AN INTEGRAL INVOLVING AN E-FUNCTION AND AN ASSOCIATED LEGENDRE FUNCTION OF THE FIRST KIND

by T. M. MACROBERT

(Received 9th October, 1950)

§ 1. *Introductory.* The formula to be proved is

$$\sin(l\pi)\int_{-1}^{1}\left[\begin{array}{l}(1-\lambda^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\lambda)\,(1-\lambda)^{-l-m-1}\\ \times E\left\{\begin{array}{l}l+m+1,\alpha_1,\alpha_1,\ldots,\alpha_p:\ z(1-\lambda)\\ \rho_1,\rho_2,\ldots,\rho_q\end{array}\right\}\end{array}\right]d\lambda$$

$$=(-1)^n\sin(m-l)\pi\;2^{-l}\,E\begin{pmatrix}l+m+n+1,\,l-m-n,\,\alpha_1,\alpha_2,\ldots,\alpha_p:\ 2z\\ l+1,\,\rho_1,\rho_2,\ldots,\rho_q\end{pmatrix}$$

$$-(-1)^n\sin(m\pi)z^l\,E\begin{pmatrix}m+n+1,\ -m-n,\,\alpha_1-l,\alpha_2-l,\ldots,\alpha_p-l:\ 2z\\ 1-l,\,\rho_1-l,\rho_2-l,\ldots,\rho_q-l\end{pmatrix}, \quad\ldots\ldots\ldots\ldots(1)$$

where n is zero or a positive integer, $R(m)>0$, $R(\alpha_s-l)>0$, $s=1,2,\ldots,p$, $p\geqq q$.

The following formulae are required in the proof:

$$E(\alpha,\beta::z)=\Gamma(\alpha)\int_0^\infty e^{-\lambda}\lambda^{\beta-1}(1+\lambda/z)^{-\alpha}\,d\lambda, \quad\ldots\ldots\ldots\ldots(2)$$

where $R(\beta)>0$, $|\operatorname{amp} z|<\pi$;

$$\sqrt{(2\pi)}\sin(l\pi)\,\Gamma(\alpha)\int_0^\infty e^{-\mu}I_{m+n+\frac{1}{2}}(\mu)\,\mu^{l-\frac{1}{2}}(z+\mu)^{-\alpha}\,d\mu$$

$$=(-1)^n\sin(m-l)\pi\;2^{-l}z^{-\alpha}E(l+m+n+1,\,l-m-n,\,\alpha:\ l+1:\ 2z)$$

$$-(-1)^n\sin(m\pi)z^{l-\alpha}E(m+n+1,\ -m-n,\,\alpha-l:\ 1-l:\ 2z), \quad\ldots\ldots\ldots\ldots(3)$$

where n is integral, $R(l+m+n)>-1$, $R(\alpha-l)>0$, $|\operatorname{amp} z|<\pi$;

$$I_{m+n+\frac{1}{2}}(z)=\frac{z^{m+\frac{1}{2}}}{\sqrt{(2\pi)}}\int_{-1}^{1}e^{z\lambda}(1-\lambda^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\lambda)\,d\lambda, \quad\ldots\ldots\ldots\ldots(4)$$

where n is zero or a positive integer and $R(m)>-1$;

$$\int_0^\infty e^{-\mu}\mu^{\alpha_{p+1}-1}E(p;\ \alpha_r:\ q;\ \rho_s:\ z/\mu)\,d\mu=E(p+1;\ \alpha_r:\ q;\ \rho_s:\ z), \quad\ldots\ldots\ldots\ldots(5)$$

where $R(\alpha_{p+1})>0$;

$$\frac{1}{2\pi i}\int_C e^{\zeta}\zeta^{-\rho_{q+1}}E(p;\ \alpha_r:\ q;\ \rho_s:\ \zeta z)\,d\zeta=E(p;\ \alpha_r:\ q+1;\ \rho_s:\ z), \quad\ldots\ldots\ldots\ldots(6)$$

where the contour C starts at $-\infty$ on the real axis, passes in the positive direction round the origin, and returns to $-\infty$ on the real axis.

Formulae 2 to 6 are to be found on pages 348, 379, 377, 379, 379 respectively of the author's *Complex Variable*, third edition.

§ 2. *Proof of the formula.* On substituting from (4) in (3), the L.H.S. of (3) becomes

$$\sin(l\pi)\,\Gamma(\alpha)\int_0^\infty e^{-\mu}\mu^{l+m}(z+\mu)^{-\alpha}\,d\mu\int_{-1}^{1}e^{\lambda\mu}(1-\lambda^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\lambda)\,d\lambda.$$

https://doi.org/10.1017/S2040618500035553 Published online by Cambridge University Press

Now change the order of integration and get

$$\sin(l\pi)\int_{-1}^{1}(1-\lambda^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\lambda)\,d\lambda\,\Gamma(\alpha)\int_0^\infty e^{-\mu(1-\lambda)}\,\mu^{l+m}(z+\mu)^{-\alpha}\,d\mu$$
$$=\sin(l\pi)z^{-\alpha}\int_{-1}^{1}(1-\lambda^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\lambda)(1-\lambda)^{-l-m-1}E\{\alpha,l+m+1::z(1-\lambda)\}\,d\lambda,$$

by (2).

Hence

$$\sin(l\pi)\int_{-1}^{1}(1-\lambda^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\lambda)(1-\lambda)^{-l-m-1}E\{\alpha,l+m+1::z(1-\lambda)\}\,d\lambda$$
$$=(-1)^n\sin(m-l)\pi\,2^{-l}E(l+m+n+1,l-m-n,\alpha:l+1:2z)$$
$$-(-1)^n\sin(m\pi)z^l E(m+n+1,\,-m-n,\alpha-l:1-l:2z),\quad\ldots\ldots(7)$$

where n is zero or a positive integer, $R(\alpha-l)>0$, $R(m)>-1$.

On replacing α in (7) by α_1, z by z/μ, and applying (5) repeatedly, and then replacing z by ζz and applying (6) repeatedly, formula (1) is obtained.

If $p\geqq q$, the R.H.S. of (1) can be expressed in terms of $p+1$ generalised hypergeometric functions by using the formula

$$E(p;\ \alpha_r:q;\ \rho_s:z)=\sum_{r=1}^{p}P(\alpha_r;\ p-1;\ \alpha_s:q;\ \rho_t:z),\quad\ldots\ldots(8)$$

where $p\geqq q+1$, and

$$P(\alpha_r;\ p-1;\ \alpha_s:q;\ \rho_t:z)=\prod_{s=1}^{p}{}'\,\Gamma(\alpha_s-\alpha_r)\{\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_r)\}^{-1}$$
$$\times\Gamma(\alpha_r)z^{\alpha_r}F\begin{Bmatrix}q+1;\ \alpha_r,\alpha_r-\rho_1+1,\ldots,\alpha_r-\rho_q+1:(-1)^{p-q}z\\ p-1;\ \alpha_r-\alpha_1+1,\ldots*\ldots,\alpha_r-\alpha_p+1\end{Bmatrix};\quad\ldots\ldots(9)$$

here, if $p=q+1$, $|z|<1$.

For the coefficient of

$$2^{-l}(2z)^{l+m+n+1}F\begin{Bmatrix}l+m+n+1,m+n+1,l+m+n-\rho_1+2,\ldots,l+m+n-\rho_q+2;\ (-1)^{p-q+1}2z\\ 2m+2n+2,l+m+n-\alpha_1+2,\ldots,l+m+n-\alpha_p+2\end{Bmatrix}$$

is

$$\frac{\Gamma(l+m+n+1)\,\Gamma(m+n+1)\prod_{s=1}^{p}\Gamma(\alpha_s-l-m-n-1)}{\Gamma(2m+2n+2)\prod_{t=1}^{q}\Gamma(\rho_t-l-m-n-1)}$$
$$\times(-1)^n\{-\sin(m-l)\pi\sin(m+n)\pi+\sin(m\pi)\sin(l+m+n)\pi\}\operatorname{cosec}(2m+2n)\pi,$$

where the last line has the value $\sin(l\pi)$; next, the coefficient of

$$2^{-l}(2z)^{l-m-n}F\begin{Bmatrix}l-m-n,\ -m-n,l-m-n-\rho_1+1,\ldots,l-m-n-\rho_q+1;\ (-1)^{p-q+1}2z\\ -2m-2n,l-m-n-\alpha_1+1,\ldots,l-m-n-\alpha_p+1\end{Bmatrix}$$

is

$$\frac{\Gamma(2m+2n+1)\prod_{s=1}^{p}\Gamma(\alpha_s-l+m+n)}{\Gamma(m+n+1)\,\Gamma(1-l+m+n)\prod_{t=1}^{q}\Gamma(\rho_t-l+m+n)}$$
$$\times(-1)^n\pi\left\{\frac{\sin(m-l)\pi}{\sin(l-m-n)\pi}+\frac{\sin(m\pi)}{\sin(m+n)\pi}\right\}=0;$$

and finally, the coefficient of

$$2^{-l}(2z)^{\alpha_1} F\left\{\begin{matrix}\alpha_1, \alpha_1-l, \alpha_1-\rho_1+1, \ldots, \alpha_1-\rho_q+1 ; \ (-1)^{p-q+1}2z \\ \alpha_1-l-m-n, \alpha_1-l+m+n+1, \alpha_1-\alpha_2+1, \ldots, \alpha_1-\alpha_p+1\end{matrix}\right\}$$

is

$$\frac{\Gamma(l+m+n-\alpha_1+1)\prod_{s=2}^{p}\Gamma(\alpha_s-\alpha_1)}{\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_1)\,\Gamma(m+n+\alpha_1-l+1)}\Gamma(\alpha_1)\Gamma(\alpha_1-l)$$
$$\times(-1)^n\{\sin(m-l)\pi\sin(\alpha_1-l)\pi-\sin(m\pi)\sin(\alpha_1\pi)\}\operatorname{cosec}(m+n-l+\alpha_1+1)\pi,$$

the value of the last line being $\sin l\pi$.

Hence, if $R(m)>0$, $R(\alpha_s-l)>0$, $s=1, 2, \ldots, p$, $p\geqq q$, $n=1, 2, 3, \ldots$,

$$\int_{-1}^{1}\left[\begin{matrix}(1-\lambda^2)^{\frac{1}{2}m}T_{m+n}^{-m}(\lambda)(1-\lambda)^{-l-m-1} \\ \times E\left\{\begin{matrix}l+m+1, \alpha_1, \alpha_1, \ldots, \alpha_p : z(1-\lambda) \\ \rho_1, \rho_1, \ldots, \rho_q\end{matrix}\right\}\end{matrix}\right]d\lambda$$
$$=\frac{\Gamma(l+m+n+1)\Gamma(m+n+1)\prod_{s=1}^{p}\Gamma(\alpha_s-l-m-n-1)}{\Gamma(2m+2n+2)\prod_{t=1}^{q}\Gamma(\rho_t-l-m-n-1)}2^{-l}(2z)^{l+m+n+1}$$
$$\times F\left\{\begin{matrix}l+m+n+1, m+n+1, l+m+n-\rho_1+2, \ldots, l+m+n-\rho_q+2 ; \ (-1)^{p-q+1}2z \\ 2m+2n+2, l+m+n-\alpha_1+2, \ldots, l+m+n-\alpha_p+2\end{matrix}\right\}$$
$$+\sum_{r=1}^{p}\frac{\Gamma(l+m+n-\alpha_r+1)\prod_{s=1}^{p}{}'\,\Gamma(\alpha_s-\alpha_r)}{\Gamma(m+n-l+\alpha_r+1)\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_r)}\Gamma(\alpha_r)\Gamma(\alpha_r-l)2^{\alpha_r-l}z^{\alpha_r}$$
$$\times F\left\{\begin{matrix}\alpha_r, \alpha_r-l, \alpha_r-\rho_1+1, \ldots, \alpha_r-\rho_q+1 ; \ (-1)^{p-q+1}2z \\ \alpha_r-l-m-n, \alpha_r-l+m+n+1, \alpha_r-\alpha_1+1, \ldots * \ldots, \alpha_r-\alpha_p+1\end{matrix}\right\}, \quad\ldots\ldots(10)$$

where, if $p=q$, $|2z|<1$.

Note 1. When $p\geqq q$, formula (10) can be verified by expanding the E-function in the integral by means of (8) and using the integrals

$$\int_{-1}^{1}(1-\lambda^2)^{\frac{1}{2}m}T_{m+n}^{-m}(\lambda)F\{p ; \ \alpha_r : q ; \ \rho_s : (1-\lambda)z\}d\lambda$$
$$=(-1)^n\frac{\prod_{s=1}^{p}(\alpha_r ; \ n)\Gamma(m+n+1)}{\prod_{t=1}^{q}(\rho_t ; \ n)\Gamma(2m+2n+2)}2^{m+n+1}z^n$$
$$\times F\left\{\begin{matrix}m+n+1, \alpha_1+n, \alpha_2+n, \ldots, \alpha_p+n ; \ 2z \\ 2m+2n+2, \rho_1+n, \rho_2+n, \ldots, \rho_q+n\end{matrix}\right\}, \quad\ldots\ldots(11)$$

where $R(m)>-1$, $p\leqq q+1$,

$$(\alpha ; \ 0)\equiv 1, \qquad (\alpha ; \ n)=\alpha(\alpha+1)\ldots(\alpha+n-1) ; \quad\ldots\ldots(12)$$

and

$$\int_{-1}^{1}(1-\lambda^2)^{\frac{1}{2}m}T_{m+n}^{-m}(\lambda)(1-\lambda)^l F\{p ; \ \alpha_r : q ; \ \rho_s : (1-\lambda)z\}d\lambda$$
$$=\frac{(-l ; \ n)\Gamma(l+m+1)}{\Gamma(l+2m+n+2)}2^{l+m+1}$$
$$\times F\left\{\begin{matrix}l+1, l+m+1, \alpha_1, \alpha_2, \ldots, \alpha_p ; \ 2z \\ l-n+1, l+2m+n+2, \rho_1, \rho_2\ldots, \rho_q\end{matrix}\right\}, \quad\ldots\ldots(13)$$

where $R(m) > -1$, $R(l+m) > -1$, and $p \leqq q+1$.

Formulae (11) and (13) can be proved by substituting the extended Rodrigues' formula

$$(1-\lambda^2)^{\frac{1}{2}m} T_{m+n}^{-m}(\lambda) = \frac{(-1)^n}{2^{m+n}\Gamma(m+n+1)} \frac{d^n}{d\lambda^n}\{(1-\lambda^2)^{m+n}\} \quad \text{......(14)}$$

in the integrals, integrating by parts n times, and then integrating term by term. If $p=q+1$, $|2z|<1$.

Note 2. As above, formula (3), with $n=0$, can be written

$$\begin{aligned} &\sqrt{(2\pi)}\,\Gamma(\alpha)\int_0^\infty e^{-\mu} I_{m+\frac{1}{2}}(\mu)\,\mu^{l-\frac{1}{2}}(z+\mu)^{-\alpha}\,d\mu \\ &\quad= \frac{\Gamma(\alpha-l-m-1)\,\Gamma(m+1)\,\Gamma(l+m+1)}{\Gamma(2m+2)} 2^{m+1} z^{l+m-\alpha+1} \\ &\qquad\times F(m+1, l+m+1\,;\ 2m+2, l+m-\alpha+2\,;\ 2z) \\ &\qquad+ \frac{\Gamma(l+m-\alpha+1)\,\Gamma(\alpha)\,\Gamma(\alpha-l)}{\Gamma(\alpha+m-l+1)} 2^{\alpha-l} F\left(\begin{matrix}\alpha, \alpha-l \\ \alpha-l-m, \alpha-l+m+1\end{matrix}\,;\ 2z\right), \quad \text{...(15)} \end{aligned}$$

where $R(l+m) > -1$, $R(\alpha-l) > 0$, $|\operatorname{amp} z| < \pi$.

UNIVERSITY OF GLASGOW