{bmatrix} K & V \\ 0 & K \end{bmatrix}.$$

It can easily be checked that R is a right and left Artinian ring and by [**5**, Proposition 3.1] is a right and left SI-ring. The right socle of R is

$$\begin{bmatrix} 0 & V \\ 0 & K \end{bmatrix}$$

and the Jacobson radical of R is

$$\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}.$$

Let V have K-basis v_1, v_2, and let E be the right ideal

$$\begin{bmatrix} 0 & Kv_1 \\ 0 & K \end{bmatrix}.$$

Then $l(E) = 0$ and $rl(E) = R$. We claim that R/E is not an injective right R-module. Let A be the right ideal

$$\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$$

and define $f : A \rightarrow R/E$ by

$$\begin{bmatrix} 0 & k_1v_1 + k_2v_2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & k_1v_2 \\ 0 & 0 \end{bmatrix} + E$$

for all elements k_1, k_2 of K. It can easily be checked that f is a homomorphism. If f can be lifted to a homomorphism $g : R \rightarrow R/E$ then there exist elements a, b of K and v of V such that $f(x) = cx$ for all elements x of A, where

$$c = \begin{bmatrix} a & v \\ 0 & b \end{bmatrix} + E.$$

In this case,

$$\begin{bmatrix} 0 & v_2 \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} a & v \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & v_1 \\ 0 & 0 \end{bmatrix} \pmod{E}$$

and hence $v_2 - av_1 \in Kv_1$, a contradiction. Thus f cannot be lifted to g and R/E is not injective as we claimed. By Lemma 4.7 R is not a right CDPI-ring.

Finally we look briefly at right Artinian rings which are right SI-rings or right CDPI-rings. Recall that if R is a right Noetherian ring then the maximal members of the set of ideals I of R with non-zero left annihilator are prime ideals of R, called the *maximal right annihilator prime ideals* of R.

THEOREM 4.13. *A ring R is a right Artinian right* SI*-ring if and only if R is semiprime Artinian or there exist semiprime Artinian rings S, T and a left S-, right T-bimodule M such that M is a faithful left S-module and a finitely generated right T-module and R is isomorphic to the ring*

$$\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}.$$

Proof. Semiprime Artinian rings are clearly right Artinian right SI-rings. Therefore suppose that S, T and M have the stated properties and

$$R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}.$$

Since M is a faithful left S-module it can easily be checked that R has right socle A given by

$$A = \begin{bmatrix} 0 & M \\ 0 & T \end{bmatrix}.$$

Since M is a finitely generated right T-module it follows that A is a finitely generated right R-module. But the ring $R/A \cong S$ and hence R is right Artinian and by [**5**, Theorem 3.11] R is a right SI-ring.

Conversely, let R be a right Artinian right SI-ring. Let J be the Jacobson radical of R and suppose that $J \neq 0$. By [**5**, Proposition 3.3(b)], $J^2 = 0$. Thus $l(J) \neq 0$ and by Lemma 2.10, J is not an essential right ideal of R. This implies that there exists a maximal ideal M of R such that M is not an essential right ideal. Let E be a non-zero right ideal of R such that $E \cap M = 0$. Then $EM = 0$ and hence $l(M) \neq 0$. There exist positive integers $1 \leqq k \leqq n$ such that the distinct maximal ideals of R are $M = M_1, M_2, \ldots, M_n$ and $B_i = l(M_i) \neq 0 (1 \leqq i \leqq k)$, $l(M_i) = 0 (k + 1 \leqq i \leqq n)$. Let $N = \bigcap_{i=1}^{k} M_i$. Then M_i is a maximal ideal implies that $M_i = r(B_i)(1 \leqq i \leqq k)$ and hence $N = r(B)$ where $B = B_1 + B_2 + \ldots + B_k$. By [**5**, Proposition 3.3(d)] R is right hereditary and by [**9**, Theorem 1] there exists a non-trivial idempotent element e such that $N = eR$. It follows that $(1 - e)Re = 0$ and

$$R \cong \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$$

where $S = eRe$, $T = (1 - e)R(1 - e) = (1 - e)R \cong R/eR = R/N$ is semiprime Artinian and $M = eR(1 - e)$ is a left S-, right T-bimodule. Since R is right Artinian it can easily be checked that S is right Artinian and M is a finitely generated right T-module. Now suppose that $s \in S$ satisfies $sM = 0$. Then there exists an element r of R such that $s = ere$ and $ere(eR(1 - e)) = 0$. That is, $(ereR)(1 - e) = 0$ and hence $(RereR)(1 - e) = 0$. If $RereR \neq 0$ then $1 - e$ belongs to a maximal right annihilator prime ideal of R. But this means that $1 - e \in M_i$ for some $1 \leqq i \leqq k$, and this contradicts the fact that $e \in M_i$. Thus $RereR = 0$ and hence $s = 0$. It follows that M is a faithful left S-module.

Now clearly

$$J = \begin{bmatrix} J_1 & M \\ 0 & 0 \end{bmatrix}$$

where J_1 is the Jacobson radical of S. Since $J^2 = 0$ it follows that $J_1M = 0$ and hence $J_1 = 0$. Thus S is semiprime and the theorem is proved.

LEMMA 4.14. *Let R be a right hereditary right Artinian ring. Then R is a right* CDPI*-ring if and only if $rl(E)/E$ is injective for every right ideal E contained in the right socle of R.*

Proof. The necessity is a consequence of Lemma 4.7. Conversely, suppose that $rl(E)/E$ is injective for every right ideal E contained in the right socle S of R. Let I be a right ideal of R and $J = I \cap S$. Then J is an essential submodule of I. It is well known that because R is right hereditary the right singular ideal $Z(R)$ of R is zero. For each element x of I there exists an essential right ideal F of R such that $xF \subseteq J$. Since $Z(R) = 0$ it follows that $l(J) = l(I)$ and hence $rl(J) = rl(I)$. By hypothesis $rl(I)/J$ is injective. Since R is right hereditary it follows that $rl(I)/I$ is injective. By Lemma 4.7 R is a right CDPI-ring.

THEOREM 4.15. *Let S be a semiprime Artinian ring and R be the ring*

$$\begin{bmatrix} S & S \\ 0 & S \end{bmatrix}.$$

Then R is a right CDPI*-ring.*

Proof. It can easily be checked that the right socle A of R and the Jacobson radical J of R are given by

$$A = \begin{bmatrix} 0 & S \\ 0 & S \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}.$$

Let E be a right ideal of R such that $E \subseteq A$. If $E \cap J = 0$ then, since S is semiprime Artinian, $E = xR$ for some idempotent element x of R. Thus in this case $E = rl(E)$ and $rl(E)/E$ is injective. Now suppose that $E \cap J \neq 0$. There exists a right ideal F contained in E such that $J \cap F = 0$ and $E = (E \cap J) \oplus F$. It can easily be checked that there exist idempotent elements e, g of S and an element f of S such that $f = fg$ and

$$E \cap J = \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix} R \quad \text{and} \quad F = \begin{bmatrix} 0 & f \\ 0 & g \end{bmatrix} R.$$

If

$$c = \begin{bmatrix} 1 - e & -(1 - e)f \\ 0 & 1 - g \end{bmatrix}$$

then it can be checked that c is an idempotent element of R and $l(E) = Rc$. Thus $rl(E) = (1 - c)R$. Another routine verification shows that if $B = \{r \in R : (1 - c)r \in E\}$ then $A \subseteq B$. Thus B is an essential right ideal of R. But R is a right SI-ring by Theorem 4.13 and hence R is a right RIC-ring. Thus $rl(E)/E \cong R/B$ is injective. By Lemma 4.14 it follows that R is a right CDPI-ring.

Example 4.16. Let K be a field and n a positive integer. Let $S = K_n$ be the

https://doi.org/10.4153/CJM-1979-011-2 Published online by Cambridge University Press

complete ring of nxn matrices with entries in K. Consider the ring

$$R = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}.$$

By Theorem 4.15 R is a right and left Artinian right CDPI-ring. If A and J denote the right socle and Jacobson radical of R, respectively, then

$$A = \begin{bmatrix} 0 & S \\ 0 & S \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}.$$

Thus the Goldie dimension of the right R-module A is 2n and J is the direct sum of n minimal right ideals of R. Contrast this example with Theorems 4.9 and 4.13.

Remark. Since submitting this paper I have discovered that Theorem 2.12 has also been proved by S. C. Goel, S. K. Jain and S. Singh in "Rings whose cyclic modules are injective or projective," Proc. Amer. Math. Soc. *53* (1975), 16–18.

References

1. J. H. Cozzens, *Homological properties of the ring of differential polynomials*, Bull. Amer. Math. Soc. *76* (1970), 75–79.

2. J. H. Cozzens and C. Faith, *Simple Noetherian rings* (Cambridge University Press, 1975).

3. C. Faith, *When are proper cyclics injective*? Pacific. J. Math. *45* (1973), 97–112.

4. A. W. Goldie, *Semiprime rings with maximum condition*, Proc. London Math. Soc. (3) *10* (1960), 201–220.

5. K. R. Goodearl, *Singular torsion and the splitting properties*, Amer. Math. Soc. Memoirs No. *124* (1972).

6. G. O. Michler and O. E. Villamayor, *On rings whose simple modules are injective*, J. Algebra *25* (1973), 185–201.

7. B. L. Osofsky, *Non-injective cyclic modules*, Proc. Amer. Math. Soc. *19* (1968), 1383–1384.

8. A. Rosenberg and D. Zelinsky, *Finiteness of the injective hull*, Math. Zeit. *70* (1959), 372–380.

9. L. W. Small, *Semiheriditary rings*, Bull. Amer. Math. Soc. *73* (1967), 656–658.

University of Glasgow
Glasgow, Scotland