J. Plasma Phys. (2023), *vol.* 89, 905890202
Published by Cambridge University Press
doi:10.1017/S0022377823000156

Quasi-parallel propagating solitons in magnetised relativistic electron–positron plasmas

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(Received 6 August 2022; revised 8 February 2023; accepted 9 February 2023)

In this article, we study nonlinear waves propagating along the background magnetic field in relativistic electron–positron plasmas. Using the reductive perturbation method, we derive a three-dimensional equation describing these waves. When the perturbations do not vary in the directions orthogonal to the background magnetic field this equation reduces to the vector modified Kortewed–de Vries equation. We present solutions of the obtained equation in the form of planar solitary waves and describe the results of study of their stability with respect to transverse perturbations. We also study numerically non-planar solitary waves.

Key words: astrophysical plasmas, plasma nonlinear phenomena, plasma waves

1. Introduction

The wave propagation in electron–positron plasmas has been studied extensively by theorists in relation to both astrophysical as well as laboratory applications. The electron–positron plasmas exist in pulsar magnetospheres (Sturrock 1971; Ruderman & Sutherland 1975; Chian & Kennel 1983; Arons & Barnard 1986; Aharonian, Bogovalov & Khangulyan 2012; Cerutti & Beloborodov 2017), active galactic nuclei (Ruffini, Vereshchagin & Xue 2010; El-Labany *et al.* 2013; Kawakatu, Kino & Takahara 2016) and the early universe (Gailis, Frankel & Dettmann 1995; Shukla 2003; Tatsuno *et al.* 2003). In the laboratory, electron–positron plasmas are created in inertial confinement fusion devises using ultra-intense lasers (Begelman, Blandford & Rees 1984; Liang, Wilks & Tabak 1998; Gahn *et al.* 2000). Another example is a semi-conductor plasma, where holes

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behave like positive charges with the mass equal to that of electrons (Shukla *et al.* 1986). The plasma can be considered as relativistic when either its bulk velocity is close to the velocity of light or when the averaged kinetic energy of particles is of the order of or greater than the electron rest energy.

The linear theory of wave propagation in non-relativistic and relativistic electron–positron plasmas was developed using both hydrodynamic as well as kinetic description (Sakai & Kawata 1980*a*; Arons & Barnard 1986; Stewart & Laing 1992; Iwamoto 1993; Zank & Greaves 1995). The nonlinear theory of waves in electron–positron plasmas has been also developed. The nonlinear Schrödinger (NLS) equation was derived and used to study the modulational instability and envelope solitons (Chian & Kennel 1983; Cattaert, Kourakis & Shukla 2005; Rajib, Sultana & Mamun 2015). The Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) equations were obtained and the dependence of width and amplitude of solitons described by these equations on parameters of an unperturbed state was studied (Verheest & Lakhina 1996; Lakhina & Verheest 1997; Rajib *et al.* 2015).

We aim to study the propagation of nonlinear waves parallel to the equilibrium magnetic field. In the case of electron–ion plasmas, this problem has been studied intensively for a few decades. It was shown that the one-dimensional quasi-parallel propagation of nonlinear waves is described by the derivative nonlinear Schrödinger (DNLS) equation (Rogister 1971; Mjølhus 1976; Mio *et al.* 1976*a*; Ruderman 2002). This equation was used to study the modulational instability of circularly polarised Alfvén waves (Mjølhus 1976; Mio *et al.* 1976*b*). The DNLS equation describes a few kinds of solitons as well as the generation of rogue waves (Ichikawa *et al.* 1980; Mjølhus & Hada 1997; Fedun, Ruderman & Erdélyi 2008). It was shown that the DNLS equation is completely integrable, the Lax pair for this equation was found and the inverse scattering method was used to obtain exact solutions for this equation (Kaup & Newell 1978; Kawata & Inoue 1978).

Later an extension of the DNLS equations to two and three dimensions (3D DNLS) was derived (Mjølhus & Wyller 1986; Ruderman 1987; Mjølhus & Hada 1997). This extension is similar to that obtained by Kadomtsev and Petviashvili (KP equation) for the KdV equation (Kadomtsev & Petviashvili 1970). The 3D DNLS was used to study the stability of solitons of the DNLS equation with respect to transverse perturbations (Ruderman 1987; Mjølhus & Hada 1997).

The propagation of nonlinear Alfvén waves parallel to the external magnetic field has been also studied in an electron–positron plasma (Sakai & Kawata 1980*a*,*b*; Mikhailovskii, Onishchenko & Smolyakov 1985*a*; Mikhailovskii, Onishchenko & Tatarinov 1985*b*,*c*; Verheest 1996; Lakhina & Verheest 1997). It was shown that, in contrast to the electron–proton plasma, nonlinear waves propagating parallel to the magnetic field are described by the vector form of the mKdV equation. Recently Ruderman (2020) derived the three-dimensional extension of the vector mKdV equation (3D vector mKdV) similar to the 3D DNLS for nonlinear Alfvén waves propagation parallel to the external magnetic field in a non-relativistic electron–positron plasma. He then used this equation to study the transverse stability of planar solitons propagating at small non-zero angles with respect to the equilibrium magnetic field.

In this article we aim to derive the 3D vector mKdV equation for nonlinear Alfvén waves propagating parallel to the external magnetic field in a relativistic electron–positron plasma, and study non-planar solitary waves. The paper is organised as follows. In the next section we formulate the problem and present the governing equations. In § 3 we briefly describe the linear theory of waves propagating along the magnetic field in relativistic electron–positron plasmas. The 3D vector mKdV equation for nonlinear Alfvén waves propagating parallel to the external magnetic field is derived in § 4. In § 5 we present solutions describing linearly polarised (planar) one-dimensional solitary waves propagating at small non-zero angles with respect to the equilibrium magnetic field, and the results of study of their stability with respect to transverse perturbations. In § 6 we describe the results of the numerical study of non-planar solitary waves. In § 7 we present the summary of the results and our conclusions.

2. Problem formulation and governing equation

We consider the propagation of nonlinear waves along the equilibrium magnetic field in a plasma that consists of electrons and positrons. We treat the electron and positron components as two charged fluids. We do not consider the annihilation or pair creation meaning that the particle number is conserved. To write down the equation describing the particle number conservation we introduce the particle flux four-vector n_s^i (Landau & Lifshitz 1966), where $s = +$ and $s = -$ correspond to positrons and electrons, respectively. The time component of the four-vector n_s^i is the particle number density times the speed of light c, and its three spatial components form the three-dimensional particle flux vector. The four-vector n_s^i is proportional to the four-velocity u_s^i, $n_s^i = n_s u_s^i$, where the Lorentz scalar n_s is the particle density in the rest frame. Now the particle conservation equation is written as

$$\frac{\partial (n_s u_s^i)}{\partial x^i} = 0, \tag{2.1}$$

where the summation with the repeating index is from 0 to 3, and $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$. Using the relation between the four-velocity and the three-dimensional velocity vector $\boldsymbol{v}$, $u_s^i = \gamma_s(c, \boldsymbol{v}_s)$, where $\gamma_s = (1 - v_s^2/c^2)^{-1/2}$ is the gamma-factor and c is the speed of light, we rewrite (2.1) as

$$\frac{\partial (\gamma_s n_s)}{\partial t} + \nabla \cdot (\gamma_s n_s \boldsymbol{v}_s) = 0, \tag{2.2}$$

where ∇ is the three-dimensional gradient.

We assume that the positron and electron fluids are inviscid and the pressure in these fluids is isotropic. Then the energy–momentum tensors of the positron and electron fluids are given by (Landau & Lifshitz 1966; Weinberg 1972)

$$T_s^{ik} = c^{-2}(e_s + p_s) u_s^i u_s^k + p_s g^{ik}, \tag{2.3}$$

where p_s is the pressure, $e_s = mc^2 n_s$ the proper energy density, m the rest mass of the electron and positron, and g^{ik} is the metric tensor with the components $g^{00} = -1$, $g^{11} = g^{22} = g^{33} = 1$, and $g^{ij} = 0$ when $i \neq j$.

We now introduce the anti-symmetric tensor F^{ik} related to the components of the electric, $\boldsymbol{E}$, and magnetic, $\boldsymbol{B}$, fields by (Weinberg 1972; Landau & Lifshitz 1975)

$$F^{0\alpha} = c^{-1} E_\alpha, \quad \alpha = 1, 2, 3, \tag{2.4a}$$

$$F^{12} = B_3, \quad F^{13} = -B_2, \quad F^{23} = B_1. \tag{2.4b}$$

We also introduce the four-current $J_s^i = (c\rho_s, \boldsymbol{j}_s)$ where ρ_s is the density of the electrical charge and $\boldsymbol{j}_s$ is the electrical current density defined by $\boldsymbol{j}_s = \rho_s \boldsymbol{v}_s$ (Landau & Lifshitz 1975). In the reference frame where the charged fluid is at rest ($\boldsymbol{v}_s = 0$) $\rho_s = q_s n_s$, where $q_+ = q$, $q_- = -q$ and q is the elementary charge. Then, in the laboratory

https://doi.org/10.1017/S0022377823000156 Published online by Cambridge University Press

reference frame,

$$\rho_s = \gamma_s q_s n_s. \tag{2.5}$$

The energy–momentum tensor is related to the four-current by (Weinberg 1972)

$$\frac{\partial T_s^{ij}}{\partial x^j} = g_{kl} F^{il} J_s^k, \tag{2.6}$$

where $g_{kl} = g^{kl}$. We assume that the motion is adiabatic and take

$$p_s = p_0 \left(\frac{n_s}{n_0}\right)^{\kappa}, \quad e_s = \frac{p_s}{\kappa - 1} + mc^2 n_s, \tag{2.7a,b}$$

where n_0 and p_0 are the unperturbed number density and pressure (the same for the electrons and positrons), and κ is the adiabatic index (equal to 5/3 for non-relativistic plasmas and 4/3 for ultra-relativistic plasmas). The electric and magnetic fields are governed by the Maxwell equations,

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \tag{2.8a}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.8b}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.8c}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \tag{2.8d}$$

where ε_0 is the permittivity of free space, μ_0 is the permeability of free space and the total electrical charge and current densities are determined by

$$\rho = \rho_+ + \rho_- = q\gamma(n_+ - n_-), \tag{2.9a}$$

$$\boldsymbol{j} = \boldsymbol{j}_+ + \boldsymbol{j}_- = q\gamma(n_+ \boldsymbol{v}_+ - n_- \boldsymbol{v}_-). \tag{2.9b}$$

Recall that $\varepsilon_0 \mu_0 = c^{-2}$.

Using (2.3)–(2.5), and the relations $J_s^i = (c\rho_s, \boldsymbol{j}_s)$ and $\boldsymbol{j}_s = \rho_s \boldsymbol{v}_s$ we obtain from (2.6) with $i = 1, 2, 3$

$$\frac{\partial[\gamma_s^2(e_s + p_s)\boldsymbol{v}_s]}{\partial t} + \nabla \cdot [\gamma_s^2(e_s + p_s)\boldsymbol{v}_s\boldsymbol{v}_s] + c^2\nabla p_s = c^2\gamma_s q_s n_s(\boldsymbol{E} + \boldsymbol{v}_s \times \boldsymbol{B}). \tag{2.10}$$

Using (2.2) and (2.7*a*,*b*) we transform this equation to

$$\frac{\partial(\gamma_s h_s \boldsymbol{v}_s)}{\partial t} + (\boldsymbol{v}_s \cdot \nabla)(\gamma_s h_s \boldsymbol{v}_s) + \frac{c^2 \nabla p_s}{\gamma_s n_s} = c^2 q_s(\boldsymbol{E} + \boldsymbol{v}_s \times \boldsymbol{B}), \tag{2.11}$$

where

$$h_s = mc^2 + \frac{\kappa p_0}{(\kappa - 1)n_0}\left(\frac{n_s}{n_0}\right)^{\kappa - 1}. \tag{2.12}$$

It is straightforward to show that the equation given by (2.6) with $i = 0$ follows from (2.2) and (2.11). Hence, it is not an independent equation and, consequently, it is not used in the following. Hence, the two-fluid description of an electron–positron plasma is given by (2.2), (2.8), (2.9), (2.11) and (2.12).

In the following, we assume that in the equilibrium $n_+ = n_- = n_0$, $\boldsymbol{v}_+ = \boldsymbol{v}_- = 0$, $\boldsymbol{E} = 0$ and $\boldsymbol{B} = B_0 \boldsymbol{e}_x$, where $\boldsymbol{e}_x$ is the unit vector along the x-axis of Cartesian coordinates x, y, z.

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3. Linear theory

Here we briefly describe the linear theory of wave propagation because in the following we use it as a guide for scaling when deriving the equation governing the propagation of nonlinear waves. As we only study the longitudinal propagation of nonlinear waves, we only consider linear wave propagation along the equilibrium magnetic field. Hence, we linearise (2.2), (2.8), (2.9), (2.11) and (2.12), and then take perturbations of all quantities proportional to $\exp[\mathrm{i}(kx-\omega t)]$. As a result, we obtain two disconnected systems of algebraic equations. The first system is for the perturbation of the number density, pressure and x-components of the velocity and electric field. It describes the longitudinal wave mode. The second system is for the y- and z-components of the velocity, electric field and magnetic field perturbation. It describes transversal wave modes. In the following, we derive the equation describing the nonlinear transversal waves. However, for completeness we also present the dispersion equation for the longitudinal waves.

The system describing transversal waves reads

$$\omega h_0 \boldsymbol{v}_{\perp s} = \mathrm{i}c^2 q_s(\boldsymbol{E}_\perp - B_0 \boldsymbol{e}_x \times \boldsymbol{v}_{\perp s}), \tag{3.1a}$$

$$k\boldsymbol{E}_\perp = -\omega \boldsymbol{e}_x \times \boldsymbol{B}_\perp, \tag{3.1b}$$

$$\mathrm{i}k\boldsymbol{e}_x \times \boldsymbol{B}_\perp = \mu_0 q n_0(\boldsymbol{v}_{\perp +} - \boldsymbol{v}_{\perp -}) - \frac{\mathrm{i}\omega}{c^2}\boldsymbol{E}_\perp, \tag{3.1c}$$

where

$$h_0 = mc^2 + \frac{\kappa p_0}{(\kappa-1)n_0}, \quad \boldsymbol{v}_{\perp s} = (0, v_{ys}, v_{zs}), \tag{3.2a}$$

$$\boldsymbol{E}_\perp = (0, E_y, E_z), \quad \boldsymbol{B}_\perp = (0, B_y, B_z). \tag{3.2b}$$

Equation (3.1) is a system of linear homogeneous algebraic equations. It only has non-trivial solutions when its determinant is zero. This condition gives the dispersion equation

$$\omega^2 = V^2k^2 + \frac{h_0\omega^2(\omega^2 - c^2k^2)}{2\mu_0 q^2 n_0 c^4(1+\sigma_0)}, \tag{3.3}$$

where

$$\sigma_0 = \frac{B^2}{2\mu_0 n_0 h_0}, \quad V = c\sqrt{\frac{\sigma_0}{\sigma_0+1}}. \tag{3.4a,b}$$

Considered as a quadratic equation for ω^2, this equation has two positive roots. The solution to this equation is

$$\frac{\omega^2\ell^2}{V^2} = \frac{1}{4\sigma_0}\left\{1 + 2(1+\sigma_0)k^2\ell^2 \pm \sqrt{1 + 4(1-\sigma_0)k^2\ell^2 + 4(1+\sigma_0)^2k^4\ell^4}\right\}, \tag{3.5}$$

where ℓ is the dispersion length given by

$$\ell = \frac{1}{2cq(\sigma_0+1)}\sqrt{\frac{h_0}{\mu_0 n_0}}. \tag{3.6}$$

When the wavelength is of the order of or smaller than ℓ, that is, $k\ell \gtrsim 1$, the wave dispersion is strong. The condition that the dispersion is weak is written as $k\ell \ll 1$. As $k\ell^2 \to 0$ the root given by (3.5) with the plus sign tends to a non-zero value. This value,

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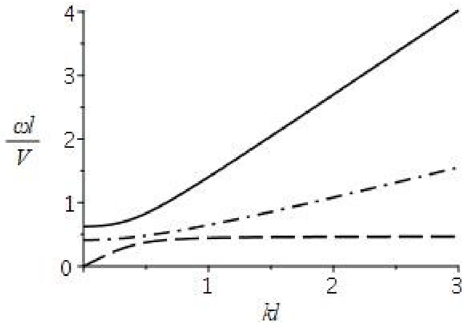

FIGURE 1. Dispersion curves defined by (3.5) and (3.8*a–c*). To plot these curves we took $a_0/V = 1/2$ and $\sigma_0 = 9/7$. The solid and dashed curves correspond to the dispersion equations given by (3.5) with the plus and minus signs, respectively. The dash-dotted curve corresponds to the dispersion equation given by (3.8*a–c*).

in turn, tends to infinity in the non-relativistic limit, that is, when $c \to \infty$. The root given by (3.5) with the minus sign tends to zero. We are only interested in the second root. For $k\ell \ll 1$ the dispersion relation determined by this root is given by the approximate expression

$$\omega = kV(1 - k^2\ell^2). \tag{3.7}$$

Although we do not need the information concerning longitudinal waves for the derivation of the nonlinear equation in § 4, we give the dispersion relation for these waves because it is used for the classification of wave modes. This dispersion relation reads

$$\omega^2 = \omega_0^2 + a_0^2 k^2, \quad \omega_0^2 = \frac{2c^2q^2n_0}{\varepsilon_0 h_0}, \quad a_0^2 = \frac{c^2\kappa p_0}{\gamma^2 n_0 h_0}. \tag{3.8a–c}$$

When $\ell k \to 0$ this dispersion equation corresponds to electrostatic waves with the frequency ω_0 independent of the wavelength. On the other hand, when $a_0 k \gg \omega_0$ the dispersion equation for longitudinal waves reduces to the approximate form $\omega \approx a_0 k$. This dispersion equation corresponds to sound waves. This sound wave is the slow mode when $a_0 < V$ and fast mode when $a_0 > V$. Transverse waves represent merged Alfvén mode and either fast mode when $a_0 < V$ or slow mode when $a_0 > V$. The dispersion curves defined by (3.5) and (3.8*a–c*) are shown in figure 1. To plot these curves we took $a_0/V = 1/2$ and $\sigma_0 = 9/7$, which corresponds to $V = 3c/4$. The dispersion curves are qualitatively the same for other values of a_0/V and σ_0. In particular, it can be shown that the dispersion curve defined by (3.8*a–c*) is always in between the two dispersion curves defined by (3.5). As we have stated previously, in the following we are studying nonlinear evolution of waves with the dispersion relation shown by the dashed curve in figure 1.

In the non-relativistic limit $p_0 \ll mc^2$ and $B^2/\mu_0 \ll mc^2$, that is, both the density of the internal plasma energy and the magnetic energy density are much smaller than the rest energy density of the plasma. The second condition implies that $\sigma_0 \ll 1$. In this case $V \approx c\sqrt{\sigma_0} \approx B_0(2\mu_0 m n_0)^{-1/2}$. We can recognise in this expression the Alfvén speed in a non-relativistic electron–positron plasma. We note that the term describing the wave dispersion (the second term in the brackets in (3.7)) is proportional to k^2. In the case of electron–ion plasma it is proportional to ik. This difference is related to the fact that the masses of positively and negatively charged particles are the same in an electron–positron plasma, whereas the mass of positively charged particles is much larger than the mass of negatively charged particles in an electron–ion plasma.

4. Derivation of equation for small-amplitude nonlinear waves

We consider nonlinear waves propagating along the equilibrium magnetic field. To derive the nonlinear equation describing the parallel propagation of nonlinear waves we use the reductive perturbation method (Kakutani *et al.* 1968; Taniuti & Wei 1968). In accordance with this method we introduce the dimensionless wave amplitude $\epsilon \ll 1$. Ruderman (2020) (referred to as Paper I in the following) showed that these waves in a non-relativistic electron–positron plasma are described by the 3D vector mKdV equation. We expect that the propagation of these waves in a relativistic electron–positron plasma is described by the same equation with the only difference that the expressions of the coefficients of this equation in terms of equilibrium quantities will be different. Hence, we introduce the same scaling variables as in Paper I,

$$\xi = \epsilon(x - Vt), \quad \eta = \epsilon^2 y, \quad \zeta = \epsilon^2 z, \quad \tau = \epsilon^3 t. \tag{4.1a–d}$$

In these new variables (2.2), (2.8), (2.9) and (2.11) are reduced to

$$\epsilon^2 \frac{\partial(\gamma_s n_s)}{\partial \tau} - V\frac{\partial(\gamma_s n_s)}{\partial \xi} + \frac{\partial(\gamma_s n_s v_{xs})}{\partial \xi} + \epsilon \boldsymbol{\nabla}_\perp \boldsymbol{\cdot} (\gamma_s n_s \boldsymbol{v}_{\perp s}) = 0, \tag{4.2a}$$

$$\begin{aligned}\epsilon^2 \frac{\partial(\gamma_s h_s v_{xs})}{\partial \tau} &+ (v_{xs} - V)\frac{\partial(\gamma_s h_s v_{xs})}{\partial \xi} + \epsilon(\boldsymbol{v}_{\perp s} \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp)(\gamma_s h_s v_{xs}) \\ &+ \frac{c^2}{\gamma_s n_s}\frac{\partial p_s}{\partial \xi} = \epsilon^{-1} c^2 q_s \left[E_x + \boldsymbol{e}_x \boldsymbol{\cdot} (\boldsymbol{v}_{\perp s} \times \boldsymbol{B}_\perp)\right],\end{aligned} \tag{4.2b}$$

$$\begin{aligned}\epsilon^2 \frac{\partial(\gamma_s h_s \boldsymbol{v}_{\perp s})}{\partial \tau} &+ (v_{xs} - V)\frac{\partial(\gamma_s h_s \boldsymbol{v}_{\perp s})}{\partial \xi} + \epsilon(\boldsymbol{v}_{\perp s} \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp)(\gamma_s h_s \boldsymbol{v}_{\perp s}) \\ &+ \frac{\epsilon c^2 \boldsymbol{\nabla}_\perp p_s}{\gamma_s n_s} = \epsilon^{-1} c^2 q_s [\boldsymbol{E}_\perp + \boldsymbol{e}_x \times (v_{xs}\boldsymbol{B}_\perp - B_x \boldsymbol{v}_{\perp s})],\end{aligned} \tag{4.2c}$$

$$\frac{\partial E_x}{\partial \xi} + \epsilon \boldsymbol{\nabla}_\perp \boldsymbol{\cdot} \boldsymbol{E}_\perp = \epsilon^{-1}\frac{q}{\varepsilon_0}(\gamma_+ n_+ - \gamma_- n_-), \tag{4.2d}$$

$$\frac{\partial B_x}{\partial \xi} + \epsilon \boldsymbol{\nabla}_\perp \boldsymbol{\cdot} \boldsymbol{B}_\perp = 0, \tag{4.2e}$$

$$\epsilon^2 \frac{\partial B_x}{\partial \tau} - V\frac{\partial B_x}{\partial \xi} = -\epsilon\, \boldsymbol{e}_x \boldsymbol{\cdot} (\boldsymbol{\nabla}_\perp \times \boldsymbol{E}_\perp), \tag{4.2f}$$

$$\epsilon^2 \frac{\partial \boldsymbol{B}_\perp}{\partial \tau} - V\frac{\partial \boldsymbol{B}_\perp}{\partial \xi} = -\boldsymbol{e}_x \times \left(\frac{\partial \boldsymbol{E}_\perp}{\partial \xi} - \epsilon \boldsymbol{\nabla}_\perp E_x\right), \tag{4.2g}$$

$$\epsilon^2 \frac{\partial E_x}{\partial \tau} - V\frac{\partial E_x}{\partial \xi} = \epsilon c^2 \boldsymbol{e}_x \boldsymbol{\cdot} (\boldsymbol{\nabla}_\perp \times \boldsymbol{B}_\perp) - \epsilon^{-1}\frac{q}{\varepsilon_0}(\gamma_+ n_+ v_{x+} - \gamma_- - n_- - v_{x-}), \tag{4.2h}$$

$$\begin{aligned}\epsilon^2 \frac{\partial \boldsymbol{E}_\perp}{\partial \tau} - V\frac{\partial \boldsymbol{E}_\perp}{\partial \xi} &= c^2 \boldsymbol{e}_x \times \left(\frac{\partial \boldsymbol{B}_\perp}{\partial \xi} - \epsilon \boldsymbol{\nabla}_\perp B_x\right) \\ &- \epsilon^{-1}\frac{q}{\varepsilon_0}(\gamma_+ n_+ \boldsymbol{v}_{\perp +} - \gamma_- n_- \boldsymbol{v}_{\perp -}),\end{aligned} \tag{4.2i}$$

where

$$\nabla_\perp = \left(0, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right). \tag{4.3}$$

Now we look for the solution in the form of expansions in the power series with respect to ϵ,

$$\left.\begin{aligned}
p_s &= p_0 + \epsilon p_s^{(1)} + \epsilon^2 p_s^{(2)} + \epsilon^3 p_s^{(3)} + \cdots, \\
n_s &= n_0 + \epsilon n_s^{(1)} + \epsilon^2 n_s^{(2)} + \epsilon^3 n_s^{(3)} + \cdots, \\
v_{xs} &= \epsilon v_{xs}^{(1)} + \epsilon^2 v_{xs}^{(2)} + \epsilon^3 v_{xs}^{(3)} + \cdots, \\
\boldsymbol{v}_{\perp s} &= \epsilon \boldsymbol{v}_{\perp s}^{(1)} + \epsilon^2 \boldsymbol{v}_{\perp s}^{(2)} + \epsilon^3 \boldsymbol{v}_{\perp s}^{(3)} + \cdots, \\
B_x &= B_0 + \epsilon B_x^{(1)} + \epsilon^2 B_x^{(2)} + \epsilon^3 B_x^{(3)} + \cdots, \\
\boldsymbol{B}_\perp &= \epsilon \boldsymbol{B}_\perp^{(1)} + \epsilon^2 \boldsymbol{B}_\perp^{(2)} + \epsilon^3 \boldsymbol{B}_\perp^{(3)} + \cdots \\
E_x &= \epsilon E_x^{(1)} + \epsilon^2 E_x^{(2)} + \epsilon^3 E_x^{(3)} + \cdots, \\
\boldsymbol{E}_\perp &= \epsilon \boldsymbol{E}_\perp^{(1)} + \epsilon^2 \boldsymbol{E}_\perp^{(2)} + \epsilon^3 \boldsymbol{E}_\perp^{(3)} + \cdots.
\end{aligned}\right\} \tag{4.4}$$

We impose the boundary conditions at $\xi \to \infty$,

$$\left.\begin{aligned}
n_s \to n_0, \quad v_{xs} \to 0, \quad B_x \to B_0, \quad E_x \to 0, \\
\boldsymbol{v}_{\perp s} \to 0, \quad \boldsymbol{B}_\perp \to 0, \quad \boldsymbol{E}_\perp \to 0.
\end{aligned}\right\} \tag{4.5}$$

In the following, we use the expansion

$$\gamma_s = 1 + \frac{\epsilon^2}{2c^2}\left[v_{xs}^{(1)2} + \left|\boldsymbol{v}_{\perp s}^{(1)}\right|^2 + 2\epsilon\left(v_{xs}^{(1)} v_{xs}^{(2)} + \boldsymbol{v}_{\perp s}^{(1)} \cdot \boldsymbol{v}_{\perp s}^{(2)}\right)\right] + \cdots. \tag{4.6}$$

4.1. *The zero-order approximation*

Substituting the expansions given by (4.4) in (2.7*a*,*b*), (2.12) and (4.2), and collecting terms of the order of unity we obtain

$$n_+^{(1)} = n_-^{(1)} = n^{(1)}, \quad v_{x+}^{(1)} = v_{x-}^{(1)} = v_x^{(1)}, \quad \boldsymbol{v}_{\perp +}^{(1)} = \boldsymbol{v}_{\perp -}^{(1)} = \boldsymbol{v}_\perp^{(1)}, \tag{4.7a}$$

$$E_x^{(1)} = 0, \quad \boldsymbol{E}_\perp^{(1)} = B_0 \boldsymbol{e}_x \times \boldsymbol{v}_\perp^{(1)}. \tag{4.7b}$$

4.2. *The first-order approximation*

Collecting terms of the order of ϵ in (2.7*a*,*b*) and (4.2), and using (2.12), (4.6) and (4.7) yields

$$V\frac{\partial n^{(1)}}{\partial\xi} - n_0\frac{\partial v_x^{(1)}}{\partial\xi} = 0, \quad p_s^{(1)} = \kappa p_0 \frac{n^{(1)}}{n_0}, \quad \frac{\partial B_x^{(1)}}{\partial\xi} = 0, \tag{4.8a}$$

$$h_0 V\frac{\partial v_x^{(1)}}{\partial\xi} - \frac{c^2}{n_0}\frac{\partial p_s^{(1)}}{\partial\xi} = -c^2 q_s\left[E_x^{(2)} + \boldsymbol{e}_x \cdot \left(\boldsymbol{v}_\perp^{(1)} \times \boldsymbol{B}_\perp^{(1)}\right)\right], \tag{4.8b}$$

$$h_0 V\frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial\xi} = -c^2 q_s\left[\boldsymbol{E}_\perp^{(2)} - \boldsymbol{e}_x \times \left(B_0 \boldsymbol{v}_{\perp s}^{(2)} - v_x^{(1)} \boldsymbol{B}_\perp^{(1)} + B_x^{(1)} \boldsymbol{v}_\perp^{(1)}\right)\right], \tag{4.8c}$$

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$$V\frac{\partial \boldsymbol{E}_\perp^{(1)}}{\partial \xi} = -c^2\boldsymbol{e}_x \times \frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi} + \frac{qn_0}{\varepsilon_0}\left(\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}\right), \tag{4.8d}$$

$$V\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi} = \boldsymbol{e}_x \times \frac{\partial \boldsymbol{E}_\perp^{(1)}}{\partial \xi}, \quad n_+^{(2)} = n_-^{(2)} = n^{(2)}, \quad v_{x+}^{(2)} = v_{x-}^{(2)} = v_x^{(2)}. \tag{4.8e}$$

Each of (4.8*b*) and (4.8*c*) represent two equations, one for $s = +$ and the other for $s = -$. Adding and subtracting these equations we obtain

$$h_0 V\frac{\partial v_x^{(1)}}{\partial \xi} - \frac{c^2}{n_0}\frac{\partial p_s^{(1)}}{\partial \xi} = 0, \quad E_x^{(2)} = -\boldsymbol{e}_x \boldsymbol{\cdot} \left(\boldsymbol{v}_\perp^{(1)} \times \boldsymbol{B}_\perp^{(1)}\right), \tag{4.9a}$$

$$\frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial \xi} = \frac{c^2 qB_0}{2Vh_0}\boldsymbol{e}_x \times \left(\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}\right), \tag{4.9b}$$

$$2\boldsymbol{E}_\perp^{(2)} = \boldsymbol{e}_x \times \left[B_0\left(\boldsymbol{v}_{\perp+}^{(2)} + \boldsymbol{v}_{\perp-}^{(2)}\right) - 2v_x^{(1)}\boldsymbol{B}_\perp^{(1)} + 2B_x^{(1)}\boldsymbol{v}_\perp^{(1)}\right]. \tag{4.9c}$$

It follows from the boundary conditions (4.5), as well as (4.8*a*) and (4.9*a*), that

$$n^{(1)} = 0, \quad p_s^{(1)} = 0, \quad v_x^{(1)} = 0, \quad B_x^{(1)} = 0. \tag{4.10a–d}$$

Using (4.7*b*) and (4.8*e*) yields

$$\boldsymbol{E}_\perp^{(1)} = -V\boldsymbol{e}_x \times \boldsymbol{B}_\perp^{(1)}, \quad \boldsymbol{v}_\perp^{(1)} = -\frac{V}{B_0}\boldsymbol{B}_\perp^{(1)}. \tag{4.11a,b}$$

It follows from (4.8*b*), (4.8*d*) and (4.11*a,b*)

$$E_x^{(2)} = 0, \quad \boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)} = \frac{1}{qn_0\mu_0}\left(1 - \frac{V^2}{c^2}\right)\boldsymbol{e}_x \times \frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi}. \tag{4.12a,b}$$

Equation (4.9*b*), the second equation in (4.12*a,b*) and the equation obtained by differentiating the second equation in (4.11*a,b*) with respect to ξ constitute a linear homogeneous system of equations for $\partial \boldsymbol{v}_\perp^{(1)}/\partial\xi$, $\partial \boldsymbol{B}_\perp^{(1)}/\partial\xi$, and $\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}$. It only has non-trivial solutions when its determinant is zero. This condition determines that V is given by (3.6).

4.3. *The second-order approximation*

Now we collect terms of the order of ϵ^2 in (2.7*a,b*) and (4.2), and using (2.12) and (4.6), and the results obtained in the previous two subsections, we obtain

$$\frac{\partial v_x^{(2)}}{\partial \xi} - \frac{V}{n_0}\frac{\partial n^{(2)}}{\partial \xi} - \frac{V}{2c^2}\frac{\partial |\boldsymbol{v}_\perp^{(1)}|^2}{\partial \xi} + \boldsymbol{\nabla}_\perp \boldsymbol{\cdot} \boldsymbol{v}_\perp^{(1)} = 0, \quad p_s^{(2)} = \kappa p_0\frac{n^{(2)}}{n_0}, \tag{4.13a}$$

$$\frac{c^2}{n_0}\frac{\partial p^{(2)}}{\partial \xi} - h_0 V\frac{\partial v_x^{(2)}}{\partial \xi} = c^2 q_s\left[E_3^{(3)} + \boldsymbol{e}_x \boldsymbol{\cdot} \left(\boldsymbol{v}_\perp^{(1)} \times \boldsymbol{B}_\perp^{(2)} + \boldsymbol{v}_{\perp s}^{(2)} \times \boldsymbol{B}_\perp^{(1)}\right)\right], \tag{4.13b}$$

$$\frac{\partial \boldsymbol{v}_{\perp s}^{(2)}}{\partial \xi} = -\frac{c^2 q_s}{Vh_0}\left[\boldsymbol{E}_\perp^{(3)} - \boldsymbol{e}_x \times \left(B_0\boldsymbol{v}_{\perp s}^{(3)} - v_x^{(2)}\boldsymbol{B}_\perp^{(1)} + B_x^{(2)}\boldsymbol{v}_\perp^{(1)}\right)\right], \tag{4.13c}$$

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$$\nabla_\perp \cdot \boldsymbol{E}_\perp^{(1)} = \frac{q}{\varepsilon_0}\left[\frac{n_0}{c^2}\boldsymbol{v}_\perp^{(1)} \cdot \left(\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}\right) + n_+^{(3)} - n_-^{(3)}\right], \tag{4.13d}$$

$$\frac{\partial B_x^{(2)}}{\partial \xi} + \nabla_\perp \cdot \boldsymbol{B}_\perp^{(1)} = 0, \quad V\frac{\partial B_x^{(2)}}{\partial \xi} = \boldsymbol{e}_x \cdot \left(\nabla_\perp \times \boldsymbol{E}_\perp^{(1)}\right), \tag{4.13e}$$

$$V\frac{\partial \boldsymbol{B}_\perp^{(2)}}{\partial \xi} = \boldsymbol{e}_x \times \frac{\partial \boldsymbol{E}_\perp^{(2)}}{\partial \xi}, \quad \boldsymbol{e}_x \cdot \nabla_\perp \times \boldsymbol{B}_\perp^{(1)} = qn_0\mu_0\left(v_{x+}^{(3)} - v_{x-}^{(3)}\right), \tag{4.13f}$$

$$\frac{V}{c^2}\frac{\partial \boldsymbol{E}_\perp^{(2)}}{\partial \xi} + \boldsymbol{e}_x \times \frac{\partial \boldsymbol{B}_\perp^{(2)}}{\partial \xi} = qn_0\mu_0\left(\boldsymbol{v}_{\perp+}^{(3)} - \boldsymbol{v}_{\perp-}^{(3)}\right). \tag{4.13g}$$

Using the second equation in (4.11*a,b*) we transform the first equation in (4.13*a*) to

$$\frac{\partial v_x^{(2)}}{\partial \xi} - \frac{V}{n_0}\frac{\partial n^{(2)}}{\partial \xi} = \frac{V^3}{2c^2B_0^2}\frac{\partial |\boldsymbol{B}_\perp^{(1)}|^2}{\partial \xi} + \frac{V}{B_0}\nabla_\perp \cdot \boldsymbol{B}_\perp^{(1)}. \tag{4.14}$$

Equation (4.13*b*) represents two equations, one for $s = +$ and the other for $s = -$. Adding these equations and using the second equation in (4.13*a*) we obtain

$$\frac{\kappa c^2 p_0}{n_0^2}\frac{\partial n^{(2)}}{\partial \xi} - h_0 V\frac{\partial v_x^{(2)}}{\partial \xi} = \frac{c^2 q}{2}\boldsymbol{e}_x \cdot \left[\left(\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}\right) \times \boldsymbol{B}_\perp^{(1)}\right]. \tag{4.15}$$

Using (4.12*a,b*) we transform this equation to

$$\frac{\kappa c^2 p_0}{n_0^2}\frac{\partial n^{(2)}}{\partial \xi} - h_0 V\frac{\partial v_x^{(2)}}{\partial \xi} = -\frac{c^2 - V^2}{4n_0\mu_0}\frac{\partial |\boldsymbol{B}_\perp^{(1)}|^2}{\partial \xi}. \tag{4.16}$$

We find from (4.14) and (4.16)

$$\frac{\partial n^{(2)}}{\partial \xi} = \frac{h_0 n_0^2 V^2}{B_0(\kappa c^2 p_0 - n_0 h_0 V^2)}\left(\nabla_\perp \cdot \boldsymbol{B}_\perp^{(1)} - \frac{\mu_0 n_0 h_0 V^2}{c^2 B_0^3}\frac{\partial |\boldsymbol{B}_\perp^{(1)}|^2}{\partial \xi}\right), \tag{4.17a}$$

$$\frac{\partial v_x^{(2)}}{\partial \xi} = \frac{\kappa c^2 p_0 V}{B_0(\kappa c^2 p_0 - n_0 h_0 V^2)}\left(\nabla_\perp \cdot \boldsymbol{B}_\perp^{(1)} + \frac{V^2(\kappa p_0 - n_0 h_0)}{2\kappa c^2 p_0 B_0}\frac{\partial |\boldsymbol{B}_\perp^{(1)}|^2}{\partial \xi}\right). \tag{4.17b}$$

Using (4.11*a,b*) and (4.12*a,b*) we obtain from (4.13*d*)

$$n_+^{(3)} - n_-^{(3)} = \frac{\varepsilon_0 V}{q}\boldsymbol{e}_x \cdot \left(\nabla_\perp \times \boldsymbol{B}_\perp^{(1)} - \frac{c^2 - V^2}{c^2 B_0}\boldsymbol{B}_\perp^{(1)} \times \frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi}\right). \tag{4.18}$$

Finally, (4.13*c*) represents two equation, one for $s = +$ and the other for $s = -$. Subtracting the second equation from the first and using (4.9*b*) and (4.11*a,b*) yields

$$\boldsymbol{E}_\perp^{(3)} - \frac{B_0}{2}\boldsymbol{e}_x \times \left(\boldsymbol{v}_{\perp+}^{(3)} + \boldsymbol{v}_{\perp-}^{(3)}\right) = \boldsymbol{e}_x \times \left(B_x^{(2)}\boldsymbol{v}_\perp^{(1)} - v_x^{(2)}\boldsymbol{B}_\perp^{(1)} - \frac{2V^3 h_0^2}{q^2 c^4 B_0^2}\frac{\partial^2 \boldsymbol{B}_\perp^{(1)}}{\partial \xi^2}\right). \tag{4.19}$$

4.4. *The third-order approximation*

In the third-order approximation we collect the terms of the order of ϵ^3 in (4.2*c*), (4.2*g*) and (4.2*i*), and use (2.12) and the second equation in (4.13*a*) to obtain

$$\frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial \tau} + v_x^{(2)}\frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial \xi} - V\frac{\partial \boldsymbol{v}_{\perp s}^{(3)}}{\partial \xi} - \frac{V}{2c^2}\frac{\partial(\boldsymbol{v}_\perp^{(1)}|\boldsymbol{v}_\perp^{(1)}|^2)}{\partial \xi} - \frac{\kappa p_0 V}{h_0 n_0^2}\frac{\partial(n^{(2)}\boldsymbol{v}_\perp^{(1)})}{\partial \xi}$$

$$+\left(\boldsymbol{v}_\perp^{(1)}\cdot\nabla_\perp\right)\boldsymbol{v}_\perp^{(1)} + \frac{\kappa c^2 p_0}{h_0 n_0^2}\nabla_\perp n^{(2)} = \frac{c^2 q_s}{h_0}\left[\boldsymbol{E}_\perp^{(4)}\right.$$

$$\left.+\boldsymbol{e}_x\times\left(v_x^{(2)}\boldsymbol{B}_\perp^{(2)} + v_{xs}^{(3)}\boldsymbol{B}_\perp^{(1)} - B_0\boldsymbol{v}_{\perp s}^{(4)} - B_x^{(2)}\boldsymbol{v}_{\perp s}^{(2)} - B_x^{(3)}\boldsymbol{v}_\perp^{(1)}\right)\right], \tag{4.20a}$$

$$\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \tau} - V\frac{\partial \boldsymbol{B}_\perp^{(3)}}{\partial \xi} = -\boldsymbol{e}_x\times\frac{\partial \boldsymbol{E}_\perp^{(3)}}{\partial \xi}, \tag{4.20b}$$

$$\frac{\partial \boldsymbol{E}_\perp^{(1)}}{\partial \tau} - V\frac{\partial \boldsymbol{E}_\perp^{(3)}}{\partial \xi} = c^2\boldsymbol{e}_x\times\left(\frac{\partial \boldsymbol{B}_\perp^{(3)}}{\partial \xi} - \nabla_\perp B_x^{(2)}\right)$$

$$-\frac{q}{\varepsilon_0}\left\{n_0\left(\boldsymbol{v}_{\perp+}^{(4)} - \boldsymbol{v}_{\perp-}^{(4)}\right) + \left(\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}\right)\left(n^{(2)} + \frac{n_0}{2c^2}|\boldsymbol{v}_\perp^{(1)}|^2\right)\right.$$

$$\left.+\boldsymbol{v}_\perp^{(1)}\left(n_+^{(3)} - n_-^{(3)}\right) + \frac{n_0}{c^2}\boldsymbol{v}_\perp^{(1)}\left[\boldsymbol{v}_\perp^{(1)}\cdot\left(\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}\right)\right]\right\}. \tag{4.20c}$$

Equation (4.20*a*) represents two equations, one for $s = +$ and the other for $s = -$. Adding these equations yields

$$\frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial \tau} + v_x^{(2)}\frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial \xi} - \frac{V}{2}\frac{\partial(\boldsymbol{v}_{\perp+}^{(3)} + \boldsymbol{v}_{\perp-}^{(3)})}{\partial \xi} - \frac{V}{2c^2}\frac{\partial(\boldsymbol{v}_\perp^{(1)}|\boldsymbol{v}_\perp^{(1)}|^2)}{\partial \xi}$$

$$-\frac{\kappa p_0 V}{h_0 n_0^2}\frac{\partial(n^{(2)}\boldsymbol{v}_\perp^{(1)})}{\partial \xi} + (\boldsymbol{v}_\perp^{(1)}\cdot\nabla_\perp)\boldsymbol{v}_\perp^{(1)} + \frac{\kappa c^2 p_0}{h_0 n_0^2}\nabla_\perp n^{(2)}$$

$$= \frac{c^2 q}{2h_0}\boldsymbol{e}_x\times\left[\boldsymbol{B}_\perp^{(1)}\left(v_{x+}^{(3)} - v_{x-}^{(3)}\right) - B_0\left(\boldsymbol{v}_{\perp+}^{(4)} - \boldsymbol{v}_{\perp-}^{(4)}\right) - B_x^{(2)}\left(\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}\right)\right]. \tag{4.21}$$

Using (4.12*a*,*b*), (4.13*f*), (4.18) and (4.19) we transform (4.20*c*) and (4.21) to

$$\frac{\partial \boldsymbol{B}_\perp^{(3)}}{\partial \xi} - \frac{V}{c^2}\boldsymbol{e}_x\times\frac{\partial \boldsymbol{E}_\perp^{(3)}}{\partial \xi} + n_0 q\mu_0\boldsymbol{e}_x\times\left(\boldsymbol{v}_{\perp+}^{(4)} - \boldsymbol{v}_{\perp-}^{(4)}\right) = \nabla_\perp B_x^{(2)} - \frac{V}{c^2}\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \tau}$$

$$+\frac{c^2 - V^2}{n_0 c^2}\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi}\left(n^{(2)} + \frac{n_0 V^2}{2c^2 B_0^2}|\boldsymbol{B}_\perp^{(1)}|^2\right) + \frac{V^2}{c^2 B_0}\left(\boldsymbol{e}_x\times\boldsymbol{B}_\perp^{(1)}\right)\boldsymbol{e}_x\cdot\nabla_\perp\times\boldsymbol{B}_\perp^{(1)}, \tag{4.22}$$

$$\boldsymbol{e}_x\times\frac{\partial \boldsymbol{E}_\perp^{(3)}}{\partial \xi} + \frac{c^2 q B_0^2}{2h_0 V}\boldsymbol{e}_x\times\left(\boldsymbol{v}_{\perp+}^{(4)} - \boldsymbol{v}_{\perp-}^{(4)}\right) = \frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \tau} - \frac{h_0^2 V^3}{c^2 q^2 B_0^2}\frac{\partial^3 \boldsymbol{B}_\perp^{(1)}}{\partial \xi^3}$$

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$$
+\left(v_x^{(2)}+\frac{B_0(c^2-V^2)}{2n_0h_0\mu_0V}B_x^{(2)}\right)\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial\xi}+\frac{\partial}{\partial\xi}\left[\boldsymbol{B}_\perp^{(1)}\left(v_x^{(2)}-\frac{\kappa p_0Vn^{(2)}}{h_0n_0^2}\right.\right.
$$

$$
\left.\left.+\frac{V}{B_0}B_x^{(2)}-\frac{V^3}{2c^2B_0^2}\left|\boldsymbol{B}_\perp^{(1)}\right|^2\right)\right]+\frac{c^2B_0}{2n_0h_0\mu_0V}\left(\boldsymbol{e}_x\times\boldsymbol{B}_\perp^{(1)}\right)\boldsymbol{e}_x\cdot\nabla_\perp\times\boldsymbol{B}_\perp^{(1)}
$$

$$
-\frac{V}{B_0}\left(\boldsymbol{B}_\perp^{(1)}\cdot\nabla_\perp\right)\boldsymbol{B}_\perp^{(1)}-\frac{\kappa c^2p_0B_0}{h_0n_0^2V}\nabla_\perp n^{(2)}. \tag{4.23}
$$

Equations (4.20b), (4.22) and (4.23) constitute a system of linear inhomogeneous algebraic equations for $\partial\boldsymbol{B}_\perp^{(3)}/\partial\xi$, $\partial\boldsymbol{E}_\perp^{(3)}/\partial\xi$ and $\boldsymbol{v}_{\perp+}^{(4)}-\boldsymbol{v}_{\perp-}^{(4)}$. Now we obtain the condition of compatibility for this system. To do this we eliminate the variables $\boldsymbol{B}_\perp^{(3)}$, $\boldsymbol{E}_\perp^{(3)}$ and $\boldsymbol{v}_{\perp+}^{(4)}-\boldsymbol{v}_{\perp-}^{(4)}$ from the system of (4.20b), (4.22) and (4.23). First we use (4.20b) to eliminate $\boldsymbol{B}_\perp^{(3)}$ from (4.22) and then divide the obtained equation by

$$
\frac{c^2-V^2}{c^2V}=\frac{2\mu_0n_0h_0V}{c^2B_0^2}=\frac{V}{\sigma_0c^2}. \tag{4.24}
$$

Then we obtain the equation with the left-hand side coinciding with the left-hand side of (4.23). Subtracting the obtained equation from (4.23) we obtain

$$
\frac{\partial\boldsymbol{B}_\perp^{(1)}}{\partial\tau}+\frac{c^2-V^2}{2c^2}\left(v_x^{(2)}-\frac{V}{n_0}n^{(2)}+\frac{V}{B_0}B_x^{(2)}-\frac{V^3}{2c^2B_0^2}|\boldsymbol{B}_\perp^{(1)}|^2\right)\frac{\partial\boldsymbol{B}_\perp^{(1)}}{\partial\xi}
$$

$$
+\frac{c^2-V^2}{2c^2}\frac{\partial}{\partial\xi}\left[\boldsymbol{B}_\perp^{(1)}\left(v_x^{(2)}-\frac{\kappa p_0V}{h_0n_0^2}n^{(2)}+\frac{V}{B_0}B_x^{(2)}-\frac{V^3}{2c^2B_0^2}|\boldsymbol{B}_\perp^{(1)}|^2\right)\right]
$$

$$
+\frac{V(c^2-V^2)}{2c^2B_0}\left[\left(\boldsymbol{e}_x\times\boldsymbol{B}_\perp^{(1)}\right)\boldsymbol{e}_x\cdot\nabla_\perp\times\boldsymbol{B}_\perp^{(1)}-\left(\boldsymbol{B}_\perp^{(1)}\cdot\nabla_\perp\right)\boldsymbol{B}_\perp^{(1)}\right]
$$

$$
-\frac{\kappa\mu_0p_0V}{n_0B_0}\nabla_\perp n^{(2)}-\frac{V}{2}\nabla_\perp B_x^{(2)}+\frac{h_0^2V^3(c^2-V^2)}{2c^6q^2B_0^2}\frac{\partial^3\boldsymbol{B}_\perp^{(1)}}{\partial\xi^3}=0. \tag{4.25}
$$

In the following, we use two identities that can be verified by the direct calculation,

$$
\left(\boldsymbol{e}_x\times\boldsymbol{B}_\perp^{(1)}\right)\boldsymbol{e}_x\cdot\nabla_\perp\times\boldsymbol{B}_\perp^{(1)}=-\boldsymbol{B}_\perp^{(1)}\times\left(\nabla_\perp\times\boldsymbol{B}_\perp^{(1)}\right), \tag{4.26a}
$$

$$
\left(\boldsymbol{B}_\perp^{(1)}\cdot\nabla_\perp\right)\boldsymbol{B}_\perp^{(1)}=\frac{1}{2}\nabla_\perp|\boldsymbol{B}_\perp^{(1)}|^2-\boldsymbol{B}_\perp^{(1)}\times\left(\nabla_\perp\times\boldsymbol{B}_\perp^{(1)}\right). \tag{4.26b}
$$

It follows from (3.6) that

$$
\frac{h_0^2V^3(c^2-V^2)}{2c^6q^2B_0^2}=V\ell^2. \tag{4.27}
$$

Using (3.6), (4.13e), (4.17) and (4.24) we obtain

$$
v_x^{(2)}-\frac{V}{n_0}n^{(2)}+\frac{V}{B_0}B_x^{(2)}-\frac{V^3}{2c^2B_0^2}|\boldsymbol{B}_\perp^{(1)}|^2=0, \tag{4.28a}
$$

$$
v_x^{(2)}-\frac{\kappa p_0V}{h_0n_0^2}n^{(2)}+\frac{V}{B_0}B_x^{(2)}-\frac{V^3}{2c^2B_0^2}|\boldsymbol{B}_\perp^{(1)}|^2
$$

$$
=\frac{V^3(n_0h_0-\kappa p_0)}{n_0h_0V^2-\kappa c^2p_0}\left(\frac{c^2-V^2}{2c^2B_0^2}|\boldsymbol{B}_\perp^{(1)}|^2-\frac{\varPhi}{B_0}\right), \tag{4.28b}
$$

https://doi.org/10.1017/S0022377823000156 Published online by Cambridge University Press

where

$$\frac{\partial \Phi}{\partial \xi} = \nabla_\perp \cdot \boldsymbol{B}_\perp^{(1)}, \quad \Phi \to 0 \quad \text{as } \xi \to \infty. \tag{4.29}$$

Using (4.24), (4.26), (4.27), (4.28) and (4.29) we transform (4.25) to

$$\begin{aligned} &\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \tau} + \frac{\alpha(c^2 - V^2)}{c^2}\frac{\partial}{\partial \xi}\left[\boldsymbol{B}_\perp^{(1)}\left(\frac{c^2-V^2}{c^2}|\boldsymbol{B}_\perp^{(1)}|^2 - 2B_0\Phi\right)\right] \\ &-\alpha B_0 \nabla_\perp\left(\frac{c^2-V^2}{c^2}|\boldsymbol{B}_\perp^{(1)}|^2 - 2B_0\Phi\right) + V\ell^2\frac{\partial^3 \boldsymbol{B}_\perp^{(1)}}{\partial \xi^3} = 0, \end{aligned} \tag{4.30}$$

where

$$\alpha = \frac{V^3(n_0h_0 - \kappa p_0)}{4n_0h_0B_0^2(V^2 - a_0^2)}. \tag{4.31}$$

We note that, in accordance with (4.6), $\gamma = 1$ in the leading-order approximation. Hence, the expression for the sound speed given by (4.6) is simplified to $a_0^2 = \kappa c^2 p_0 (n_0 h_0)^{-1}$. It follows from (3.2*a*) that $n_0h_0 > \kappa p_0$. Hence, $\alpha > 0$ when $V > a_0$, that is, when transverse waves are the merged Alfvén and fast waves, whereas $\alpha < 0$ when $V < a_0$, that is, when transverse waves are the merged Alfvén and slow waves.

Now we return to the original independent variables and introduce the notation

$$\boldsymbol{b} = \frac{\epsilon(c^2 - V^2)}{c^2}\boldsymbol{B}_\perp^{(1)}, \quad \varphi = \frac{\epsilon^2(c^2-V^2)}{c^2}\Phi. \tag{4.32}$$

As a result we reduce (4.29) and (4.30) to

$$\frac{\partial \varphi}{\partial x} = \nabla_\perp \cdot \boldsymbol{b}, \quad \varphi \to 0 \quad \text{as } x \to \infty, \tag{4.33}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} + V\frac{\partial \boldsymbol{b}}{\partial x} + \alpha\frac{\partial}{\partial x}\left[\boldsymbol{b}(|\boldsymbol{b}|^2 - 2B_0\varphi)\right] - \alpha\nabla_\perp(|\boldsymbol{b}|^2 - 2B_0\varphi) + V\ell^2\frac{\partial^3 \boldsymbol{b}}{\partial x^3} = 0, \tag{4.34}$$

where we changed the notation and put

$$\nabla_\perp = \left(0, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{4.35}$$

Formally (4.34) is the same as the equation derived in Paper I for non-relativistic plasmas. Only the expressions for α and V in terms of equilibrium quantities are more involved. In the limit of $c \to \infty$ these expressions are reduced to those obtained in Paper I.

5. Solitary waves

In this section, we consider solutions to the system of (4.33) and (4.34) that depend on t and $X = x + k_y y + k_z z$, where k_y and k_z are constant, and $k_y \ll 1$, $k_z \ll 1$. Then (4.33) and (4.34) reduce to

$$\frac{\partial \varphi}{\partial X} = \frac{\partial(\boldsymbol{k}_\perp \cdot \boldsymbol{b})}{\partial X}, \quad \varphi \to 0 \quad \text{as } X \to \infty, \tag{5.1}$$

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$$\frac{\partial \boldsymbol{b}}{\partial t} + V\frac{\partial \boldsymbol{b}}{\partial X} + \alpha\frac{\partial}{\partial X}\left[\boldsymbol{b}(|\boldsymbol{b}|^2 - 2B_0\varphi)\right]$$
$$-\alpha B_0\boldsymbol{k}_\perp\frac{\partial}{\partial X}(|\boldsymbol{b}|^2 - 2B_0\varphi) + V\ell^2\frac{\partial^3 \boldsymbol{b}}{\partial X^3} = 0, \tag{5.2}$$

where $\boldsymbol{k}_\perp = (0, k_y, k_z)$. It follows from (5.1) that $\varphi = \boldsymbol{k}_\perp \cdot \boldsymbol{b}$. Substituting this result into (5.2) yields

$$\frac{\partial \boldsymbol{h}}{\partial t} + W\frac{\partial \boldsymbol{h}}{\partial X} + \alpha\frac{\partial(|\boldsymbol{h}|^2\boldsymbol{h})}{\partial X} + V\ell^2\frac{\partial^3 \boldsymbol{h}}{\partial X^3} = 0, \tag{5.3}$$

where

$$\boldsymbol{h} = \boldsymbol{b} - B_0\boldsymbol{k}_\perp, \quad W = V - \alpha B_0^2 k_\perp^2. \tag{5.4a,b}$$

Equation (5.3) is the non-integrable vector mKdV equation. Previously this equation was derived for waves propagating along the magnetic field in electron–positron plasmas (Sakai & Kawata 1980*b*; Verheest 1996). It was also derived for the description of transverse perturbations in a chain of interacting particles (Gorbacheva & Ostrovsky 1983), nonlinear waves in micropolar media (Erbay & Suhubi 1989*a*,*b*), in generalised elastic solids (Erbay 1999) and deformed hyperelastic dispersive solids (Destrade & Saccomandi 2008). As (5.2) was derived under assumption that all perturbations decay at infinity, it follows that $\boldsymbol{b} \to 0$ and, consequently, $\boldsymbol{h} \to -B_0\boldsymbol{k}_\perp$ as $X \to \infty$.

We note that there is also the integrable vector mKdV equation. In differs from (5.3) in that the nonlinear term is proportional to $|\boldsymbol{h}|^2(\partial\boldsymbol{h}/\partial X)$.

Now we look for solutions describing solitary waves. It is more convenient to use not (5.3) but (5.2) with $\varphi = \boldsymbol{k}_\perp \cdot \boldsymbol{b}$. We look for the solutions to this equation that depend on $\theta = X - (C + V)t$, where C is a constant. Then we obtain

$$C\boldsymbol{b} - \alpha\left[(\boldsymbol{b} - B_0\boldsymbol{k}_\perp)(|\boldsymbol{b}|^2 2B_0\boldsymbol{k}_\perp \cdot \boldsymbol{b})\right] - V\ell^2\frac{\partial^2 \boldsymbol{b}}{\partial \theta^2} = 0. \tag{5.5}$$

This is the system of two second-order differential equations for b_y and b_z. It was shown in Paper I that this system can be written in the Hamiltonian form as

$$\dot{g}_y = -\frac{\partial \mathcal{H}}{\partial b_y}, \quad \dot{g}_z = -\frac{\partial \mathcal{H}}{\partial b_z}, \quad \dot{b}_y = \frac{\partial \mathcal{H}}{\partial g_y}, \quad \dot{b}_z = \frac{\partial \mathcal{H}}{\partial g_z}, \tag{5.6a–d}$$

where the Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}\left(g_y^2 + g_z^2\right) + \frac{1}{4V\ell^2}\left[\alpha(b^2 - 2B_0\boldsymbol{k}_\perp \cdot \boldsymbol{b})^2 - 2Cb^2\right], \tag{5.7}$$

$g_y = \dot{b}_y$, $g_z = \dot{b}_z$, $b^2 = b_y^2 + b_z^2$ and the dot indicates the derivative with respect to θ.

When $\boldsymbol{k}_\perp = 0$ the waves propagate exactly along the equilibrium magnetic field. In this case the two equations for b_y and b_z defined by (5.5) are identical. Hence, we can take one of the two variables equal to zero. We take $b_z = 0$. Then the expression for the soliton propagating along the magnetic field is (see, e.g., Verheest 1996)

$$b_y = \frac{\sqrt{2C}}{\sqrt{\alpha}\cosh(\theta/L)}, \quad L = \ell\sqrt{\frac{V}{C}}. \tag{5.8a,b}$$

It is important that this solution only exists when $\alpha > 0$.

Now we consider the case where $\boldsymbol{k}_\perp \neq 0$. Without loss of generality we can take $k_z = 0$ and $k_y > 0$. In the following, we write k instead of k_y. First we consider planar solitons with $b_z = 0$. We have $\mathcal{H} = 0$ for solitons. Then it follows from (5.7) that b_y satisfies the equation

$$2V\ell^2 \dot{b}_y^2 = b_y^2 \left[2C - \alpha(b_y - 2B_0 k)^2\right]. \tag{5.9}$$

Solutions to this equation describing solitons satisfy the condition $b_y \to 0$ as $|\theta| \to \infty$. It is shown in Paper I that these solutions only exist for

$$C > 2\alpha B_0^2 k^2 \tag{5.10}$$

when $\alpha > 0$. These solutions are given by

$$b_y = \frac{\pm 2(C - 2\alpha B_0^2 k^2)}{\sqrt{2\alpha C}\cosh(\theta/L) \mp 2\alpha B_0 k}, \tag{5.11}$$

where

$$L = \ell\sqrt{\frac{V}{C - 2\alpha B_0^2 k^2}}. \tag{5.12}$$

This expression is slightly different from that given in Paper I. However, it is easily obtained from the latter by the linear substitution of the independent variable. The solitons described by (5.7) with the upper and lower signs are called bright and dark, respectively.

When $\alpha < 0$ solutions describing solitons exist only when C satisfies the inequality

$$2\alpha B_0^2 k^2 < C < 0. \tag{5.13}$$

In this case there is only one soliton described by (5.7) with the upper sign. As now there is only one soliton, we do not use the terms 'bright' and 'dark' when $\alpha < 0$.

It is shown in Paper I that bright solitons are stable and dark solitons are unstable with respect to transverse perturbations. When $\alpha < 0$ the solitons are unstable with respect to these perturbations.

Finally, (5.9) also posses a solution in the form of an algebraic soliton. It can be verified by the direct substitution that

$$b_y = \frac{4kB_0}{1 + (\theta/\lambda)^2}, \quad \lambda = \frac{\ell}{kB_0}\sqrt{\frac{V}{2\alpha}} \tag{5.14a,b}$$

is a solution to (5.9) when $C = 2\alpha B_0^2 k^2$. It can be also obtained from (5.11) with the upper sign by taking $C \to 2\alpha B_0^2 k^2$.

6. Non-planar solitary waves

In the previous section we described planar solitary waves where the magnetic field perturbation is in the plane defined by the direction of solitary wave propagation and the unperturbed magnetic field, that is, the plane containing the vectors $\boldsymbol{e}_x$ and $\boldsymbol{k}_\perp$. These solutions can be called solitons because they are the solutions to the integrable mKdV equation. In this section we consider non-planar solitary waves where the magnetic field perturbation is not in the plane defined by the vectors $\boldsymbol{e}_x$ and $\boldsymbol{k}_\perp$. As they are solutions to the non-integrable vector mKdV equation we call them solitary waves rather than solitons. We only consider the case where $\alpha > 0$.

Although we cannot rigorously prove this, it seems that the Hamiltonian system defined by (5.6*a–d*) and (5.7) is non-integrable, that is, there is only one first integral of this system, $\mathcal{H} = \text{const}$. Hence, we only can find non-planar solitary waves numerically. For the numerical study, we introduce the dimensionless variables

$$\sigma = \frac{kB_0\theta\sqrt{\alpha}}{\ell\sqrt{V}}, \quad u_y = \frac{b_y}{kB_0}, \quad u_z = \frac{b_z}{kB_0}, \quad U = \frac{C}{\alpha k^2 B_0^2}. \tag{6.1a–d}$$

We note that U is the dimensionless soliton speed in the reference frame moving with the velocity of infinitely long small perturbations V

Without loss of generality, we always can take $\boldsymbol{k}_\perp = k\boldsymbol{e}_y$ with $k > 0$, where $\boldsymbol{e}_y$ is the unit vector of the y-axis. Then the Hamiltonian system of equations (5.6*a–d*) reduces to

$$\frac{\mathrm{d}^2 u_y}{\mathrm{d}\sigma^2} = -(u^2 - 2u_y)(u_y - 1) + Uu_y, \tag{6.2}$$

$$\frac{\mathrm{d}^2 u_z}{\mathrm{d}\sigma^2} = -u_z(u^2 - 2u_y - U), \tag{6.3}$$

where $u^2 = u_y^2 + u_z^2$. The dimensionless form of the Hamiltonian is

$$\tilde{\mathcal{H}} = \frac{2V\ell^2}{\alpha k^4 B_0^4}\mathcal{H} = \left(\frac{\mathrm{d}u_y}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}u_z}{\mathrm{d}\sigma}\right)^2 + \frac{1}{2}\left[(u^2 - 2u_y)^2 - 2Uu^2\right]. \tag{6.4}$$

For a solitary wave we have $\tilde{\mathcal{H}} = 0$. This condition was used for controlling the accuracy of numerical solutions. In the dimensionless variables, (5.11) describing planar solitary waves is rewritten as

$$u_y = \frac{\pm 2(U - 2)}{\sqrt{2U}\cosh(\sigma\sqrt{U - 2}) \mp 2}. \tag{6.5}$$

The solutions describing these solitons exist for $U > 2$. The asymptotic behaviour of solutions to the system of (6.2) and (6.3) decaying at infinity is given by

$$u_y \sim \exp\left(-\sqrt{U - 2}\,|\sigma|\right), \quad u_z \sim \exp\left(-\sqrt{U}\,|\sigma|\right). \tag{6.6a,b}$$

It follow from these asymptotic expressions that non-planar solitary waves also can exist only for $U > 2$.

The system of (6.2) and (6.3) is very complicated and the full study of properties of its solutions deserves a separate paper. Here we only present its particular solutions showing that non-planar solitary waves exist. We looked for solutions where $b_y(\sigma)$ is an even function and $b_z(\sigma)$ is an odd function. In accordance with this we impose the initial conditions at $\sigma = 0$

$$u_y = 2 + a\sqrt{2U}, \quad \frac{\mathrm{d}u_y}{\mathrm{d}\sigma} = 0, \quad \frac{\mathrm{d}u_z}{\mathrm{d}\sigma} = -\left(2 + a\sqrt{2U}\right)\sqrt{(1 - a^2)U}, \quad u_z = 0, \tag{6.7a–d}$$

where $a \in (0, 1]$. Taking $a = 1$ we obtain the solution given by (6.5) with the upper sign and $b_z = 0$. We note that the expressions for u_y and $\mathrm{d}u_z/\mathrm{d}\sigma$ at $\sigma = 0$ are related by the condition $\tilde{\mathcal{H}} = 0$. Obviously if $u_y(\sigma)$ and $u_z(\sigma)$ are solutions to the system of (6.2) and (6.3), then $u_y(\sigma)$ and $-u_z(\sigma)$ are also solutions. Hence, the solitary waves exist in pairs.

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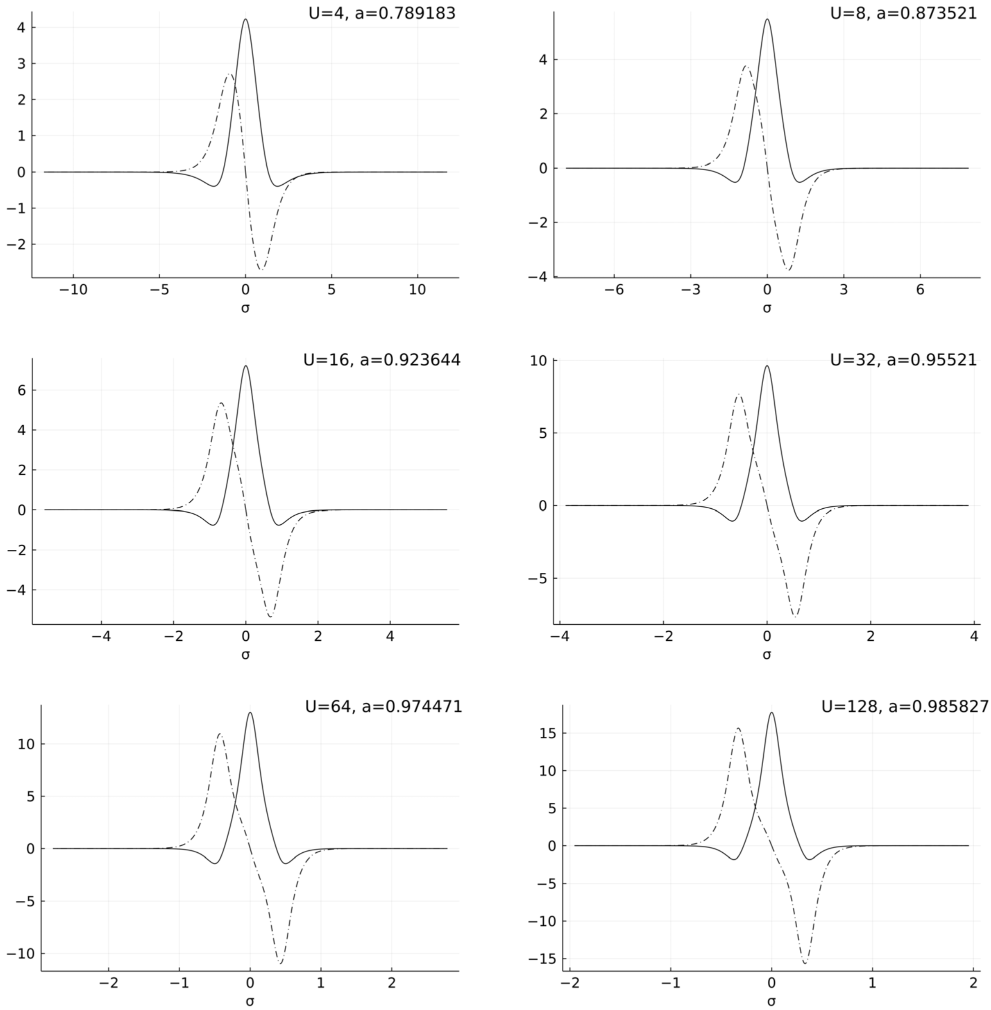

FIGURE 2. Dependence of u_y (solid line) and u_z (dash-dotted line) on σ for various values of U and a given in the figures.

In general, solutions to the system of (6.2) and (6.3) with the initial conditions (6.7*a–d*) are very complicated with both $u_y(\sigma)$ and $u_z(\sigma)$ having many maxima and minima. However, for any values of U we found a particular value of a when $u_y(\sigma)$ closely resembles this function defined by (6.5) with the upper sign. These solutions are shown in figure 2 for various values of U.

The solitary wave half-width $\bar{\sigma}$ can be defined as the half-width of the graph of $u_y(\sigma)$ at the half of its height, that is, $u_y(\bar{\sigma}) = \frac{1}{2}u_y(0)$. We calculated the dependences of U and $\bar{\sigma}$ on the solitary wave amplitude $A = u_y(0)$ for non-planar solitary waves. They are shown in figure 3 by solid curves. For comparison, we calculated the same quantities for planar solitons. It follows from (6.5) that

$$U = \frac{1}{2}(A-2)^2, \quad \bar{\sigma} = \frac{\sqrt{2}}{\sqrt{A(A-4)}} \ln \frac{2(A-3)+\sqrt{(A-4)(3A-8)}}{A-2}. \qquad (6.8a,b)$$

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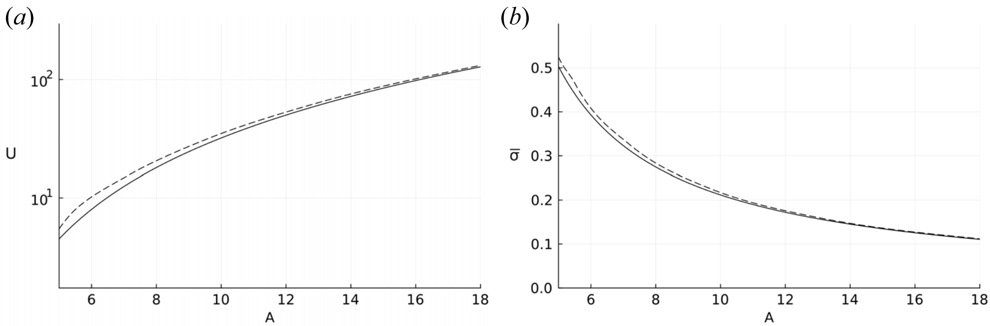

FIGURE 3. Dependence of (*a*) the dimensionless relative soliton velocity U and (*b*) the dimensionless soliton width $\bar{\sigma}$ on the dimensional soliton amplitude A. The solid lines correspond to non-planar solitons and the dashed lines to planar solitons described by (6.5) with the upper sign.

The curves defined by these equations are shown in figure 3 by dashed lines. We can see that the difference between the solid and dashed curves is very small.

7. Summary and conclusions

In this article, we have studied the propagation of nonlinear waves along the background magnetic field in a relativistic electron–positron plasma. We assumed that the characteristic spatial scale of variation of all quantities along the background magnetic field, L_{ch}, is much larger than the dispersion length, whereas the characteristic spatial scale of variation of all quantities in the directions orthogonal to the background magnetic field is much larger than L_{ch}. Using the reductive perturbation method, we have derived an equation describing the temporary evolution of the magnetic field perturbation in these waves. This equation is a generalisation of a similar equation previously derived for waves in a non-relativistic electron–positron plasma, and it coincides with the non-relativistic equation when the speed of light tends to infinity. When the magnetic field perturbation does not vary in the directions orthogonal to the background magnetic field this equation reduces to the vector mKdV equation.

As the relativistic equation differs from the non-relativistic equation only in the expressions for coefficients, we immediately used the results obtained previously for the non-relativistic equation. In particular, we have presented the expression for quasi-parallel planar solitons and then recalled the results on the stability of these solitons with respect to transverse perturbations.

We have then studied non-planar solitary waves. They are described by a Hamiltonian system of equations with two degrees of freedom, which is an autonomous system of two second-order differential equations. Although we cannot prove this rigorously, it seems that this system is not integrable. Therefore, we looked for non-planar solitary waves numerically. In general, these solitary waves have very complex properties; however, for some values of parameters we found solutions where the behaviour of one of the two components of the magnetic field perturbation resembles very closely the behaviour of the magnetic field perturbation in planar solitons. The numerical results clearly show that non-planar solitary waves exist.

Acknowledgements

Editor D. Uzdensky thanks the referees for their advice in evaluating this article.

Funding

E.P. is grateful to the RSF (grant 19-12-00253) for financial support.

Declaration of interests

The authors report no conflict of interest.

REFERENCES

AHARONIAN, F.A., BOGOVALOV, S.V. & KHANGULYAN, D. 2012 Abrupt acceleration of a 'cold' ultrarelativistic wind from the Crab pulsar. *Nature* **482**, 507–509.

ARONS, J. & BARNARD, J.J. 1986 Wave propagation in pulsar magnetospheres: dispersion relations and normal modes of plasmas in superstrong magnetic fields. *Astrophys. J.* **302**, 120–155.

BEGELMAN, M.C., BLANDFORD, R.D. & REES, M.J. 1984 Theory of extragalactic radio-sources. *Rev. Mod. Phys.* **56**, 255–351.

CATTAERT, T., KOURAKIS, I. & SHUKLA, P.K. 2005 Envelope solitons associated with electromagnetic waves in a magnetized pair plasma. *Phys. Plasmas* **12**, 012319.

CERUTTI, B. & BELOBORODOV, A.M. 2017 Electrodynamics of pulsar magnetospheres. *Space Sci. Rev.* **207**, 111–136.

CHIAN, A.C.-L. & KENNEL, C.F. 1983 Self-modulational formation of pulsar microstructures. *Astrophys. Space Sci.* **97**, 9–18.

DESTRADE, M. & SACCOMANDI, G. 2008 Nonlinear transverse waves in deformed dispersive solids. *Wave Motion* **43**, 325–336.

EL-LABANY, S.K., EL-SHAMY, E.F., SABRY, R. & KHEDR, D.M. 2013 The interaction of two nonplanar solitary waves in electron–positron–ion plasmas: an application in active galactic nuclei. *Phys. Plasmas* **20**, 012105.

ERBAY, S. 1999 Coupled modified Kadomtsev–Petviashvili equations in dispersive elastic media. *Intl J. Non-Linear Mech.* **34**, 289–297.

ERBAY, S. & SUHUBI, E.S. 1989*a* Nonlinear-wave propagation in micropolar media. 1. The general-theory. *Intl J. Engng Sci.* **27**, 895–914.

ERBAY, S. & SUHUBI, E.S. 1989*b* Nonlinear-wave propagation in micropolar media. 2. Special cases, solitary waves and Painleve analysis. *Intl J. Engng Sci.* **27**, 915–919.

FEDUN, V.M., RUDERMAN, M.S. & ERDÉLYI, R. 2008 Generation of short-lived large-amplitude magnetohydrodynamic pulses by dispersive focusing. *Phys. Lett.* A **372**, 6107–6110.

GAHN, C., TSAKIRIS, G.D., PRETZLER, G., WITTE, K.J., DELFIN, C., WAHLSTRÖM, C.G. & HABS, D. 2000 Generating positrons with femtosecond-laser pulses. *Appl. Phys. Lett.* **77**, 2662–2664.

GAILIS, R.M., FRANKEL, N.E. & DETTMANN, C.P. 1995 Magnetohydrodynamics in the expanding Universe. *Phys. Rev.* D **52**, 6901–6917.

GORBACHEVA, O.B. & OSTROVSKY, L.A. 1983 Non-linear vector waves in a mechanical model of a molecular chain. *Physica* D **8**, 223–228.

ICHIKAWA, Y.-H., KONNO, K., WADATI, M. & SANUKI, H. 1980 Spiky soliton in circular polarized Alfvén wave. *J. Phys. Soc. Japan* **48**, 279–286.

IWAMOTO, N. 1993 Collective modes in nonrelativistic electron–positron plasmas. *Phys. Rev.* E **47**, 604–611.

KADOMTSEV, B.B. & PETVIASHVILI, V.I. 1970 On the stability of solitary waves in weakly dispersing media. *Sov. Phys. Dokl.* **15**, 539–541. [Translated from Russian: Kadomtsev, B. B. & Petviashvili, V. I. 1970 Ob ustoychivosti uedinyonnyh voln v slabo dispergiruyushchih sredah. *Doklady Akademii Nauk SSSR* **192**, 753–756.]

KAKUTANI, T., ONO, H., TANIUTI, T. & WEI, C.-C. 1968 Reductive perturbation method in nonlinear wave propagation II. Application to hydromagnetic waves in cold plasma. *J. Phys. Soc. Japan* **24**, 1159–1166.

KAUP, D.J. & NEWELL, A.C. 1978 An exact solution for a derivative nonlinear Schrödinger equation. *J. Math. Phys.* **19**, 798–801.

KAWAKATU, N., KINO, M. & TAKAHARA, F. 2016 Evidence for a significant mixture of electron/positron pairs in FRII jets constrained by cocoon dynamics. *Mon. Not. R. Astron. Soc.* **457**, 1124–1136.

KAWATA, T. & INOUE, H. 1978 Exact solutions of the derivative nonlinear Schrödinger equation under the nonvanishing conditions. *J. Phys. Soc. Japan* **44**, 1968–1976.

LAKHINA, G.S. & VERHEEST, F. 1997 Alfvénic solitons in ultrarelativistic electron–positron plasmas. *Astrophys. Space Sci.* **253**, 97–106.

LANDAU, L.D. & LIFSHITZ, E.M. 1966 *Course of Theoretical Physics*, vol. 6, 3rd edn., Fluid Mechanics. Pergamon Press.

LANDAU, L.D. & LIFSHITZ, E.M. 1975 *Course of Theoretical Physics*, vol. 2, 4th edn., The Classical Theory of Fields. Butterworth Heinemann.

LIANG, E.P., WILKS, S.C. & TABAK, M. 1998 Pair production by ultraintense lasers. *Phys. Rev. Lett.* **91**, 4887–4890.

MIKHAILOVSKII, A.B., ONISHCHENKO, O.G. & SMOLYAKOV, A.I. 1985*a* Theory of low-frequency electromagnetic solitons in a relativistic electron–positron plasma. *Sov. J. Plasma Phys.* **11**, 215–219. [Translated from Russian: Mikhailovskii, A. B., Onishchenko, O. G., and Smolyakov, A. I. 1985a Teoriya nizkochastotnyh elektromagnitnyh solitonov v relativistskoi elektronno-positronnoy plasme. *Fizika Plazmy* **11**, 369-375.]

MIKHAILOVSKII, A.B., ONISHCHENKO, O.G. & TATARINOV, E.G. 1985*b* Alfvén solitons in a relativistic electron-position plasma. I. Hydrodynamic theory. *Plasma Phys. Control. Fusion* **27**, 527–537.

MIKHAILOVSKII, A.B., ONISHCHENKO, O.G. & TATARINOV, E.G. 1985*c* Alfvén solitons in a relativistic electron-position plasma. II. Kinetic theory. *Plasma Phys. Control. Fusion* **27**, 539–556.

MIO, K., OGINO, T., MINAMI, K. & TAKEDA, S. 1976*a* Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas. *J. Phys. Soc. Japan* **41**, 265–271.

MIO, K., OGINO, T., MINAMI, K. & TAKEDA, S. 1976*b* Modulational instability and envelope-solutions for nonlinear Alfvén waves propagating along magnetic-field in plasmas. *J. Phys. Soc. Japan* **41**, 667–673.

MJØLHUS, E. 1976 On the modulational instability of hydromagnetic waves parallel to the magnetic field. *J. Plasma Phys.* **16**, 321–334.

MJØLHUS, E. & HADA, T. 1997 In *Nonlinear Waves and Chaos in Space Plasmas* (ed. T. Hada & H. Matsumoto), p. 121. Terrapub.

MJØLHUS, E. & WYLLER, J. 1986 Alfvén solitons. *Phys. Scr.* **33**, 442–451.

RAJIB, T.I., SULTANA, S. & MAMUN, A.A. 2015 Solitary waves in rotational pulsar magnetosphere. *Astrophys. Space Sci.* **357**, 52.

ROGISTER, A. 1971 Parallel propagation of nonlinear low-frequency waves in high-β plasma. *Phys. Fluids* **14**, 2733–2739.

RUDERMAN, M.S. 1987 Quasilongitudinall propagating solitons in a plasma with Hall dispersion. *Fluid Dyn.* **22**, 299–305. [Translated from Russian: Ruderman, M. S. 1987 Ustoichivost' kvaziprodol'no rasprostranyayushchihsya solitonov v plasme s Hollovskoi dispersiey. *Izvestiya Akademii Nauk SSSR, Mekhanika Zhidkosti i Gaza*, No. 2, 159-165.]

RUDERMAN, M.S. 2002 DNLS equation for large-amplitude solitons propagating in an arbitrary direction in a high-β Hall plasma. *J. Plasma Phys.* **67**, 271–276.

RUDERMAN, M.S. 2020 Quasi-parallel propagation of solitary waves in magnetised non-relativistic electron–positron plasmas. *J. Plasma Phys.* **86**, 905860311.

RUDERMAN, M.A. & SUTHERLAND, P.G. 1975 Theory of pulsars: polar gaps, sparks, and coherent microwave radiation. *Astrophys. J.* **196**, 51–72.

RUFFINI, R., VERESHCHAGIN, G & XUE, S.-S. 2010 Electron-positron pairs in physics and astrophysics: from heavy nuclei to black holes. *Phys. Rep.* **487**, 1–140.

SAKAI, J. & KAWATA, T. 1980*a* Waves in an ultra-relativistic electron–positron plasma. *J. Phys. Soc. Japan* **49**, 747–752.

SAKAI, J. & KAWATA, T. 1980*b* Non-linear Alfvén-wave in an ultra-relativistic electron–positron plasma. *J. Phys. Soc. Japan* **49**, 753–758.

SHUKLA, R.K. 2003 Generation of magnetic fields in the early universe. *Phys. Lett.* A **310**, 182–186.

SHUKLA, P.K., RAO, N.N., YU, M.Y. & TSINTSADZE, N.L. 1986 Relativistic nonlinear effects in plasmas. *Phys. Rep.* **138**, 1–149.

STEWART, G.A. & LAING, E.W. 1992 Wave propagation in equal-mass plasmas. *J. Plasma Phys*. **47**, 295–319.

STURROCK, P.A. 1971 A model of pulsars. *Astrophys. J*. **164**, 529–556.

TANIUTI, T. & WEI, C.-C. 1968 Reductive perturbation method in nonlinear wave propagation. I. *J. Phys. Soc. Japan* **24**, 941–946.

TATSUNO, T., BEREZHIANI, V.I., PEKKER, M. & MAHAJAN, S.M. 2003 Angular momenta creation in relativistic electron–positron plasma. *Phys. Rev.* E **68**, 016409.

VERHEEST, F. 1996 Solitary Alfvén modes in relativistic electron–positron plasmas. *Phys. Lett.* A **213**, 177–182.

VERHEEST, F. & LAKHINA, G.S. 1996 Oblique solitary Alfvén modes in relativistic electron–positron plasmas. *Astrophys. Space Sci*. **240**, 215–224.

WEINBERG, S. 1972 *Gravitation and cosmology: Principles and applications of the general theory of relativity*. John Wiley & Sons.

ZANK, G.P. & GREAVES, R.G. 1995 Linear and nonlinear modes in nonrelativistic electron–positron plasmas. *Phys. Rev.* E **51**, 6079–6090.