COARSE DYNAMICS AND FIXED-POINT THEOREM

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Abstract. We study semigroup actions on a coarse space and the induced actions on the Higson corona from a dynamical point of view. Our main theorem states that if an action of an abelian semigroup on a proper coarse space satisfies certain conditions, the induced action has a fixed point in the Higson corona. As a corollary, we deduce a coarse version of Brouwer's fixed-point theorem.

§1. Introduction

A metric space X is *proper* if closed, bounded sets in X are compact. Let X and Y be proper metric spaces, and let $f\colon X \to Y$ be a map (not necessarily continuous). We define the following.

(a) The map f is *proper* if for each bounded subset B of Y, $f^{-1}(B)$ is a bounded subset of X.

(b) The map f is *bornologous* if for every $R > 0$ there exists $S > 0$ such that for each $x, y \in X$, $d(x, y) < R$ implies that $d(f(x), f(y)) < S$.

(c) The map f is *coarse* if it is proper and bornologous.

Let $f, g\colon X \to Y$ be maps. We define f as *close* to g, denoted $f \simeq g$, if there exists $R > 0$ such that $d(f(x), g(x)) < R$ for all $x \in X$. We define X and Y as *coarsely equivalent* if there exist coarse maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f$ and $f \circ g$ are close to the identity maps of X and Y, respectively. A *coarse space* is a coarsely equivalent class of proper metric spaces. The category of coarse spaces consists of coarse spaces and coarse maps.

Let $\varphi\colon X \to \mathbb{C}$ be a bounded continuous map. For each $r > 0$, we define a map $V_r\varphi\colon X \to \mathbb{R}$ by

$$V_r\varphi(x) := \sup\bigl\{|\varphi(y) - \varphi(x)| : d(x, y) < r\bigr\}.$$

We define φ as a *Higson function* if for each $r > 0$, $V_r\varphi$ vanishes at infinity. The Higson functions on a proper metric space X form a unital C^*-algebra, denoted by $C_h(X)$. It follows from the Gelfand-Naimark theorem that there exists a compactification hX of X such that $C(hX) = C_h(X)$. The compactification hX is called the *Higson compactification*. Its boundary $hX \setminus X$ is denoted by νX and called the *Higson corona*. The Higson corona is a functor from the category of coarse spaces into the category of compact Hausdorff spaces. Namely, a coarse map $f\colon X \to Y$ induces a unique continuous map $\nu f\colon \nu X \to \nu Y$, and moreover, if coarse maps $f, g\colon X \to Y$ are close, then $\nu f = \nu g$. It follows that the Higson coronae of two proper metric spaces which are coarsely equivalent to each other are homeomorphic. We note that the Higson corona of an unbounded proper metric space is never second countable and that its cardinality is greater than or equal to $2^{2^{\aleph_0}} = 2^{\mathfrak{c}}$. This means that the Higson corona is a huge, extremely complicated space. We refer to [5] for a general reference of coarse geometry and the Higson compactification.

Let X be a proper metric space, and let G be a finitely generated semigroup acting on X. Here G is a proper metric space with a left-invariant word metric. A coarse action, defined below, of G on X induces a continuous action of G on the Higson corona νX. The main subject of this article is to study fixed-point properties of these actions.

DEFINITION 1.1. An action of G on X is a *coarse action* if for each element g of G, the map $\Psi_g\colon X \to X$ defined by $x \mapsto g \cdot x$ is a coarse map.

DEFINITION 1.2. For a point x_0 of X, the orbit map $\Phi_{x_0}\colon G \to X$ is defined by $g \mapsto g \cdot x_0$. We define the following.

(a) The orbit of x_0 is *proper* if Φ_{x_0} is proper.
(b) The orbit of x_0 is *bornologous* if Φ_{x_0} is bornologous.
(c) The orbit of x_0 is *coarse* if Φ_{x_0} is coarse.

The action of G on itself is a typical example of the coarse action (see Lemma 2.1).

THEOREM 1.3. *Assume that $G = \mathbb{N}^k$ or $\mathbb{Z}^k$ and that G acts on X as a coarse action. Suppose that there exists a point x_0 of X whose orbit is coarse. Then the induced action of G on the Higson corona νX has a fixed point. Namely, there exists a point x of νX such that $g \cdot x = x$ for any element $g \in G$.*

Moreover, let $\overline{X}$ be a coarse compactification of X (see Definition 2.2). We suppose that the action of G on X is continuous and extends to a continuous action on $\overline{X}$. Then the action on the boundary $\partial X = \overline{X} \setminus X$ has a fixed point. Namely, there exists a point z of ∂X such that $g \cdot z = z$ for any element $g \in G$.

EXAMPLE 1.4. Let G be a finitely generated group with an element h of infinite order. Then a group action of $\mathbb{Z}$ on G by $(n, g) \mapsto h^n g$ is a coarse action, and the orbit of $e \in G$ is coarse. Thus, the action of $\mathbb{Z}$ on the Higson corona νG has a fixed point. Moreover, if G is a hyperbolic group, this action extends to the Gromov boundary $\partial_g G$ (see Proposition 2.4 and Example 2.5). Then this action of $\mathbb{Z}$ on $\partial_g G$ has a fixed point. This is a well-known fact on the boundary of hyperbolic groups (see [2, Chapter 8, Proposition 10 and Theorem 30]).

EXAMPLE 1.5. The wreath product $\mathbb{Z} \wr \mathbb{Z}$ contains $\mathbb{Z}^n$ as a subgroup for any positive integer n (see [5, p. 135]). Thus, the action of $\mathbb{Z}^n$ on $\mathbb{Z} \wr \mathbb{Z}$ is coarse, and the induced action of $\mathbb{Z}^n$ on $\nu(\mathbb{Z} \wr \mathbb{Z})$ has a fixed point.

REMARK 1.6. Let F_2 be a free group of rank 2. By Lemma 2.1, the action of F_2 on F_2 is a coarse action and has a coarse orbit. However, the induced action of F_2 on the Higson corona νF_2 does not have any fixed point (see Proposition 2.8).

Let $f \colon X \to X$ be a coarse map. We call a point x of X a *coarse fixed point* of f if the orbit of x, $\{f^n(x) \colon n \in \mathbb{N}\}$, is bounded.

COROLLARY 1.7 (coarse version of Brouwer's fixed-point theorem). *Let X be a proper metric space. Suppose that $f \colon X \to X$ is an isometry and that $\overline{X}$ is a coarse compactification of X such that f extends to a continuous map $\bar{f} \colon \bar{X} \to \bar{X}$. If f has no coarse fixed point in X, then $\bar{f}$ has a fixed point in ∂X.*

In this paper, *isometries* does not mean *surjective isometries*. This corollary says that an isometry f always has a coarse fixed point in X or a fixed point in ∂X.

EXAMPLE 1.8. The Gromov boundary of the hyperbolic plane $\mathbb{H}^2$ is S^1. Let $f \colon \mathbb{H}^2 \to \mathbb{H}^2$ be a continuous map such that f extends to the Gromov boundary. Then Brouwer's fixed-point theorem says that $f \colon \mathbb{H}^2 \cup S^1 \to \mathbb{H}^2 \cup S^1$ has a fixed point. Let Γ be a discrete group of isometries acting freely on $\mathbb{H}^2$ with quotient a compact surface. Here Γ is coarsely equivalent to $\mathbb{H}^2$,

and its Gromov boundary is also S^1. Let $f \colon \Gamma \to \Gamma$ be an isometry. Then Corollary 1.7 says that $f \colon \Gamma \cup S^1 \to \Gamma \cup S^1$ has a coarse fixed point in Γ, or a fixed point in S^1.

EXAMPLE 1.9. Let M be a compact path metric space. The *cone* CM on M is the quotient space $M \times [0, \infty)/\sim$, where $(x, t) \sim (x', t')$ if and only if either $x = x'$ and $t = t'$ or $t = t' = 0$. We define a continuous function $\tau \colon [0, \infty) \to [0, 1)$ by

$$\tau(t) = \frac{t}{1+t}.$$

Set $\overline{CM} = M \times [0, 1]/\sim$. The embedding $CM \to \overline{CM} : (x, t) \mapsto (x, \tau(t))$ gives us a compactification of CM. We can define an appropriate metric on CM such that CM is a proper metric space. Then $\overline{CM}$ becomes a coarse compactification (see Appendix). Let $f \colon CM \to CM$ be an isometry, and we suppose that f extends to a continuous map $\bar{f} \colon \overline{CM} \to \overline{CM}$. Then f has a coarse fixed point in CM, or $\bar{f}$ has a fixed point on the boundary $\partial(CM) \cong M$. If M is Euclidean neighborhood retract (ENR), then the Lefschetz fixed-point theorem implies that $\bar{f}$ has a fixed point since $\overline{CM}$ is contractible. However, let g be an isometry on a *punctured cone* $CM^\times = M \times [1/2, \infty)$. We suppose that g extends to a continuous map $\bar{g}$ on a coarse compactification $\overline{CM^\times} = M \times [1/2, 1]$. Then g also has a coarse fixed point in $CM^\times$, or $\bar{g}$ has a fixed point in the boundary $\partial(CM) \cong M$. Since $CM^\times$ is homotopic to M, the Lefschetz fixed-point theorem does not imply this.

EXAMPLE 1.10. Set $M = S^{n-1}$. Then CM is homeomorphic to the n-dimensional Euclidean space $\mathbb{R}^n$, and $\overline{CM}$ is homeomorphic to unit ball $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$. Any isometry $f \colon \mathbb{R}^n \to \mathbb{R}^n$ which can be extended to continuous map $\bar{f} \colon B^n \to B^n$ has a coarse fixed point in $\mathbb{R}^n$ or a fixed point in $\partial(\mathbb{R}^n) = S^{n-1}$.

REMARK 1.11. In Corollary 1.7, the assumption that the map f is an isometry is essential. See Section 4.

§2. Coarse action

LEMMA 2.1. *Let G be a finitely generated group or $G = \mathbb{N}^k$ with left-invariant word metric for some generating set. The action of G on G given by the left-translation $(g, h) \mapsto gh$ is a coarse action. Furthermore, any orbit of $h \in G$ has a coarse orbit.*

Proof. For given $g \in G$, the map $\Psi_g \colon G \to G$ given by $h \mapsto gh$ is an isometry, so it is a coarse map. Let $h \in G$ be given; we consider the orbit map $\Phi_h \colon G \to G$ given by $g \mapsto gh$. We denote the word length by $|\cdot|$. For any $g, g' \in G$, we have

$$d\big(\Phi_h(g), \Phi_h(g')\big) = d(gh, g'h) = |h^{-1}g^{-1}g'h| \leq |g^{-1}g'| + 2|h|.$$

This shows that Φ_h is a large-scale Lipschitz map and hence a bornologous map. Let D be a bounded subset of G, which is a finite set. Since Φ_g is injective, $\sharp\Phi_g^{-1}(D) \leq \sharp D < \infty$. This shows that Φ_h is a proper map. It follows that Φ_h is a coarse map and that the orbit of h is coarse. □

The coarse action of G on X induces the continuous action on the Higson corona of X. Since the Higson corona is an extremely complicated space, it is useful to find more manageable compactifications. The following show the relation between the Higson corona and such compactifications. We say that $E \subset X \times X$ is a controlled set if there exists $R > 0$ such that any $(x, y) \in E$ satisfies $d(x, y) < R$ (see [5, Chapter 2]).

DEFINITION 2.2. Let X be a proper metric space, and let $\overline{X}$ be a compactification of X. We call $\overline{X}$ a *coarse compactification* if the following condition is satisfied: for any controlled set $E \subset X \times X$ of X, its closure $\overline{E}$ in $\overline{X} \times \overline{X}$ meets the boundary $\partial(X \times X) = \overline{X} \times \overline{X} \setminus (X \times X)$ only at the diagonal; that is,

$$\overline{E} \cap \partial(X \times X) \subset \Delta_{\partial X} = \big\{(\omega, \omega) \colon \omega \in \partial X\big\}.$$

PROPOSITION 2.3. *The Higson compactification hX is a coarse compactification. Moreover, it is a universal coarse compactification, in the sense that, for any coarse compactification $\overline{X}$ of X, the identity map $\mathrm{id} \colon X \to X$ extends uniquely to a continuous surjective map $\iota \colon hX \to \overline{X}$.*

PROPOSITION 2.4. *Let $f \colon X \to X$ be a continuous coarse map, and let $\overline{X}$ be a coarse compactification. Suppose that f extends to a continuous map $\bar{f} \colon \overline{X} \to \overline{X}$. Then $\bar{f}(\partial X) \subset \partial X$ and the following diagram commutes:*

$$
\begin{array}{ccc}
hX & \xrightarrow{\,{}_h f = f \cup_\nu f\,} & hX \\
{\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\
\overline{X} & \xrightarrow{\quad \bar{f} \quad} & \overline{X}
\end{array}
$$

Proof. By the properness of f, we have $\bar{f}(\partial X) \subset \partial X$. Since f is a continuous coarse map, f induces $f^* \colon C_h(X) \to C_h(X)$. Because f extends to a continuous map on $\overline{X}$, f^* maps $C(\overline{X})$ to itself. It is clear that f^* and the inclusion $\iota^* \colon C(\overline{X}) \hookrightarrow C_h(X)$ commute. $\square$

EXAMPLE 2.5. Let X be a Gromov hyperbolic space. Then the Gromov compactification of X is a coarse compactification (see [5, Lemma 6.23]). If G is a hyperbolic group and $\gamma \in G$ has an infinite order, then the left-translation $\gamma \cdot \colon G \to G$ extends to a continuous map on the Gromov boundary $\partial_g G$ (see [2, Chapters 7, 8]).

LEMMA 2.6. *Let G be $\mathbb{N}^k$ or $\mathbb{Z}^k$. The coarse action of G on G by $(g, n) \mapsto g + n$ extends to the trivial action on the Higson corona of G. Namely, for any $g \in G$, the induced map ${}_\nu\Psi_g \colon \nu G \to \nu G$ is equal to the identity map.*

Proof. Let $g \in G$ be given. For any $n \in G$, $d(n, \Psi_g(n)) = d(n, g + n) = |g|$. Then Ψ_g is close to the identity $\mathrm{id}_G \colon G \to G$. It follows that ${}_\nu\Psi_g = \mathrm{id}_{\nu G}$. $\square$

Proof of Theorem 1.3. Let x_0 be a point in X whose orbit is coarse. The G-equivariant map $\Phi_{x_0} \colon G \to X$ given by $g \mapsto g \cdot x_0$ is a coarse map. Thus, Φ_{x_0} extends to the continuous map ${}_\nu\Phi_{x_0} \colon \nu G \to \nu X$. For each $g \in G$, the coarse map $\Psi_g \colon X \to X$ extends to the continuous map ${}_\nu\Psi_g \colon \nu X \to \nu X$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
\nu G & \xrightarrow{\ {}_\nu\Psi_g = \mathrm{id}\ } & \nu G \\[2pt]
{}_\nu\Phi_{x_0} \downarrow & & \downarrow {}_\nu\Phi_{x_0} \\[2pt]
\nu X & \xrightarrow[\ {}_\nu\Psi_g\ ]{} & \nu X
\end{array}
$$

By Lemma 2.6, the upper right arrow is the identity map. We choose any $x' \in \nu G$, and we put $x = {}_\nu\Phi_{x_0}(x')$. Then by the above commutative diagram, we have ${}_\nu\Psi_g(x) = x$.

Moreover, let $\overline{X}$ be a coarse compactification of X. We suppose that the action of G on X is continuous and extends to a continuous action on $\overline{X}$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
\nu X & \xrightarrow{\ {}_\nu\Psi_g\ } & \nu X \\[2pt]
\iota \downarrow & & \downarrow \iota \\[2pt]
\partial X & \xrightarrow[\ \overline{\Psi_g}\ ]{} & \partial X
\end{array}
$$

Set $z = \iota(x)$. We have $\overline{\Psi}_g(z) = z$. □

REMARK 2.7. Suppose that X is a proper metric space and that Φ_{x_0} is a continuous map. By the results of Dranishnikov, Keesling, and Uspenskij [1], if Φ_{x_0} is a coarse embedding and $\Phi_{x_0}(G)$ is a closed subset, we can show that the map $_\nu\Phi_{x_0}$ in the proof of Theorem 1.3 is an embedding. Indeed, [1, Theorem 1.4] states that the closure of $\Phi_{x_0}(G)$ in hX is homeomorphic to the Higson corona $\nu(\Phi_{x_0}(G))$ of $\Phi_{x_0}(G)$, which is a metric space with the induced metric from X. Since Φ_{x_0} is a coarse embedding, νG is homeomorphic to $\nu(\Phi_{x_0}(G))$. Thus, $_\nu\Phi_{x_0} \colon \nu G \to \nu X$ is an embedding. It follows that there are $2^{2^{\aleph_0}}$ fixed points on the Higson corona νX.

We cannot generalize Theorem 1.3 to a free group action, as is shown by the following.

PROPOSITION 2.8. *The action of F_2 on νF_2 induced by the left-translation $F_2 \times F_2 \to F_2$ has no fixed point. Namely, there exists no point x of νF_2 such that $g \cdot x = x$ for any element g of F_2.*

Proof. Gromov compactification is a coarse compactification, and isometric action on X induces the continuous action on the boundary. If the induced action of F_2 on νF_2 has a fixed point, the induced action of F_2 on the Gromov boundary $\partial_g F_2$ also has a fixed point. However, one can easily see that for any point $z \in \partial_g F_2$, there exists an element g of F_2 such that $g \cdot z \neq z$. □

§3. Coarse fixed points

DEFINITION 3.1. Let G be a finitely generated semigroup acting on X. We call a point x of X a *coarse fixed point* if its orbit $G \cdot x = \{g \cdot x \colon g \in G\} \subset X$ is a bounded set.

If G is an infinite group and x is a coarse fixed point, then the orbit of x is not proper. In the following two propositions, the converse holds.

PROPOSITION 3.2. *Let X be a metric space such that any bounded subset $D \subset X$ is a finite set. Suppose that $\mathbb{N}$ acts on X. Then a point of X whose orbit is not proper is a coarse fixed point.*

Proof. Let x_0 be a point in X whose orbit is not proper. Then there exists a bounded set $D \subset X$ such that $\{n \in \mathbb{N} \colon n \cdot x_0 \in D\}$ is an infinite set. Because D is a finite set, there exist positive integers $m > n$ such that $m \cdot x_0 = n \cdot x_0$.

For any integer $l > m$, there exist integers $k > 0$ and $r = 0, \ldots, m - n - 1$ satisfying $l - n = k(m - n) + r$. Thus, we have $l \cdot x_0 = (n + r) \cdot x_0$. It follows that $\mathbb{N} \cdot x_0 \subset \{x_0, 1 \cdot x_0, 2 \cdot x_0, \ldots, (m - 1) \cdot x_0\}$. $\qquad\square$

PROPOSITION 3.3. *Let X be a proper metric space. Suppose that $\mathbb{N}$ acts on X by isometries. Then each point of X whose orbit is not proper is a coarse fixed point.*

Proof. Suppose that the orbit of x_0 is not proper. Then there exists a bounded set $D \subset X$ such that $\sharp\{n \in \mathbb{N} \colon n \cdot x_0 \in D\} = \infty$. We can assume that x_0 lies in D. We notice that for any point x of the orbit $\mathbb{N} \cdot x_0$, there exist $n(x) \in \mathbb{N}$ such that $n(x) \cdot x$ lies in D. We define a bounded subset $K \subset X$ by

$$K = B(D, 1) \cap \mathbb{N} \cdot x_0.$$

Here $B(D, 1) = \{x \in X \colon {}^{\exists}y \in D, d(x, y) < 1\}$ is the 1-neighborhood of D. Since $\overline{K}$ is a compact set, there exist $x_1, \ldots, x_N \in K$ such that

$$\overline{K} \subset \bigcup_{i=1}^{N} B(x_i, 1).$$

For $0 \leq i \leq N$, we denote $T_i = n(x_i)$. The point $T_i \cdot x_i$ lies in D. We define a positive number L by

$$L = \max_{0 \leq i \leq N} \max_{0 \leq a \leq T_i} d(x_0, a \cdot x_i).$$

We define inductively a sequence $\{i_k\}_{k=0}^{\infty}$ consisting of integers in $\{1, \ldots, N\}$ and an increasing sequence $\{S_j\}_{j=0}^{\infty}$ as follows. Let $S_0 = T_0$. Since $S_0 \cdot x_0 = T_0 \cdot x_0 \in K$, there exists an integer $i_0 \in \{1, \ldots, N\}$ such that $S_0 \cdot x_0 \in B(x_{i_0}, 1)$. Assume that we have defined $i_0, \ldots, i_n$ and $S_0, \ldots, S_n$ such that, for $0 \leq j \leq n$ and $0 \leq a < T_{i_j}$, they satisfy

$$S_j = \sum_{k=0}^{j-1} T_{i_k},$$

$$S_j \cdot x_0 \in B(x_{i_j}, 1),$$

$$(a + S_j) \cdot x_0 \in B(x_0, L + 1).$$

Set $S_{n+1} = T_{i_n} + S_n$. Since $T_{i_n} \cdot x_{i_n} \in D$ and

$$d(S_{n+1} \cdot x_0, T_{i_n} \cdot x_{i_n}) = d(S_n \cdot x_0, x_{i_n}) < 1,$$

we have $S_{n+1} \cdot x_0 \in B(D, 1)$, and thus $S_{n+1} \cdot x_0$ lies in K. Hence, there exists an integer i_{n+1} such that $S_{n+1} \cdot x_0$ lies in $B(x_{i_{n+1}}, 1)$. For $0 \leq a < T_{i_{n+1}}$, we have $d(x_0, a \cdot x_{i_{n+1}}) \leq L$ and

$$d\big((a + S_{n+1}) \cdot x_0, a \cdot x_{i_{n+1}}\big) = d(S_{n+1} \cdot x_0, x_{i_{n+1}}) < 1.$$

It follows that $(a + S_{n+1}) \cdot x_0$ lies in $B(x_0, L + 1)$. This shows that for any integer $l > 0$, $l \cdot x_0$ lies in $B(x_0, L + 1)$, and thus x_0 is a coarse fixed point. □

The hypothesis that X is a proper space is necessary (see [3, Exercise 5.2.26]).

REMARK 3.4. Under the assumption of Proposition 3.3, if x_0 is a coarse fixed point, then any point x of X is a coarse fixed point.

If the orbit is *not coarse*, there are two possibilities; namely, the orbit is either nonproper or nonbornologous. However, if the action is an isometry, any orbit is bornologous.

LEMMA 3.5. *Let X be a proper metric space with an isometric action of $\mathbb{N}$. Then the action is a coarse action, and any orbit is bornologous.*

Proof. An isometric action is a coarse action. For any given point x of X, put $L = d(1 \cdot x, x)$. Then we have $d((i + 1) \cdot x, i \cdot x) = L$ for all integers $i > 0$. Hence, for any integers $m \geq n > 0$, we have

$$d\big(\Phi_x(m), \Phi_x(n)\big) = d(m \cdot x, n \cdot x) \leq \sum_{i=n}^{m-1} d\big((i + 1) \cdot x, i \cdot x\big) = L|m - n|.$$

Thus, Φ_x is a bornologous map. □

Proof of Corollary 1.7. An isometric action of $\mathbb{N}$ on X is defined by $(n, x) \mapsto f^n(x)$. If f has no coarse fixed point, then by Proposition 3.3 and Lemma 3.5, any orbit of this action is coarse. Thus, Theorem 1.3 implies that $\bar{f}$ has a fixed point. □
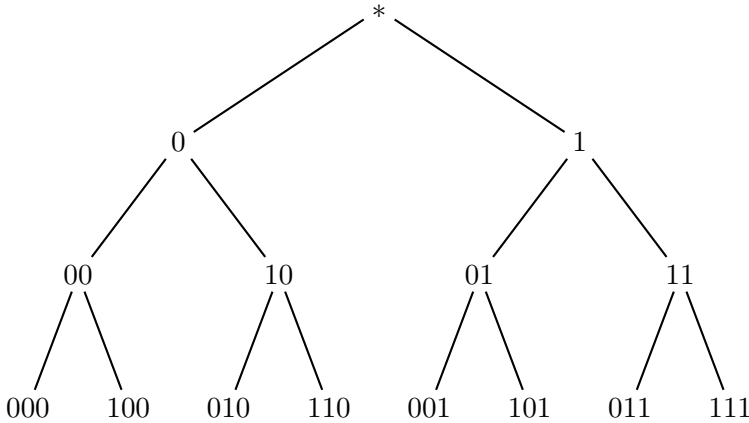
§4. Another example of coarse action

There exists a coarse map which has no coarse fixed points on the coarse space or fixed points on its boundary.

Adding machine

Let $T_2 = (V_2, E_2)$ be a binary tree with the set of vertices V_2 and the set of edges E_2. We identify V_2 with the following sets:

$$V_2 \cong \{*\} \sqcup \bigsqcup_{n \geq 1} \{0, 1\}^n.$$

Here $*$ denotes the base point:

$*$ and 0 (resp., 1) can be joined by the edge $e_{*,0}$ (resp., $e_{*,1}$). Let $x = (i_{n-1}, \ldots, i_0)$ and $y = (j_n, \ldots, j_0)$ be two vertices. Both x and y can be joined by the edge $e_{x,y}$ if and only if $i_0 = j_0, \ldots, i_{n-1} = j_{n-1}$. T_2 has the usual metric and is a geodesic space. V_2 has the induced metric from T_2.

We construct a coarse action of $\mathbb{N}$ on V_2. Let x be a vertex. If $x = (i_n, \ldots, i_0) \neq (1, \ldots, 1)$, we define $1 \cdot x = (j_{n-1}, \ldots, j_0)$ with

$$\sum_{k=0}^{n-1} j_k 2^k = \sum_{k=0}^{n-1} i_k 2^k + 1.$$

If $x = (\overbrace{1, \ldots, 1}^{n})$, we define $1 \cdot x = (\overbrace{1, 0, \ldots, 0}^{n+1})$. Finally, we define $1 \cdot * = (0)$.

The Gromov product of $x = (i_{n-1}, \ldots, i_0)$ and $y = (j_{m-1}, \ldots, j_0)$ with a base point $*$ is defined to be

$$(x \mid y) = \frac{1}{2} \big\{ d(x, *) + d(y, *) - d(x, y) \big\}.$$

We notice that $(x \mid y) = r$ if and only if $i_0 = j_0, \ldots, i_{r-1} = j_{r-1}$ and $i_r \neq j_r$.

LEMMA 4.1. $1\cdot: V_2 \to V_2$ is a coarse map.

Proof. Let $x = (i_{n-1}, \ldots, i_0)$ and $y = (j_{m-1}, \ldots, j_0)$ be two vertices. Set $(x \mid y) = r$. Then $i_0 = j_0, \ldots, i_{r-1} = j_{r-1}$. It is easy to see that $(1 \cdot x \mid 1 \cdot y) \geq r$. Hence, we have

$$d(1 \cdot x, 1 \cdot y) = d(1 \cdot x, *) + d(1 \cdot y, *) - 2(1 \cdot x \mid 1 \cdot y)$$
$$\leq (n+1) + (m+1) - 2r$$
$$= d(x, y) + 2.$$

Thus, $x \mapsto 1 \cdot x$ is a large-scale Lipschitz map. Since the map $1\cdot$ is injective and every bounded set of V_2 is a finite set, the map $1\cdot$ is proper. □

This action can be extended to a continuous action on the Gromov boundary $\partial_g V_2$ of V_2. Here we can identify $\partial_g V_2$ as the Cantor set $\{0,1\}^{\mathbb{N}}$ with a metric d defined to be

$$d(x, y) = 2^{-(x|y)}.$$

Here $x = (i_k)_{k=0}^{\infty}$, and $y = (j_k)_{k=0}^{\infty}$.

PROPOSITION 4.2. *The action of $\mathbb{N}$ on $\partial_g V_2$ is minimal. Namely, for any point x of $\partial_g V_2$, its orbit $\mathbb{N} \cdot x$ is dense in $\partial_g V_2$.*

Proof. Let $x \in \partial_g V_2$ be given. For any point y of $\partial_g V_2$ and $\epsilon > 0$, we will show that there exists an integer n such that $d(n \cdot x, y) < \epsilon$. Set $N > -\log \epsilon / \log 2$, and set $x = (i_k)_{k=0}^{\infty}$. We choose $a = 2^{N+1} - \sum_{k=0}^{N} i_k 2^k$; then we have

$$a \cdot x = (\ldots, \overbrace{0, \ldots, 0}^{N+1}).$$

Set $y = (j_k)_{k=0}^{\infty}$, and set $b = \sum_{k=0}^{N} j_k 2^k$. Then we have

$$(a + b) \cdot x = (\ldots, j_N, \ldots, j_0).$$

It follows that $d((a + b) \cdot x, y) < 2^{-N} < \epsilon$. □

In particular, the action of $\mathbb{N}$ on $\partial_g V_2$ has no fixed point. Moreover, the action has no coarse fixed point in V_2. Thus, by Proposition 3.2 and Theorem 1.3, the orbit of $x \in V_2$ can never be bornologous.

Appendix

Metric of the cone and its compactification

Let M be a compact path metric space. The cone CM on M is the quotient space $M \times [0, \infty)/\sim$, where $(x, t) \sim (x', t')$ if and only if either $x = x'$ and $t = t'$ or $t = t' = 0$. According to Roe [4, Section 3.6], we construct a metric on CM. Let $\lambda\colon [0, \infty) \to [0, \infty)$ be a continuous function with $\lambda(t) = 0$ if and only if $t = 0$. If γ is a path in CM, we define its λ-length, $l_\lambda(\gamma)$, to be

$$\sup\left\{\sum_{j=0}^{n-1}\big(|t_j - t_{j+1}| + \max\big\{\lambda(t_j), \lambda(t_{j+1})\big\}d(x_j, x_{j+1})\big)\right\},$$

where the supremum is taken over all finite sequences $(x_j, t_j)_{j=0}^n$ of points on the path γ (with (x_0, t_0) and (x_n, t_n) being two endpoints). We define a metric d_λ on CM by $d_\lambda((x, t), (x', t')) = \inf l_\lambda(\gamma)$, where the infimum is taken over all paths joining (x, t) and (x', t'). The metric d_λ is compatible with the topology of CM, and it becomes a proper metric space (see [4, Proposition 3.47]).

PROPOSITION A.1. *Suppose that λ is an increasing, unbounded function; then the compactification $\overline{CM} = M \times [0, 1]/\sim$ is a coarse compactification.*

Proof. Let $E \subset CM \times CM$ be a controlled set, and let r_E be a positive number such that

$$\sup\big\{d_\lambda(\mathbf{x}, \mathbf{y})\colon (\mathbf{x}, \mathbf{y}) \in E\big\} < r_E.$$

Let $\{(\mathbf{x}_n, \mathbf{y}_n)\} \subset E$ be a sequence such that $\mathbf{x_n}$ and $\mathbf{y_n}$ tend to x and y in $\partial(CM) \cong M$, respectively. We denote $\mathbf{x}_n = (x_n, t_n)$ and $\mathbf{y}_n = (y_n, s_n)$. It is enough to show that $x = y$ (see [5, Theorem 2.27]). Suppose that $x \neq y$. Since x_n and y_n converge to x and y, respectively, for any $\epsilon > 0$ there exists $N_1 > 0$ such that, for all $n > N_1$, x_n and y_n satisfy $d(x_n, y_n) > d(x, y) - \epsilon$. Because λ is an increasing, unbounded function, there exists $N_2 > 0$ such that, for all $t > N_2$, we have

$$\lambda(t) > \frac{r_E}{d(x, y) - \epsilon}.$$

Since $\mathbf{x_n}$ tends to infinity, there exists $N_3 > 0$ such that, for all $n > N_3$, the second component t_n of $\mathbf{x_n}$ satisfies $t_n > N_2$. Thus, if we choose an integer n satisfying $n > \max\{N_1, N_3\}$, then we have $d_\lambda(\mathbf{x}_n, \mathbf{y}_n) > r_E$; this contradicts the definition of r_E. $\qquad\square$

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