The second chapter revolves around the Calkin algebra $\mathcal{Q}(H)$, the quotient of $\mathcal{B}(H)$ modulo the ideal of compact operators $\mathcal{K}(H)$, for a separable, infinite-dimensional Hilbert space H. The Calkin algebra has been object of intense studies by the researchers in operator algebras for almost 80 years. Over the last 15 years, it has also become fertile ground for applications of set theory in C*-algebras, due to its structural similarities with the boolean algebra $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$, of which it is considered the noncommutative analogue. We investigate some structural properties of $\mathcal{Q}(H)$, focusing on the question 'What C*-algebras embed into $\mathcal{Q}(H)$?'. We prove that every C*-algebra, regardless of its density character, can be embedded into $\mathcal{Q}(H)$ in a ccc forcing extension of the universe (this is part of the joint work [3]). This is used to prove that the statement 'Every C*-algebra of density character less than $2^{\aleph_0}$ embeds into $\mathcal{Q}(H)$' is independent from $\mathsf{ZFC} + 2^{\aleph_0} \geq \aleph_\alpha$, for every $\alpha > 2$. In particular, such statement is implied by Martin's axiom. Chapter 2 ends with a generalization (under Martin's axiom) to nonseparable C*-algebras of Voiculescu's theorem, a classical result in the theory of extensions of separable C*-algebras (these results are also in [5]).

The last chapter concerns liftings of abelian subalgebras of coronas of C*-algebras. For a C*-algebra $\mathcal{A}$, the multiplier algebra $\mathcal{M}(\mathcal{A})$ is the 'maximal' unitization of $\mathcal{A}$, while its corona $\mathcal{Q}(\mathcal{A})$ is the quotient $\mathcal{M}(\mathcal{A})/\mathcal{A}$. They are the noncommutative analogues of the Čech–Stone compactification and the corona, respectively, of a locally compact Hausdorff space. Given a set of commuting elements in a corona of a nonabelian, nonunital C*-algebra, we study what obstructions could prevent the existence of a commutative lifting to the multiplier algebra. We show that, while for countable families the only issues arising are of K-theoretic nature, for larger families the size itself becomes an obstacle. Using a combinatorial argument which goes back to Luzin's families, we prove, for a fairly general class of separable, nonabelian, nonunital C*-algebras $\mathcal{A}$, the existence of a set of $\aleph_1$ commuting elements in $\mathcal{Q}(\mathcal{A})$ containing no uncountable subset that lifts to a set of commuting elements in $\mathcal{M}(\mathcal{A})$ (see also [6]).

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Lorenzo Galeotti, ***The Theory of the Generalised Real Numbers and Other Topics in Logic***, Universität Hamburg, Germany, 2019. Supervised by Benedikt Löwe. MSC: 03E15, 03D60, 03F30, 03C55. Keywords: descriptive set theory, generalised Baire spaces, transfinite computability, Peano arithmetics, model theoretic set theory.

Abstract

The *real line* $\mathbb{R}$ and *Baire space* ω^ω are two of the most fundamental objects in descriptive set theory and have been studied extensively. In recent years, set theorists have become increasingly interested in *generalised Baire spaces* κ^κ, i.e., the sets of functions from κ to κ for an uncountable cardinal κ. In the thesis, two κ-analogues of $\mathbb{R}$ are discussed, one

due to Sikorski [7] and one due to Galeotti [5]. We use these spaces in the context of generalised metrisability theory. In particular, we use generalised metrisability theory to define a generalised notion of Polish spaces which we compare and combine with a generalised version of Choquet games introduced in [4].

The classical Bolzano-Weierstraß and Heine-Borel theorems fail on generalisations of the real line. We study two suitable weakenings of these theorems, one due to Sikorski [7] and one introduced in this thesis. For the second weakening, we give a full characterisation in terms of large cardinals. (Cf. also [3].)

Moreover, we use generalisations of the real line to develop two new models of transfinite computability, one generalising type two Turing machines and one generalising a model of computation due to Blum, Shub, and Smale [2]. We give a full characterisation of the computational strength of the generalised Blum, Shub, and Smale machines and compare them to the classical models of transfinite computability. Finally, we use the generalised version of type two Turing machines to begin the development of a generalised version of the classical theory of Weihrauch degrees (cf. [6]).

The last two chapters of the thesis are the result of the work of the author on topics in logic which are not directly related to generalisations of the real number continuum.

First we study the possible order types of models of syntactic fragments of Peano arithmetic. In particular, we study which order types can occur in models of such fragments and show that different fragments of Peano arithmetic can be distinguished using the order types of their models.

In the final chapter of the thesis we study Löwenheim–Skolem theorems for logics extending first-order logic. We extend the work done in [1] by relating upward Löwenheim–Skolem theorems for strong logics to reflection principles in set theory. Finally, we apply our framework to the study of the large cardinal strength of the upward Löwenheim–Skolem theorem for second-order logic; we provide both upper and lower bounds.

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