THE PRACTICAL USE OF VARIATION PRINCIPLES IN THE DETERMINATION OF THE STABILITY OF NON LINEAR SYSTEMS

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Abstract

We are interested in the motion of non linear systems. In this paper we use a variation principle and an iteration procedure in order to treat the stability of free oscillations against small disturbances of the initial conditions. It is found that approximations to the low lying stability lines can be obtained using the Rayleigh-Ritz variation principle and that these approximations can be consistently improved using an iteration procedure. These approximations are compared with the tabulated values in the special case of the Mathieu Equation. The results are in general a considerable improvement on those obtained using the usual Perturbation Theory, and have a much wider range of validity.

1. Introduction

The equations of motion which arise in non-linear mechanics are frequently of such a type that an exact solution in terms of tabulated functions is impossible to obtain. The question of obtaining approximate solutions to these equations has been discussed in another paper [1]. There it is shown that a variation principle and iteration methods can be used to obtain excellent approximations. The equations which determine the *stability* of a periodic motion of a non linear system are linear, but usually involve a Hill equation, and even approximate solutions of this equation are notoriously difficult to obtain. In this paper we discuss the stability of free vibrations of a non linear two degree of freedom system.

The first three sections contain no new material. They are devoted to a brief description of the derivation of the stability criteria, and to an illustration of their application in a simple non trivial case. These sections introduce a specific problem and establish the notation.

Two particular methods are introduced for dealing with the problem.

The Rayleigh Ritz *Variation principle* is used to obtain closer approximations to the stability lines than is usual. An *iteration method* based on the variation principle is described which is capable of finding the low lying stability lines to any degree of accuracy. This iteration method may be used to obtain general results about the orientation of these lines. Elementary Group Theory is used to increase the usefulness of these methods.

As a check, these methods are applied to special systems which give rise to the Mathieu equation. The results are remarkably accurate; they represent a considerable improvement on the corresponding results using the usual perturbation expansion. The variational method is also applied to show the effect on the stability of the introduction of unsymmetric terms into the system.

Finally a slightly more general mechanical system is considered. This illustrates the general procedure and indicates the circumstances under which the Duffing approximation can be safely used when dealing with the stability problems.

Emphasis is laid on the methods and on the clear distinction between what is approximation and what is exact.

2. The Stability Equations

Let us consider a conservative mechanical system with n degrees of freedom governed by a Hamiltonian of the form

$$(2.1)\qquad H(\pi_1, \pi_2, \cdots \pi_n; x_1, x_2, \cdots x_n) = \sum k_{ij}(x_1 \cdots x_n)\pi_i\pi_j + V(x_1, x_2, \cdots x_n)$$

Here $x_1 \cdots x_n$ are a set of generalized coordinates, and $\pi_1, \pi_2, \cdots, \pi_n$ are the corresponding conjugate momenta. In the special case that the k_{ij} are independent of $x_1 \cdots x_n$, and, in addition, V is a multinomial of degree two in $x_1 \cdots x_n$, the system is termed linear. In other cases, the system is termed non linear and the general motion is usually difficult to determine. Sometimes there is a particular motion or set of possible motions of the system which is known, although the general motion may or may not be determined. We shall refer to *a one-parameter family of periodic solutions of Hamilton's equations as a 'mode' of the system. We are interested in finding out the conditions under which such a mode is stable against small perturbations of the initial conditions.*

For reasons of simplicity we restrict ourselves to systems for which it is possible to choose new canonical coordinates $q_1 \cdots q_n$ and momenta $p_1 \cdots p_n$ so that the mode under consideration is characterised by

$$(2.2)\qquad q_1 = p_1 = q_2 = p_2 = \cdots = q_{n-1} = p_{n-1} = 0$$

This results in certain restrictions on the Hamiltonian. The equations of motion are

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} \tag{2.3}$$

As these are to have (2.2) above as a possible solution we must have

$$\left(\frac{\partial H}{\partial q_i}\right)' = \left(\frac{\partial H}{\partial p_i}\right)' = 0, \qquad i = 1, 2, \cdots, n-1 \tag{2.4}$$

where the prime indicates that $q_1, \cdots, q_{n-1}$ and $p_1, \cdots, p_{n-1}$ are set equal to zero; the equations of motion of the mode are derived from $H(0, 0, \cdots, 0, p_n; 0, 0, \cdots, 0, q_n)$.

The general formulation of the conditions under which such a motion is stable has been given by Poincaré [7] and Whittaker [13]. We summarise the argument for the convenience of the reader.

The motion can be supposed to be of period T and the required solution of the equations of motion (2.3) is

$$\begin{aligned} q_n &= \varphi(t) \equiv \varphi(t+T) \\ p_n &= \psi(t) \equiv \psi(t+T) \\ q_1 &= p_1 = q_2 = p_2 = \cdots = q_{n-1} = p_{n-1} = 0 \end{aligned} \tag{2.5}$$

We consider an adjacent orbit

$$\begin{aligned} q_n &= \varphi(t) + \xi_n(t) \\ p_n &= \psi(t) + \eta_n(t) \\ q_i &= \xi_i(t), \quad i = 1, 2, \cdots, n-1 \\ p_i &= \eta_i(t), \quad i = 1, 2, \cdots, n-1 \end{aligned} \tag{2.6}$$

where ξ_i and η_i are small. The variables in this orbit satisfy Hamilton's equations of motion (2.3) with the same function H. We substitute (2.5) and (2.6) into the equations of motion (2.3), and take the difference. If we retain only first order quantities we are left with the following equations; (the prime on the derivatives refers to conditions (2.2), as before):

$$\frac{d\xi_i}{dt} = \sum_{j=1}^{n} \left\{ \left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)' \eta_j + \left(\frac{\partial^2 H}{\partial p_i \partial q_j}\right)' \xi_j \right\}, \qquad i = 1 \cdots n \tag{2.7a}$$

and

$$\frac{d\eta_i}{dt} = -\sum_{j=1}^{n} \left\{ \left(\frac{\partial^2 H}{\partial q_i \partial q_j}\right)' \xi_j + \left(\frac{\partial^2 H}{\partial q_i \partial p_j}\right)' \eta_j \right\}, \qquad i = 1 \cdots n \tag{2.7b}$$

However, the use of (2.4) simplifies these equations. The coefficients are to be evaluated on the unperturbed orbit, that of (2.4); thus $(\partial H/\partial q_i)' = 0$ when $i \neq n$ for *all* values of p_n and q_n; this yields

$$\frac{\partial}{\partial q_n}\left(\frac{\partial H}{\partial q_i}\right)' = 0$$

and so

$$\left(\frac{\partial^2 H}{\partial q_n \partial q_i}\right)' = 0 \qquad i \neq n \tag{2.8a}$$

Similarly

$$\left(\frac{\partial^2 H}{\partial p_n \partial p_i}\right)' = \left(\frac{\partial^2 H}{\partial q_n \partial p_i}\right)' = \left(\frac{\partial^2 H}{\partial p_n \partial q_i}\right)' = 0 \qquad i \neq n \tag{2.8b}$$

This means that ξ_n and η_n can be separated from (2.7) and that in the equations for ξ_i and η_i $(i = 1, \cdots, n-1)$ the summation goes only from $j = 1$ to $j = n - 1$.

The n^{th} equations of (2.7) can be combined to give a pair of linear simultaneous equations in ξ_n and η_n with periodic coefficients. These equations give a stability criterion for the unperturbed motion $q_n = \varphi(t)$. This motion is governed by the Hamiltonian $H(0, 0, \cdots, 0, p_n; 0, 0, \cdots, 0, q_n)$. It is to be observed that working directly from the potential energy $V(0, 0, \cdots 0, q_n)$ we may obtain a stability criterion corresponding to a different and less rigid definition of stability. (See Stoker [10], p. 219). We do not concern ourselves here with the n^{th} equations of (2.7).

We shall discuss the other $2n - 2$ equations (2.7), which determine the stability of the mode against small initial perturbations of the coordinates $q_1 \cdots q_{n-1}; p_1 \cdots p_{n-1}$. These equations are

$$\frac{d\xi_i}{dt} = \sum_{j=1}^{n-1}\left\{\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)' \eta_j + \left(\frac{\partial^2 H}{\partial p_i \partial q_j}\right)' \xi_j\right\} \qquad i = 1 \cdots n-1 \tag{2.9a}$$

$$\frac{d\eta_i}{dt} = -\sum_{j=1}^{n-1}\left\{\left(\frac{\partial^2 H}{\partial q_i \partial q_j}\right)' \xi_j + \left(\frac{\partial^2 H}{\partial q_i \partial p_j}\right)' \eta_j\right\} \qquad i = 1 \cdots n-1 \tag{2.9b}$$

Since $\varphi(t)$ and $\psi(t)$ are periodic, so are any functions of them. The coefficients appearing in (2.9) are functions of $\varphi(t)$ and $\psi(t)$ and are therefore periodic. The nature of the solutions of (2.9) determines the stability of the mode, which therefore depends only on the parameters occurring in $(\partial^2 H/\partial p_i \partial p_j)'$, $(\partial^2 H/\partial q_i \partial q_j)'$, $(\partial^2 H/\partial p_i \partial q_j)'$, and $\varphi(t)$. Discussion of the actual criterion is deferred to the next section.

In this paper we restrict ourselves to a two degree of freedom system in which

$$\left(\frac{\partial^2 H}{\partial p_1^2}\right)' = \text{constant}$$

In this case (2.9) reduces to

$$\frac{d^2\xi_1}{dt^2} = \left\{ - \left(\frac{\partial^2 H}{\partial p_1^2}\right)' \left(\frac{\partial^2 H}{\partial q_1^2}\right)' + \left(\frac{\partial^2 H}{\partial p_1 \partial q_1}\right)'^2 + \frac{d}{dt}\left(\frac{\partial^2 H}{\partial p_1 \partial q_1}\right)' \right\} \xi_1 \tag{2.10}$$

which is the Hill equation discussed in Whittaker [12] and Ince [4]; the quantity in brackets in [2.10] is a known periodic function of time.

3. A Particular Two Degree of Freedom System

We consider as examples, the stability of particular motions of a class of simple two degree of freedom systems. These are all special cases of the system described below. Three simple non linear springs, two of which are identical, are connected to two equal masses and two fixed points in a symmetrical way, as indicated in diagram 1. The masses are constrained to move in the

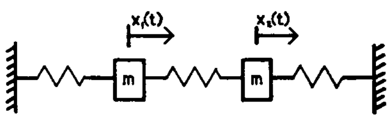

DIAGRAM 1

The simple two degree of freedom system described in Section 3.

straight line which contains the fixed points. We define $x_1(t)$ and $x_2(t)$ as the displacement of each mass from its equilibrium position. The potential energy of the outer spring in terms of its extension u is taken to be

$$\frac{ma_1u^2}{2} + \frac{ma_2u^3}{3} + \frac{ma_3u^4}{4}$$

and the corresponding quantity for each inner spring is

$$\frac{mA_1u^2}{2} + \frac{mA_2u^3}{3} + \frac{mA_3u^4}{4}$$

This system has been considered by R. M. Rosenberg and C. P. Atkinson [8] and the above notation has been taken over from their paper. Where comparable, their results agree with ours.

It is clear from the symmetry of the system that if the outer springs are symmetric, that is if $a_2 = 0$, two simple modes exist. One occurs if the masses move so that at any time they are symmetrically situated with respect to the fixed points. This mode is known as the 'out of phase mode' and has the characteristic that $x_1 + x_2$ maintains its initial value, zero. The other, or 'in phase' mode, is one in which the distance between the masses remains constant; it has the property that $x_1 - x_2$ is zero.

Following the procedure outlined in section 2 we set

$$q_1 = \frac{(x_1 + x_2)}{\sqrt{2}}$$

$$q_2 = \frac{(x_1 - x_2)}{\sqrt{2}}$$

and the Hamiltonian of the system takes the form

$$H(p_1, p_2, q_1, q_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \sum_r \sum_s m\alpha_{rs} q_1^r q_2^s \tag{3.1}$$

The non zero values of α_{rs} are:

$$\begin{aligned}
&\alpha_{20} = \frac{a_1}{2} \qquad && \alpha_{21} = \frac{a_2}{\sqrt{2}} \\
&\alpha_{02} = \frac{a_1}{2} + A_1 \qquad && \alpha_{03} = \frac{a_2}{3\sqrt{2}} - \frac{2\sqrt{2}}{3} A_2 \\
&\alpha_{40} = \frac{a_3}{8} && \\
&\alpha_{04} = \frac{a_3}{8} + A_3 && \\
&\alpha_{22} = \tfrac{3}{4} a_3 &&
\end{aligned} \tag{3.2}$$

In the case of symmetric outer spings, $a_2 = 0$ and so there are no terms linear in q_2 in (3.1). The Hamiltonian (3.1) then satisfies the conditions of equation (2.4). If *all* the springs are symmetric the form of the Hamiltonian is symmetric under interchange of q_1 and q_2. In that case, any condition obtained for the stability of one mode corresponds to a condition for the other mode.

Application of the analysis outlined in Section 2 leads directly to the pair of Hill equations

$$\frac{d^2\xi_1}{dt^2} + 2\left\{\alpha_{20} + \alpha_{21}\varphi(t) + \alpha_{22}\{\varphi(t)\}^2\right\}\xi_1 = 0 \tag{3.3a}$$

$$\frac{d^2\xi_2}{dt^2} + 2\left\{\alpha_{02} + 3\alpha_{03}\varphi(t) + 6\alpha_{04}\{\varphi(t)\}^2)\right\}\xi_2 = 0 \tag{3.3b}$$

when the equations of the orbit are

$$q_1 = 0; \qquad q_2 = \varphi(t) \tag{3.4}$$

As remarked above, equation (3.3b) refers to a rather rigid definition of stability and can be replaced by inspection of the potential energy curve so as to obtain a more practical criterion. The nature of the solutions of equation (3.3a) yield additional criteria; in particular if (3.3a) has an unbounded solution the motion is unstable.

The solutions of the Hill equations are well known. (See for example references 4 and 12). If we denote by T' the least period of the coefficient of ξ_1 in (3.3a), two independent solutions are

$$\xi^{(1)}(t) = \sigma_1{}^{t/T'} g_1(t)$$

(3.5) and

$$\xi^{(2)}(t) = \sigma_2{}^{t/T'} \left\{ g_2(t) + \delta_{\sigma_1\sigma_2} \frac{t}{\sigma_1 T'} g_3(t) \right\}$$

where $g_i(t)$ $(i = 1, 2, 3)$ are periodic functions of t of period T', $\delta_{\sigma_1\sigma_2}$ is the Kronecker delta, and σ_1 and σ_2 are the roots of the equation

$$\sigma^2 - 2\mu\sigma + 1 = 0. \tag{3.6}$$

μ is real and its value depends on the parameters occurring in the coefficient of ξ_1 in equation (3.3a). As these parameters are varied, so μ varies leading to different values of σ_1 and σ_2; however, in view of (3.6)

$$\sigma_1 \sigma_2 = 1 \tag{3.7}$$

Should a particular value of μ lead to real roots of (3.6) it is clear that one of the solutions (3.5) is unbounded. But if σ_1 and σ_2 are complex and distinct then both have modulus 1 and the solutions (3.5) are bounded. The condition for real roots is

$$|\mu| > 1$$

The transition case, that for which $|\mu|$ is one, is obtained if α_{2r} takes particular values. To fix ideas, suppose that we consider a particular value of α_{21}. If α_{20} and α_{22} are taken as the coordinates of points on a plane, the transition values of α_{20} and α_{22} lie on curves in the plane; these curves divide regions whose α_{20} and α_{22} give $|\mu| > 1$ from those regions whose α_{20} and α_{22} give $|\mu| < 1$, that is 'unstable' regions from 'stable' regions. *We are interested in determining the location of these boundary lines.*

Thus the stability line is obtained by writing down the condition

$$\mu = \pm 1. \tag{3.8}$$

Referring again to equation (3.6) we see that in these cases either

$$\sigma_1 = \sigma_2 = 1 \tag{3.9a}$$

or

$$\sigma_1 = \sigma_2 = -1 \tag{3.9b}$$

and these lead to both solutions (3.5) having the same period, either T' or $2T'$. Since other values of μ cannot lead to both solutions (3.5) having these periods we may replace the stability line condition (3.8) by an equivalent condition, namely that (3.3a) *has a periodic solution of period either T' or $2T'$.*

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4. Exact Solutions for a Simple Case

In this section we deal with a particular case of the system described in Section 3. The reason for this choice is that it leads directly to the Mathieu equation whose solutions have been discussed at length in the literature (McLachlan [6]).

We choose a case in which both springs are symmetric i.e. $a_2 = A_2 = 0$ and in which the out of phase mode is simple harmonic, that is $\alpha_{04} = 0$. This can be done by arranging the spring constants to have values satisfying

$$\begin{gathered} a_1 \neq 0 \quad A_1 \neq 0 \quad a_2 = A_2 = 0 \\ \frac{a_3}{8} = -A_3 \neq 0. \end{gathered} \tag{4.1}$$

With this choice, the Hamiltonian (3.1) takes the simple form

$$H(p_1, p_2, q_1, q_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + m\{\alpha_{20}q_1^2 + \alpha_{02}q_2^2 + \alpha_{40}q_1^4 + \alpha_{22}q_1^2q_2^2\} \tag{4.2}$$

The out of phase mode ($q_1 = 0$, $p_1 = 0$) has as its Hamiltonian

$$H(0, p_2, 0, q_2) = \frac{p_2^2}{2m} + m\alpha_{02}q_2^2 \tag{4.3}$$

with the solution

$$q_2 = \varphi(t) = A \sin(\omega t) \tag{4.4}$$

where

$$\omega = \sqrt{2\alpha_{02}} = \sqrt{(a_1 + 2A_1)} \tag{4.5}$$

Substituting these parameters into the Hill equation (3.3b) gives

$$\frac{d^2\xi_r}{dt^2} + [a + bA^2 \sin^2(\omega t)]\, \xi_r = 0 \tag{4.6}$$

where

$$a = 2\alpha_{20} = a_1, \quad b = 2\alpha_{22} = \tfrac{3}{2}a_3 = -12A_3 \tag{4.7}$$

Equation (4.6) is the Mathieu Equation. This is the simplest non-trivial case of the Hill equation. Putting $\omega t = \theta$ in (4.6), one has

$$\frac{d^2\xi_r}{d\theta^2} + \left\{\frac{a}{\omega^2} + \frac{bA^2}{\omega^2}\sin^2\theta\right\} \xi_r = 0 \tag{4.8}$$

Thus the stability of the system depends on the two dimensionless quantities a/ω^2 and bA^2/ω^2.

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In view of the discussion at the end of the previous section we are interested in those solutions $\xi(t)$ of (4.8) which are periodic with a period of either T' or $2T'$, where

$$T' = \tfrac{1}{2}T = \frac{\pi}{\omega} \tag{4.9}$$

A solution of (4.6) with either of these periods is known as a Mathieu function and the corresponding value of a/ω^2 as its eigenvalue.

The Mathieu functions are discussed by Whittaker and Watson [12] who give expansions for the eigenvalues. They have been extensively tabulated by Goldstein [3]. Their application to non-linear mechanics is discussed by Stoker. [10]

In Diagram 2, a/ω^2 is plotted against bA^2/ω^2. If $bA^2/\omega^2 = 0$ it is clear that (4.8) has harmonic solutions for positive a/ω^2 and exponential solutions for negative a/ω^2. This means that the positive half of the a/ω^2 axis is in a stable region, and the negative half of the a/ω^2 axis is in an unstable region. As the curves denote boundaries between stable and unstable regions, it is clear that the shaded portions of the diagram represent the stable regions, and the unshaded portions, the unstable regions.

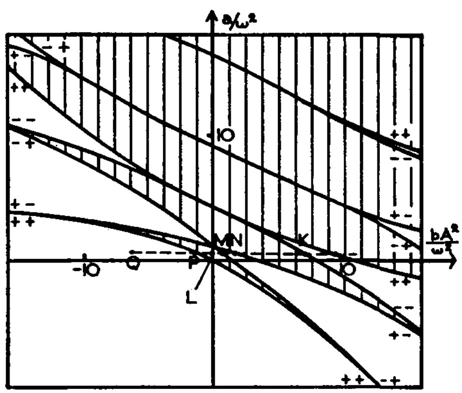

DIAGRAM 2

The stability diagram for the Mathieu Equation

$$\frac{d^2\xi}{dt^2} + [a + bA^2 \sin^2 \omega t]\xi = 0$$

The shaded regions correspond to stable motions. The method of using this diagram is discussed in Section 4. Each stability line has attached to it the values of (θ_1, θ_2) defined in Section 5.

The procedure for using the stability diagram is as follows, and is illustrated on diagram 2. The values of a/ω^2 and b/ω^2 for the mechanical system in question can be determined using (4.4), (4.5), and (4.7). They are α_{20}/α_{02} and $A^2 \cdot \alpha_{22}/\alpha_{02}$ respectively. A line is drawn on the stability diagram, parallel to the axis $a/\omega^2 = 0$ a distance α_{20}/α_{02} above it. If α_{22}/α_{02} is negative, the point corresponding to an out of phase motion of amplitude A lies to the left of L on this line. Thus if A is small enough so that $|\alpha_{22}A^2/\alpha_{02}| < LP$, the motion is stable and if A is larger, the motion is unstable. If α_{22}/α_{02} is positive, the representative point lies to the right of L and if A takes a small value again the motion is stable. However, increasing A leads successively to unstable, stable, and unstable motions, the type changing as A becomes large enough to make the representative point reach M, N, K, etc.

5. Symmetry Properties of the Stability Lines in the Case of General Springs

The function $\varphi(t)$ is a periodic solution of the equations of motion for a particle of mass m moving in the potential $V(0, q_2)$. As such it has certain well known symmetry properties. If the time t is measured from a moment at which $d\varphi/dt = 0$, then the path is symmetric under time reversal, that is

$$\varphi(t) = \varphi(-t) \tag{5.1}$$

There is another distinct point where $d\varphi/dt = 0$ and if the time taken for the particle to reach this point for the first time is denoted by t_0 we have, by the same argument

$$\varphi(t_0 + t) = \varphi(t_0 - t) \tag{5.2}$$

Finally we have the periodicity condition:

$$\varphi(t) = \varphi(t + T), \tag{5.3}$$

(5.2) and (5.3) result in $t_0 = \frac{1}{2}T$. These three conditions are also satisfied by any function of $\varphi(t)$. It should be pointed out that a particular function of $\varphi(t)$ might have more symmetry properties than $\varphi(t)$, and that $\varphi(t)$ in a particular case might have more symmetry properties than the three listed above. However, these three apply to any $\varphi(t)$ obtained as a periodic solution as indicated above. This point is illustrated in Diagram 6.

The Hill equation under consideration can be written in the form

$$\frac{d^2\xi}{dt^2} + \{a + c\varphi(t) + b\{\varphi(t)\}^2\}\xi = 0 \tag{5.4}$$

and this can be written

$$A(t)\xi(t) = \lambda\xi(t) \tag{5.5}$$

where $A(t)$ is the operator

$$A(t) = \frac{d^2}{dt^2} + c\varphi(t) + b\{\varphi(t)\}^2 \tag{5.6}$$

and $\lambda = -a$. Only discrete values of λ, the eigenvalues, lead to solutions $\xi(t)$ of (5.5) with period T' or $2T'$. As $\varphi(t)$ has period T, it is apparent that

$$A(t+T) = A(t) \quad (\text{all } t)$$

We have already defined the least period of $A(t)$ as T',

$$A(t+T') = A(t) \quad (\text{all } t) \tag{5.7}$$

Hence T is an integral multiple of T'. In view of equation (5.1) we also have the relation

$$A(-t) = A(t) \tag{5.8}$$

The following discussion of the symmetry properties is based on these two properties (5.7) and (5.8).

Let us suppose that $\xi_r(t)$ is a particular solution of (5.5) with λ_r the corresponding value of λ. Thus

$$A(t)\xi_r(t) = \lambda_r \xi_r(t) \tag{5.9}$$

Since $A(t)$ and λ are both real, $\xi_r(t)$ can be taken as real. We may replace t in (5.9) by $-t$ and using relation (5.8) we may write

$$A(t)\xi_r(-t) = \lambda_r \xi_r(-t) \tag{5.10}$$

Using (5.9) and (5.10) it follows that $\xi_r(t)$ can be chosen to be either symmetric or antisymmetric under time reversal. For if $\xi_r(-t)$ is not already equal to $\pm\xi_r(t)$, $\xi_r(t)$ and $\xi_r(-t)$ are linearly independent solutions of the same equation. In that case we can use instead the functions $\xi_r(t) + \xi_r(-t)$ and $\xi_r(t) - \xi_r(-t)$, and each of these has definite properties under time reversal. Thus we may restrict our attention to functions which have the property

$$\xi_r(t) = \theta_1 \xi_r(-t) \tag{5.11}$$

where θ_1 can take the values $+1$ or -1.

A second and distinct symmetry classification may be obtained by making use of the periodicity property of $A(t)$. Using (5.7) and (5.8) it can be easily shown that

$$A\left(\frac{T'}{2}+t\right) = A\left(\frac{T'}{2}-t\right) \tag{5.12}$$

Hence we may write

$$A\left(\frac{T'}{2}+t\right)\xi_r\left(\frac{T'}{2}+t\right) = \lambda_r \xi_r\left(\frac{T'}{2}+t\right) \tag{5.13}$$

and

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$$A\left(\frac{T'}{2}+t\right)\xi_r\left(\frac{T'}{2}-t\right)=\lambda_r\xi_r\left(\frac{T'}{2}-t\right) \tag{5.14}$$

These equations are analogous to equations (5.9) and (5.10) and application of precisely the same arguments as those above lead to a similar conclusion. Thus we may restrict ourselves to functions which have the property

$$\xi_r\left(\frac{T'}{2}+t\right)=\theta_2\xi_r\left(\frac{T'}{2}-t\right) \tag{5.15}$$

where θ_2 can take the values $+1$ or -1. The values of θ_1 and θ_2 which belong to a particular function $\xi_r(t)$ can be used to classify it. We shall refer to the pair of values (θ_1, θ_2) as the "representation" to which $\xi_r(t)$ belongs.

Since $\xi_r(t)$ is of period T' or $2T'$ it can be expanded in a Fourier series

$$\xi_r(t)=a_0+\sum_{r=1}^{\infty}a_r\cos\frac{\pi rt}{T'}+\sum_{r=1}^{\infty}b_r\sin\frac{\pi rt}{T'} \tag{5.16}$$

However, direct inspection shows that, depending on the representation to which $\xi_r(t)$ belongs, certain terms in the expansion are absent. For example if $\theta_1=1$, $\xi_r(t)=\xi_r(-t)$ and all the b_r vanish; if $\theta_2=1$, a_{2r+1} and b_{2r} are zero. These properties are summarised in Table 1.

In diagram 2, section 4, each stability line has, next to it, the values (θ_1, θ_2) of the periodic solution from which the line is derived. It is seen that certain lines touch, and other lines cross, on points along the vertical axis. These crossings and touchings can be predicted once and for all, without detailed knowledge of the complete stability diagram. This is done in Section 7 below.

For a rigorous discussion of eigenfunction theory, we refer the reader to Courant and Hilbert [2]. In particular, the completeness of the functions ξ_r

TABLE 1

Representation θ_1	θ_2	Fourier Expansion	Period
1	1	$a_0+\sum a_{2r}\cos\frac{2r\pi t}{T'}$	T'
1	-1	$\sum a_{2r+1}\cos\frac{(2r+1)\pi t}{T'}$	$2T'$
-1	1	$\sum b_{2r+1}\sin\frac{(2r+1)\pi t}{T'}$	$2T'$
-1	-1	$\sum b_{2r}\sin\frac{2r\pi t}{T'}$	T'

used in this paper can be established following the procedure of pages 359 et seq. of this reference. The necessary integral equations in the case of the Mathieu equation are derived in Whittaker and Watson [12].

6. Approximation Methods for Determining the Stability Lines

There exists a general method for obtaining information about the lowest eigenvalue λ_0 of an equation

$$A(t)\xi_r(t) = \lambda_r \xi_r(t), \tag{6.1}$$

known as the Rayleigh Ritz variational method. Since $\xi_r(t)$ satisfies the condition

$$\xi_r(t+2T') = \xi_r(t)$$

it can be shown that these functions form a complete normal orthogonal sequence in the interval $(0, 2T')$, that is

$$\int_0^{2T'} \xi_r(t)\xi_s(t)dt = 0 \qquad r \neq s \tag{6.2a}$$

and we may choose the arbitrary constant multiplying $\xi_r(t)$ such that

$$\int_0^{2T'} \xi_r(t)\xi_r(t)dt = 1 \tag{6.2b}$$

We suppose the eigenvalues to be ordered so that $\lambda_0 \leqq \lambda_1 \leqq \lambda_2 \cdots$

Now let $\psi(t)$ be any square integrable function for which $A(t)\psi$ exists. We form the expression

$$\Lambda = \frac{\int_0^{2T'} \psi(t)[A(t)\psi(t)]dt}{\int_0^{2T'} [\psi(t)]^2 dt.} \tag{6.4}$$

$\psi(t)$ can be expanded in terms of the set $\xi_r(t)$, i.e.,

$$\psi(t) = \sum a_r \xi_r(t) \tag{6.5}$$

where the a_r are numerical coefficients. Using (6.1), (6.2) and (6.5), it is easy to show that (6.4) can be reduced to

$$\Lambda = \frac{\sum_{r=0}^{\infty} a_r^2 \lambda_r}{\sum_{r=0}^{\infty} a_r^2} \tag{6.6}$$

Since $\lambda_0 \leqq \lambda_r$ for all r, expression (6.6) is greater than the expression formed by replacing λ_r by λ_0 in (6.6). Thus we have

$$\Lambda = \frac{\Sigma a_r^2 \lambda_r}{\Sigma a_r^2} \geqq \frac{\Sigma a_r^2 \lambda_0}{\Sigma a_r^2} = \lambda_0 \tag{6.7}$$

This relation provides a powerful tool for approximating to the lowest eigenvalue of equation (6.1). If all the a_r except a_0 are zero, the equality sign in (6.7) holds. So if we have an idea about the $\xi_0(t)$ of (6.5) we can choose $\psi(t)$ so that a_0 is appreciably greater than the other a_r, thereby making (6.7) a good approximation. We may use the variational technique: we choose $\psi(t)$ with one or more adjustable parameters μ_i. Then we minimise expression (6.4) with respect to the μ_i. Since the inequality (6.7) holds whatever the values of μ_i, this procedure gives the closest inequality possible with $\psi(t)$ of that particular form.

We use the expression (5.6) for $A(t)$, and $\lambda = -a$. Inserting these into (6.4), the inequality (6.7) becomes

$$(6.8)\quad a\int_0^T \{\psi(t)\}^2 dt \leqq \int_0^T \{-b(\varphi(t))^2 - c\varphi(t)\}\{\psi(t)\}^2 dt + \int_0^T \left(\frac{d\psi}{dt}\right)^2 dt$$

It appears that only the lowest eigenvalue λ_0 can be found by this method (i.e. only the first line on the stability diagram). However, as a result of the symmetry classification of section 5, the eigenvalues and corresponding eigenfunctions fall into four distinct classes, depending upon the values of θ_1 and θ_2; (see table 1.) Let us suppose that the "trial function" $\psi(t)$ is chosen such that it has $\theta_1 = +1$ and $\theta_2 = -1$, say. Then it is easily seen that the only non-zero coefficients a_r in the sum (6.5), and hence also in (6.6), are such that $\xi_r(t)$ has the same symmetry (θ_1, θ_2) as $\psi(t)$. Hence we have the sharper inequality

$$(6.9)\qquad \Lambda \geqq \lambda_{\min}(\theta_1, \theta_2)$$

where $\lambda_{\min}(\theta_1, \theta_2)$ is the lowest eigenvalue for the particular symmetry class θ_1, θ_2 to which the trial function $\psi(t)$ belongs.

By choosing trial functions $\psi(t)$ of the four possible symmetry types, in turn, and using (6.8) on each one, we thus obtain variational approximations to *four* low lying stability lines on the stability diagram.

The variational method described can be used to obtain closer approximations to the true eigenvalue λ_r by using trial functions of greater flexibility, that is containing more parameters. The main trouble is that the procedure for minimising a function with respect to many parameters is cumbersome (for example see Appendix 1). In view of this we describe an iteration method, based on the variational method.

We rewrite the equation

$$(6.10)\qquad A\xi_r = \lambda_r \xi_r$$

in a form which leaves the left hand side as a *simple* differential operator which is easy to invert; for example, in the case considered in Section 3 we write the differential equation

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$$\frac{d^2\xi_r}{dt^2} + bA^2 \sin^2\omega t \xi_r = - a\xi_r$$

in the form

$$\frac{d^2\xi_r}{dt^2} + a\xi_r = - bA^2 \sin^2\omega t \xi_r \cdot$$

The iteration procedure is based on obtaining a series of successively closer approximations $\xi^{[0]}$ $\xi^{[1]}, \cdots$ to the true eigenfunction ξ, and corresponding approximations to the relation between a and b. However, in this case we treat b as the dependent variable and the iteration procedure yields successive approximations $b^{[0]}, b^{[1]}, \cdots$ to the exact value of b, considered as a function of a.

We suppose, therefore, that (6.10) above has been written in the form

$$C\xi_r = bB\xi_r \tag{6.11}$$

where the equation

$$C\xi_r = \varphi$$

has straightforward solutions. The iteration procedure is based on the following recurrence relations;

$$C\xi_r^{[n]} = b^{[n-1]} B\xi_r^{[n-1]} \qquad n \geqq 1 \tag{6.12}$$

$$b^{[n]} = \frac{(\xi_r^{[n]}, C\xi_r^{[n]})}{(\xi_r^{[n]}, B\xi_r^{[n]})} \qquad n \geqq 0 \tag{6.13}$$

Here the expression (ψ, χ) is defined by

$$(\psi, \chi) = \frac{1}{2T'}\int_0^{2T'} \psi(t)\chi(t)dt \tag{6.14}$$

i.e. it is the mean value of the product of ψ and χ over one period and our attention is restricted to operators C and B which have the 'Hermitean' property, that is to say

$$(\psi, C\chi) = (\chi, C\psi) \tag{6.15}$$

where both ψ and χ are periodic in t with period T, but are otherwise unrestricted. It is easy to verify, using partial integration if necessary that the operators used in the following sections have this 'Hermitean' property.

We choose a trial function $\xi^{[0]}$ belonging to a particular representation and use (6.13) with $n = 0$ to evaluate $b^{[0]}$. We then solve the differential equation (6.12) to find $\xi^{[1]}$. The constants of integration are chosen so that $\xi^{[n]}$ belongs to the same representation as $\xi^{[0]}$.

This procedure, as remarked above, yields successive approximations $b^{[0]}(a)$, $b^{[1]}(a) \cdots$ to the exact function $b(a)$. It is convenient to choose

$\xi^{[0]}(a, t)$ so that the curve $b^{[0]}(a)$ touches the exact curve $b(a)$ at $(0, a_r)$ when b is plotted against a. It will be shown in a forthcoming publication [5] that with this choice of $\xi^{[0]}$, $b^{[1]}(a)$, $b^{[2]}(a) \cdots$ all touch the true curve $b(a)$ at this same point. Moreover $b^{[n]}(a)$ has $(n+2)$ point contact with $b(a)$. That is to say when $a = a_r$ the values of $b^{[n]}(a)$ and its first $n+1$ derivatives are the same as the corresponding values for $b(a)$ and its first $n+1$ derivatives when $x = a_r$. This result, which shows the relation between $b^{[n]}_{r\pm}(a)$ and $b_{r\pm}(a)$ at $(0, a_r)$ is used in the next section to derive a result relating $b_{r+}(a)$ and $b_{r-}(a)$ at this point.

7. The Crossing and Touching of Stability Lines

We now consider a special case of the Hill equation

$$C(a, t)\xi(a, t) = b(a)B(t)\xi(at)$$

where

$$C = \frac{d^2}{dt^2} + a$$

and B has a *finite* Fourier expansion

$$B = \sum_{s=0}^{N} \beta_s \cos 2s\omega t \tag{7.1}$$

In the following discussion we assume that all the $\beta_s (s \leqq N)$ are non zero. (See, however, the remarks in the final paragraph of this section)

The simplest non trivial case of this equation is the Mathieu equation (4.6) where $N = 1$ and $\beta_0 = -\beta_1 = -A^2/2$. We are interested in the relation satisfied by b and a such that $\xi(a, t)$ is periodic with period $2\pi/\omega$. As is the case for the Mathieu equation, we see that if $b = 0$, the equation becomes

$$\frac{d^2\xi}{dt^2} + a\xi = 0 \tag{7.2}$$

and if ξ is to be periodic with period T' or $2T'$, a takes one of a series of values $a_0, a_1, \cdots$ where

$$a_r = \omega^2 r^2$$

Except for $r = 0$, there are two independent solutions $\xi^{[0]}_{r+}$ and $\xi^{[0]}_{r-}$ corresponding to each a_r. We use the classification of section 5 where the subscript $+$ or $-$ refers to the value of θ_1 and the value of θ_2 is $(-1)^r$. Thus we have

$$\xi^{[0]}_{r+} = \cos r\omega t \qquad r > 0 \tag{7.3a}$$

$$\xi^{[0]}_{r-} = \sin r\omega t \qquad r > 0 \tag{7.3b}$$

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Determination of $b_{r\pm}^{[0]}$ is straightforward. Using (7.1) and (7.3) we find that

$$B\xi_{r+}^{[0]} = \beta_0 \cos r\omega t + \sum_{s=1}^{N} \frac{\beta_s}{2} \{\cos (r+2s)\omega t + \cos (r-2s)\omega t\} \tag{7.4}$$

and that $B\xi_{r-}^{[0]}$ has precisely the same form with sines replacing cosines throughout. For future convenience we define

$$\varepsilon_r = a - a_r = a - \omega^2 r^2 \tag{7.5}$$

and, using this notation

$$C\xi_{r\pm}^{[0]} = \varepsilon_r \xi_{r\pm}^{[0]} \tag{7.6}$$

Elementary integration shows

$$(\xi_{r\pm}^{[0]}, C\xi_{r\pm}^{[0]}) = \tfrac{1}{2}\varepsilon_r \tag{7.7}$$

and

$$(\xi_{r\pm}^{[0]}, B\xi_{r\pm}^{[0]}) = \tfrac{1}{2}(\beta_0 \pm \beta_r) \tag{7.8}$$

using the convention $\beta_r = 0$ if $r > N$.

Substitution of these values into (6.13) gives

$$b_{r\pm}^{[0]}(a) = \frac{\varepsilon_r}{(\beta_0 \pm \frac{1}{2}\beta_r)} = \frac{a - a_r}{\beta_0 \pm \frac{1}{2}\beta_r} \tag{7.9}$$

Equation (7.9) shows that the curves $b_{r\pm}^{[0]}(a)$ are straight lines through $(0, a_r)$ and that these lines are identical if $r > N$, but not otherwise. Now it is remarked above and proved in a forthcoming publication [5] that the zero order curve $b_r^{[0]}(a)$ touches the true curve $b_r(a)$ at $a = a_r$. In view of this we have the result that the *true* curves $b_{r+}(a)$ and $b_{r-}(a)$ touch at $(0, a_r)$ if $r > N$ as they both touch the same line $b_{r\pm}^{[0]}(a)$ at this point. Reference to diagram 2 shows that this result is borne out in the case of the Mathieu equation. Another point to notice which follows from (7.9) is that the gradient of $b_{r\pm}^{[0]}(a)$ at the point $(0, a_r)$ is $(\beta_0 \pm \frac{1}{2}\beta_r)$ and this is also the gradient of $b_{r\pm}(a)$. This implies that if the stability lines $b_{r+}(a)$ and $b_{r-}(a)$ touch each other at $(0, a_r)$ the gradient is β_0 and independent of r while if they intersect, the average of their gradients is β_0.

We now generalise the result about the type of contact the curves $b_{r+}(a)$ and $b_{r-}(a)$ have at the point $(0, a_r)$. We show that this depends on the 'quotient' k obtained by dividing r by N and defined

$$r = kN + m \qquad 1 \leqq m \leqq N \tag{7.10}$$

We have already shown that if $1 \leqq r \leqq N$, that is if $k = 0$ the curves $b_{r+}(a)$ and $b_{r-}(a)$ cut without touching, that is have one point contact, and that if $r > N$, i.e. $k > 1$ they touch.

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We use the recurrence relation (6.12) to find $\xi^{[1]}_{r\pm}$. Using (7.4) for $B\xi^{[0]}_{r\pm}$ we easily find that

$$(7.11)\quad \xi^{[1]}_{r+} = \cos r\omega t + \frac{\varepsilon_r}{2\beta_0}\sum_{s=1}^{N}\left\{\frac{\beta_s}{\varepsilon_{r+2s}}\cos(r+2s)\omega t + \frac{\beta_s}{\varepsilon_{r-2s}}\cos(r-2s)\omega t\right\} \qquad r > N$$

The equation for $\xi^{[1]}_{r-}$ is the same as this with sine replacing cosine throughout. Since we are interested in the shape of $b_{r\pm}(a)$ near $a = a_r$ the quantity ε_r is a first order small quantity in this region. It is straightforward but tedious to verify the following results about the form of $\xi^{[n]}_{r\pm}$. If we write

$$(7.12)\qquad \xi^{[n]}_{r\pm} = \sum_{s=0}^{r+2nN} \alpha_{s\pm}\xi^{[0]}_{s\pm}$$

where $\xi^{[0]}_{s\pm}$ are given by (1.3), then $\alpha_{s\pm}$ is clearly zero unless $|r-s|$ is even. If $r > 2nN$, $\alpha_{s+} = \alpha_{s-}$ for all s. Otherwise α_{s+} can differ from α_{s-} by terms of order $\varepsilon_r{}^k$, k being defined in equation (7.10). It also follows that

$$\begin{aligned}
(\xi^{[n]}_{r+}, B\xi^{[n]}_{r+}) - (\xi^{[n]}_{r-}, B\xi^{[n]}_{r-}) &= 0 && 2n \leqq k\\
&= 0(\varepsilon_r{}^k) && 2n \geqq k+1\\
(\xi^{[n]}_{r+}, B\xi^{[n-1]}_{r+}) - (\xi^{[n]}_{r-}, B\xi^{[n-1]}_{r-}) &= 0 && 2n \leqq k-1\\
&= 0(\varepsilon_r{}^k) && 2n \geqq k
\end{aligned}$$

Using the recurrence relations (6.12) and (6.13), it is easy to show that

$$b^{[n]} = b^{[n-1]}\frac{(\xi^{[n]}, B\xi^{[n-1]})}{(\xi^{[n]}, B\xi^{[n]})}$$

or

$$b^{[n]}_{r\pm} = b^{[0]}_{r\pm}\prod_{s=1}^{n}\frac{(\xi^{[s]}_{r\pm}, B\xi^{[s-1]}_{r\pm})}{(\xi^{[s]}_{r\pm}\ B\xi^{[s]}_{r\pm})}$$

Now each term in the product for $b^{[n]}_{+}$ is either the same as the corresponding term in the product for $b^{[n]}_{-}$ or differs from it by terms of order $\varepsilon_r{}^k$. Moreover, if $r > N$ that is $k \geqq 1$, we have from (7.9) that

$$b^{[0]}_{r+} = b^{[0]}_{r-} = \frac{\varepsilon_r}{\beta_r}.$$

Thus we see that the difference $b^{[n]}_{r+} - b^{[n]}_{r-}$ is of order ε_r^{k+1}. Now when $a = a_r$, $\varepsilon_r = 0$. This gives the result that the expression $b^{[n]}_{r+}(a) - b^{[n]}_{r-}(a)$ and its first k derivatives at $a = a_r$ are zero, establishing that the curves $b^{[n]}_{r+}(a)$ and $b^{[n]}_{r-}(a)$ have $k+1$ point contact with each other at $(0, a_r)$.

As mentioned above the n^{th} iteration curve $b^{[n]}_{r\pm}$ has $n+2$ point contact with the true curve $b_{r\pm}$ at $(0, a_r)$; so by taking an n greater than k we see that the true curve $b_{r+}(a)$ has $k+1$ point contact with $b_{r-}(a)$ at $(0, a_r)$.

This situation is illustrated in diagrams 2, 7b and 3 in the cases where $N = 1, 2$ and 3 respectively.

In the foregoing discussion we assumed that all the β_s $(s < N)$ were non zero. Lifting this restriction has the effect of increasing the degree of contact in particular cases, but cannot decrease it. Thus in general all the results

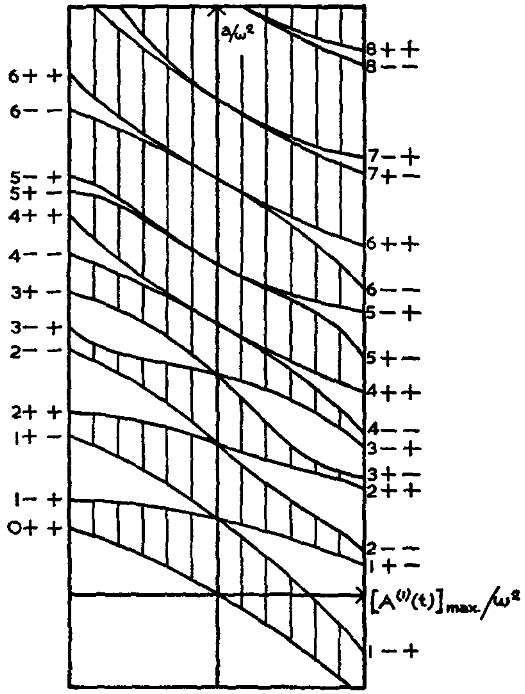

DIAGRAM 3

Schematic Stability Diagram for the case $N = 3$.

This diagram indicates schematically the positions of the stable and unstable regions corresponding to a Hill equation of the following form

$$\left(\frac{d^2}{dt^2} + A^{(1)}(t)\right)\xi_r(t) = \lambda_r \xi_r(t); \quad \xi(t + 2T') = \xi(t)$$

where

$$A^{(1)}(t) = \sum_{r=0}^{N} \beta_r \cos 2r\omega t; \quad \omega = \frac{\pi}{T'}$$

and $N = 3$.

The shaded areas indicate the stable regions. Attached to each line are the values of r, θ_1 and θ_2. As noted in the text, the pairs of lines with $r = 1, 2$ and 3 cut on the a/ω^2 axis, the average gradient at the point of intersection being β_0. Pairs of lines with $r = 4, 5$ and 6 touch each other, those with $r = 7, 8$ and 9 touch and cut each other and so on.

The diagram is not drawn to scale. In fact the intersections on the a/ω^2 axis are at $a/\omega^2 = 0, 1, 4, 9, \ldots$

referring to the degree of contact of particular pairs of curves should be interpreted in this sense; namely the degree of contact may be greater than that stated, but is not less. [1])

8. Test of the Variational Method

We now consider the problem for which we listed the exact solutions in Section 4, this time using the Rayleigh Ritz variational principle. This will give us some indication about how accurate this method is before going on and using it in cases where the exact solution has not been tabulated.

In the previous section we showed that an approximation to a stability line is obtained by inserting a trial function ψ, belonging to a particular representation (θ_1, θ_2), into the inequality (6.7). We start with the representation $(\theta_1, \theta_2) = (+1, +1)$ and use the simplest trial function belonging to this representation, namely $\psi = 1$. Substituting this into (6.7) gives

$$\frac{a}{\omega^2} \leqq -\tfrac{1}{2}\frac{bA^2}{\omega^2} \tag{8.1}$$

No variation is possible as the trial function contains no variable parameter. This result is in fact the same as is obtained using perturbation theory to first order.

We next choose as a trial function the next simplest function of this representation, that is

$$\psi = 1 + \mu \cos(2\omega t) \tag{8.2}$$

After carrying out the integrations in (6.4) we obtain

$$\frac{a}{\omega^2} \leqq \left[-\tfrac{1}{2}\frac{bA^2}{\omega^2}\left(1 + \mu + \frac{\mu^2}{2}\right) + 2\mu^2\right]\left[1 + \tfrac{1}{2}\mu^2\right]^{-1} \tag{8.3}$$

This can be written

$$\frac{a}{\omega^2} \leqq \frac{a_0 + a_1\mu + a_2\mu^2}{b_0 + b_1\mu + b_2\mu^2} \equiv g(\mu) \tag{8.4}$$

and the smallest value of a/ω^2 is required. In Appendix I it is shown that the extremum values of this expression are

$$\frac{a_1b_1 - 2a_0b_2 - 2a_2b_0 \pm \sqrt{(a_1b_1 - 2a_0b_2 - 2a_2b_0)^2 - (a_1^2 - 4a_0a_2)(b_0^2 - 4b_0b_2)}}{a_1^2 - 4a_0a_2}$$

Substitution for a_i, b_i, of the constants occurring in (8.3) gives

[1] The authors are indebted to the referee for drawing their attention to this point.

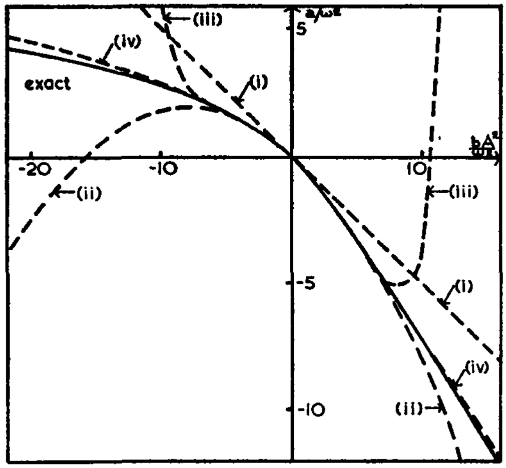

DIAGRAM 4(a)

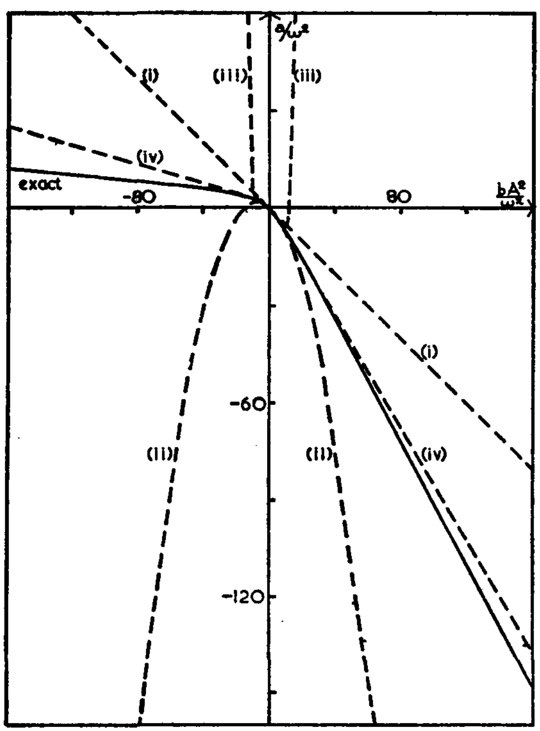

DIAGRAM 4(b)

The lowest stability line for the Mathieu Equation. The full line is obtained from the exact values tabulated by Goldstein [3]. The broken lines are approximations obtained using:

(i) First order Perturbation Theory

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$$\frac{a}{\omega^2} = -\frac{1}{2}\frac{bA^2}{\omega^2}$$

(ii) Second order Perturbation Theory

$$\frac{a}{\omega^2} = -\frac{1}{2}\frac{bA^2}{\omega^2} - \frac{1}{32}\left(\frac{bA^2}{\omega^2}\right)^2$$

(iii) Eighth order Perturbation Theory (Equation 8.6)

(iv) The Variation Method

$$\frac{a}{\omega^2} = -\frac{1}{2}\frac{bA^2}{\omega^2} + 2 - 2\sqrt{1+\frac{1}{32}\left(\frac{bA^2}{\omega^2}\right)^2}$$

$$\frac{a}{\omega^2} \leqq -\frac{1}{2}\frac{bA^2}{\omega^2} + 2\left\{1 - \left(1 + \frac{1}{32}\left(\frac{bA^2}{\omega^2}\right)^2\right)^{1/2}\right\} \tag{8.5}$$

The curves obtained by replacing the inequalities in (8.1) and (8.5) by equalities are plotted in Diagrams 4(a) and 4(b), together with the exact curve, and the curves obtained using perturbation theory.

TABLE 2

Representation	Trial Function	Resulting Inequality $\left(X = \frac{bA^2}{\omega^2}\right)$
$+\,+$	1	$\frac{a}{\omega^2} \leqq -\frac{1}{2}X$
	$1 + \mu \cos 2\omega t$	$\frac{a}{\omega^2} \leqq -\frac{1}{2}X + 2 - 2\sqrt{1+\frac{1}{32}X^2}$
$+\,-$	$\cos \omega t$	$\frac{a}{\omega^2} \leqq -\frac{1}{4}X + 1$
	$\cos \omega t + \mu \cos 3\omega t$	$\frac{a}{\omega^2} \leqq -\frac{3}{8}X + 5 - 4\sqrt{1-\frac{1}{16}X+\frac{5}{1024}X^2}$
$-\,+$	$\sin \omega t$	$\frac{a}{\omega^2} \leqq -\frac{3}{4}X + 1$
	$\sin \omega t + \mu \sin 3\omega t$	$\frac{a}{\omega^2} \leqq -\frac{5}{8}X + 5 - 4\sqrt{1+\frac{1}{16}X+\frac{5}{1024}X^2}$
$-\,-$	$\sin 2\omega t$	$\frac{a}{\omega^2} \leqq -\frac{1}{2}X + 4$
	$\sin 2\omega t + \mu \sin 4\omega t$	$\frac{a}{\omega^2} \leqq -\frac{1}{2}X + 10 - 6\sqrt{1+\frac{1}{576}X^2}$

The inequalities obtained for the four lowest stability lines of the Mathieu equation:

$$\frac{d^2\xi}{dt^2} + (a + bA^2\sin^2\omega t)\xi = 0$$

These equalities are listed in Appendix II in the ε, δ notation (Stoker).

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It is interesting to compare this approximate result with the results obtained using Perturbation Theory. Carrying out the usual perturbation procedure (McLachlan [6]) it is possible to obtain an expansion, the first terms of which are:

$$\begin{aligned}\frac{a}{\omega^2} = -\tfrac{1}{2}\frac{bA^2}{\omega^2} - 2\left(\frac{bA^2}{8\omega^2}\right)^2 + \tfrac{7}{8}\left(\frac{bA^2}{8\omega^2}\right)^4 - \tfrac{29}{36}\left(\frac{bA^2}{8\omega^2}\right)^6 \\ + \tfrac{68687}{73728}\left(\frac{bA^2}{8\omega^2}\right)^8 + \cdots\end{aligned} \tag{8.6}$$

The corresponding expansion of (8.5) is

$$\frac{a}{\omega^2} = -\tfrac{1}{2}\frac{bA^2}{\omega^2} - 2\left(\frac{bA^2}{8\omega^2}\right)^2 + \left(\frac{bA^2}{8\omega^2}\right)^4 - \cdots \tag{8.7}$$

Since (8.6) is in a sense an exact expansion, it might be inferred that (8.5) is a worse approximation than (8.6), the expansion of (8.5) differing from the true expansion by terms of order $(bA^2/\omega^2)^4$. Reference to Diagrams 4(a) and 4(b) indicates that such an inference is misleading. In the region $|bA^2/\omega^2| < +6$, both approximations are excellent and it is probably true to say that (8.6) is closer to the exact curve. However, with $6 < |bA^2/\omega^2| < 160$, the variation method approximation is reasonable while (8.6) gives an entirely false picture.

The reason for this state of affairs is not far to seek. The variation method involves considering a particular value of bA^2/ω^2 and finding the best function ξ_r of a simple type. In the perturbation expansion, a more complicated function ξ_r is used, which is arranged to be correct for small bA^2/ω^2; large powers of bA^2/ω^2 are neglected. It is clear that such a procedure is valid only for small bA^2/ω^2.

The variation iteration method is also used to find the same stability line for the Mathieu equation. In this case we treat equation (6.11)

$$C\xi_r = bB\xi_r \tag{8.8}$$

where

$$C = \frac{d^2}{dt^2} + a \tag{8.9}$$

and

$$B = \frac{A^2}{2}(-1 + \cos 2\omega t) \tag{8.10}$$

We are interested in the stability line passing through the origin, that is with $a_r = 0$. The trial function used is

$$\xi_0^{[0]} = 1 \tag{8.11}$$

and the zero order eigenvalue $b^{[0]}(a)$ is given by

$$\tfrac{1}{2}A^2 b_0^{[0]} = -a = -\varepsilon_0 \tag{8.12}$$

Remembering that $\varepsilon_r = a - \omega^2 r^2$, we find without much difficulty that the first order approximations are

$$\xi_0^{[1]} = 1 - \frac{\varepsilon_0}{\varepsilon_2}\cos 2\omega t \tag{8.13}$$

$$\tfrac{1}{2}A^2 b_0^{[1]} = -\frac{\varepsilon_0\left(1+\tfrac{1}{2}\dfrac{\varepsilon_0}{\varepsilon_2}\right)}{\left(1+\dfrac{\varepsilon_0}{\varepsilon_2}+\tfrac{1}{2}\left(\dfrac{\varepsilon_0}{\varepsilon_2}\right)^2\right)} \tag{8.14}$$

and that the second order approximations are

$$\xi_0^{[2]} = \frac{1+\tfrac{1}{2}\dfrac{\varepsilon_0}{\varepsilon_2}}{1+\dfrac{\varepsilon_0}{\varepsilon_2}+\tfrac{1}{2}\left(\dfrac{\varepsilon_0}{\varepsilon_2}\right)^2}\left\{\left(1+\tfrac{1}{2}\frac{\varepsilon_0}{\varepsilon_2}\right)- \frac{\varepsilon_0}{\varepsilon_2}\left(1+\frac{\varepsilon_0}{\varepsilon_2}\right)\cos 2\omega t + \frac{\varepsilon_0^2}{2\varepsilon_2\varepsilon_4}\cos 4\omega t\right\} \tag{8.15}$$

$$\tfrac{1}{2}A^2 b_0^{[2]} = \frac{-\varepsilon_0\left(1+\tfrac{3}{2}\dfrac{\varepsilon_0}{\varepsilon_2}+\left(\tfrac{5}{4}+\tfrac{1}{8}\dfrac{\varepsilon_0}{\varepsilon_4}\right)\dfrac{\varepsilon_0^2}{\varepsilon_2^2}+\tfrac{1}{2}\dfrac{\varepsilon_0^3}{\varepsilon_2^3}\right)}{1+\dfrac{2\varepsilon_0}{\varepsilon_2}\left(\tfrac{9}{4}+\dfrac{\varepsilon_0}{4\varepsilon_4}+\dfrac{\varepsilon_0^2}{16\varepsilon_4^2}\right)\dfrac{\varepsilon_0^2}{\varepsilon_2^2}+\left(\tfrac{3}{2}+\dfrac{\varepsilon_0}{4\varepsilon_4}\right)\dfrac{\varepsilon_0^3}{\varepsilon_2^3}+\dfrac{\varepsilon_0^4}{2\varepsilon_2^4}} \tag{8.16}$$

These functions are of the form indicated in equation (7.12), the coefficients of cos $2r\omega t$ being of order ε_0^r. The curves $b_0^{[0]}(a)$, $b_0^{[1]}(a)$ and $b_0^{[2]}(a)$ are illustrated in diagram 5.

These curves have various striking characteristics which are worthy of further discussion. One feature is the closeness of these approximations to the true curve in the region where a/ω^2 is negative; in fact when bA^2/ω^2 is 160, the true value of a/ω^2 differs from the second order approximation by only $2\frac{1}{2}$ percent. However, when a/ω^2 is positive a completely different behaviour is evident. $b_0^{[1]}(a)$ so to speak turns on its tracks, crosses the a/ω^2 axis and then recrosses it at $(0, a_2/\omega^2)$ touching the stability curve $b_{2+}(a)$ at this point. Thereafter it does not seem to follow any stability line at all closely. The curve $b_0^{[2]}(a)$ does the same sort of thing twice. In fact in general $b_0^{[n]}(a)$ crosses the a/ω^2 axis $2n+1$ times touching each of the curves $b_0(a)$, $b_{2+}(a) \cdots b_{2n+}(a)$ at the points where they cross this axis. This behaviour is

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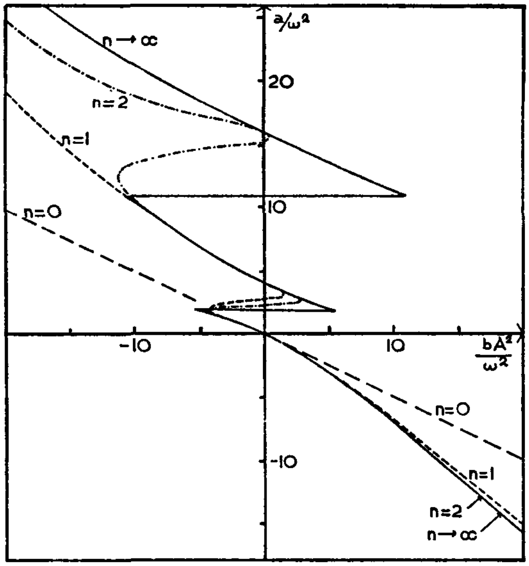

DIAGRAM 5

The curves $b_0^{[n]}$ for $n = 0, 1, 2$ given by equations (8.12), (8.14) and (8.16) are shown by broken lines. The curve to which $b_0^{[n]}$ tends as $n \to \infty$ is shown by the full line. This line is the locus of the $b_{r\pm}$ of smallest absolute magnitude for which $(\theta_1, \theta_2) = (+1, +1)$.

apparent after inspection of the form of the function $\xi_0^{[n]}$. If we consider the region near $(0, a_{2r}/\omega^2)$ where $r \leqq n$, we can treat ε_{2r} as a small quantity. Writing

$$\xi_0^{[n]} = \sum_{s=0}^{n} \alpha_{2s} \cos 2s\omega t$$

we see that the largest term is α_{2r}. Moreover the ratio

$$\frac{\alpha_{2(r\pm p)}}{\alpha_{2(r\pm p\pm 1)}} \sim \frac{1}{\varepsilon_{2r}} \qquad (\varepsilon_{2r} \text{ small})$$

Thus in this region, so long as $n \geqq r$, the function $\xi_0^{[n]}$ is essentially $\cos 2r\omega t$, i.e. the function $\xi_{2r+}^{[0]}$. This function gives rise to a curve $b_{2r+}^{[0]}$ which touches the true curve b_{2r+} at $(0, a_{2r}/\omega^2)$ as shown in section 7. So the curve $b_0^{[n]}$ also has this property.

It can be shown (Schwinger [9], Svartholm [11]) that using this iteration procedure $b_{r\pm}^{[n]}(a)$ tends to the eigenvalue $b(a)$ of the smallest absolute value, which corresponds to the same representation as $b_{r\pm}^{[0]}(a)$. The limit curve therefore consists of sections of the different curves $b_{r\pm}(a)$. There are only four such limit curves corresponding to the four representations respectively. The one corresponding to $(\theta_1, \theta_2) = (+1, +1)$ is illustrated in diagram 5.

It is apparent that this procedure indicates accurately the positions of the lowest stability line when bA^2/ω^2 is positive, but when bA^2/ω^2 is negative, only a short section of this stability line can be found.

9. Another Example

In this section we deal with a second particular case of the system described in section 3. We choose this case so that, again, the out of phase mode is simple harmonic, but now we allow the springs to be unsymmetric, i.e. $a_2 \neq 0$. This can be done by choosing a mechanical system for which

$$\begin{aligned} a_1 \neq 0; \quad & a_3/8 = -A_3 \neq 0 \\ A_1 \neq 0; \quad & a_2 = 4A_2 \neq 0 \end{aligned} \tag{9.1}$$

These lead to a Hamiltonian

$$H(p_1, p_2, q_1, q_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \sum_{r,s=1}^{4} m\alpha_{rs} q_1^r q_2^s \tag{9.2}$$

in which the non zero α_{rs} are given by (3.2). It is important to note that, as a result of (9.1) α_{02} is not zero and α_{03} and α_{04} are zero. Because of this the out of phase mode $(q_1 = p_1 = 0)$ has the Hamiltonian

$$H(0, p_2, 0, q_2) = \frac{p_2^2}{2m} + m\alpha_{02} q_2^2 \tag{9.3}$$

with the solution

$$q_2(t) = \varphi(t) = A \sin \omega t \tag{9.4}$$

where $\omega = \sqrt{2\alpha_{02}} = \sqrt{(a_1 + 2A_1)}$. Putting this into the Hill equation (3.3b) gives

$$\frac{d^2\xi_r}{dt^2} + (a + bA^2 \sin^2 \omega t + cA \sin \omega t)\, \xi_r = 0 \tag{9.5}$$

where a and b have the same values as before (see (4.7)) and c, the term which introduces the lack of symmetry, takes the value

$$c = 2\alpha_{21} = \sqrt{2}a_2 \tag{9.6}$$

The period of $(a + bA^2 \sin^2 \omega t + cA \sin \omega t)$ is clearly the same as that of $\sin \omega t$, viz. $2\pi/\omega$. So in this case the period of the solution $\xi_r(t)$ is T' or $2T'$ where

$$T' = T = \frac{2\pi}{\omega} \tag{9.7}$$

This is in contrast to the state of affairs in sections 4 and 8 where ω is related to T' by (4.9). This difference is illustrated in diagram 6. Reference to this diagram shows precisely how the introduction of the term $cA \sin \omega t$ into (9.5) affect the period T'. Putting $A(t)$ in terms of its Fourier expansion of period $2T'$ we have

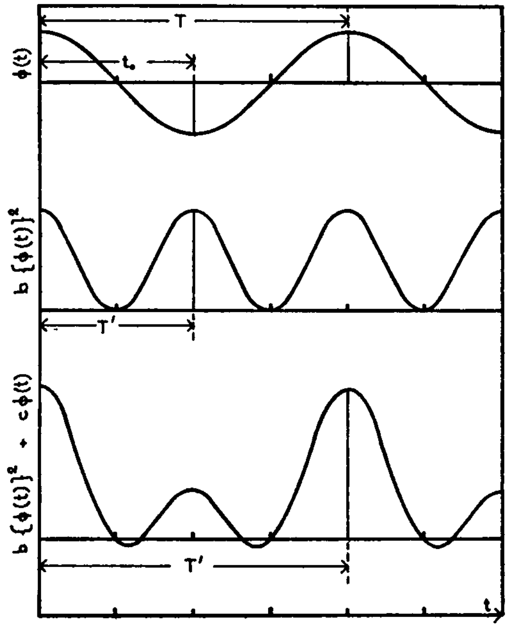

DIAGRAM 6

Examples of the function $A^{(1)}(t)$ which occurs in the operator

$$A(t) = \frac{d^2}{dt^2} + A^{(1)}(t);$$

$$A^{(1)}(t) = c\varphi(t) + b\{\varphi(t)\}^2.$$

It is apparent that the period T' of $A^{(1)}(t)$ or of $A(t)$ depends on whether or not $c = 0$. Different periods for $\xi_r(t)$ are responsible for the difference illustrated in diagrams 7(a) and 7(b). t_0, T and T' are defined in equations (5.2), (5.3) and (5.7).

$$\begin{aligned} A(t) &= bA^2 \sin^2 \omega t + cA \sin \omega t + \frac{d^2}{dt^2} \\ &= \tfrac{1}{2}bA^2 + cA \cos \frac{2\pi t}{T'} - \tfrac{1}{2}bA^2 \cos \frac{4\pi t}{T'} + \frac{d^2}{dt^2} \end{aligned} \tag{9.8}$$

In the notation of section 4, the Fourier Coefficients β_n are zero for $n > 2$ and hence $N = 2$. The orientation of the stability lines is shown in diagram 7(b).

We now carry out exactly the same analysis as we carried out in the previous section. But, as a result of the different period, the functions

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TABLE 3

The inequalities obtained for four low lying stability lines of the Hill equation

$$\frac{d^2\xi}{dt^2} + (a + bA^2 \sin^2 \omega t + cA \sin \omega t)\xi = 0$$

Trial Function	Resulting Inequality $\left(X = \frac{bA^2}{\omega^2}; Y = \frac{cA}{\omega^2}\right)$	Limit of Inequality when $Y = 0$
1	$\frac{a}{\omega^2} \leqq -\frac{1}{2}X$	$\frac{a}{\omega^2} \leqq -\frac{1}{2}X$
$1 + \mu \cos \omega t$	$\frac{a}{\omega_2} \leqq -\frac{5}{8}X + \frac{1}{2} - \frac{1}{2}\sqrt{(1 - \frac{1}{4}X)^2 + 2Y^2}$	
$\cos \frac{1}{2}\omega t$	$\frac{a}{\omega^2} \leqq -(\frac{1}{2}X + \frac{1}{2}Y) + \frac{1}{4}$	$\frac{a}{\omega^2} \leqq -\frac{1}{2}X + \frac{1}{4}$
$\cos \frac{1}{2}\omega t + \mu \cos \frac{3}{2}\omega t$	$\frac{a}{\omega^2} \leqq -(\frac{1}{2}X + \frac{1}{4}Y) + \frac{5}{4} - \sqrt{(1 + \frac{1}{4}Y)^2 + (\frac{1}{2}Y + \frac{1}{4}X)^2}$	$\frac{a}{\omega^2} \leqq -\frac{1}{2}X + \frac{5}{4} - \sqrt{1 + \frac{1}{16}X^2}$
$\sin \frac{1}{2}\omega t$	$\frac{a}{\omega^2} \leqq -(\frac{1}{2}X - \frac{1}{2}Y) + \frac{1}{4}$	$\frac{a}{\omega^2} \leqq -\frac{1}{2}X + \frac{1}{4}$
$\sin \frac{1}{2}\omega t + \mu \sin \frac{3}{2}\omega t$	$\frac{a}{\omega^2} \leqq -(\frac{1}{2}X - \frac{1}{4}Y) + \frac{5}{4} - \sqrt{(1 - \frac{1}{4}Y)^2 + (-\frac{1}{2}Y + \frac{1}{4}X)^2}$	$\frac{a}{\omega^2} \leqq -\frac{1}{2}X + \frac{5}{4} - \sqrt{1 + \frac{1}{16}X^2}$
$\sin \omega t$	$\frac{a}{\omega^2} \leqq -\frac{1}{4}X + 1$	$\frac{a}{\omega^2} \leqq -\frac{1}{4}X$
$\sin \omega t + \mu \sin 2\omega t$	$\frac{a}{\omega^2} \leqq -\frac{3}{8}X + \frac{5}{2} - \frac{3}{2}\sqrt{(1 - \frac{1}{12}X)^2 + \frac{2}{9}Y^2}$	

$\xi_r(t)$ belonging to the various representations (θ_1, θ_2) are not exactly the same as in section 8. For instance, with reference to table 1 and using (9.7) above the functions of representation $(\theta_1, \theta_2) = (1, 1)$ are

$$a_0 + \sum a_{2r} \cos r\omega t. \tag{9.9}$$

The two most elementary trial functions of this representation used are $\psi = 1$ as in the previous case and

$$\psi = 1 + \mu \cos \omega t \tag{9.10}$$

in contrast to the corresponding trial function (8.2) of the previous case.

In table 3 are presented the results of carrying out the variational procedure. The inequalities in column 2 are approximations to stability lines. In column 3 are shown the result of putting $c = 0$ in these inequalities, i.e. removing the lack of symmetry. It is interesting to study the correspondence between these stability lines and the lines in table 2. The line corresponding to the representation (1, 1) becomes a line listed in table 2 corresponding to the symmetric case (i.e., also (1, 1)). However, the two lines corresponding to the representations $(+1, -1)$ and $(-1, +1)$ become, on removal of the asymmetric term, the same line and this line does not appear at all in table 2. The positions of these lines are shown diagramatically in Diagrams 7(a) and 7(b). These diagrams indicate that *the effect of the asymmetry in the spring is to introduce additional unstable regions, not merely to displace slightly the boundaries of the existing regions.*

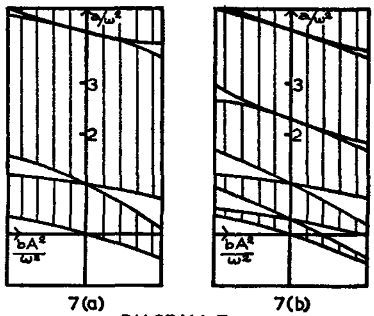

DIAGRAM 7

Both diagrams indicate the location of stability lines (near the a/ω^2 axis) of the equation

$$\frac{d^2\xi}{dt^2} + (a + bA^2 \cos^2 \omega t + cA \cos \omega t)\xi = 0$$

where ξ satisfies periodic boundary conditions.

$$\text{In (7a)} \quad c = 0 \quad \text{and} \quad \xi\left(t + \frac{2\pi}{\omega}\right) = \xi(t)$$

$$\text{In 7(b)} \quad c \neq 0 \quad \text{and} \quad \xi\left(t + \frac{4\pi}{\omega}\right) = \xi(t)$$

10. A General Case

In the two previous sections we dealt with very special cases of the system described in section 3. It should be emphasised that the only approximation in those cases is introduced as a result of not knowing the exact positions of the stability lines, and we have at our disposal methods for increasing the closeness of the approximation if this is desired. We also know that the exact line lies below any variational approximation to it when plotted as in the diagrams. This is because the variational approximation gives a definite inequality, and not only an approximate equality (see equation (6.8)).

In practice, however, it is unlikely that the mechanical system is one of these special cases, unless it is specially constructed to be one.

In this section we illustrate the general procedure by considering the problem of section 3 with the springs symmetric, i.e., $a_2 = A_2 = 0$, but dropping the restriction introduced in section 4 that $a_3/8 = -A_3$. However, for simplicity we restrict ourselves in what follows to the case in which α_{02} and α_{20} are both positive. The generalisation is straightforward. If the springs are normal a_1 and A_1 are positive and so are α_{02} and α_{20}. The Hamiltonian takes the form

$$(10.1)\quad H(p_1, p_2, q_1, q_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + m\{\alpha_{20}q_1^2 + \alpha_{02}q_2^2 + \alpha_{40}q_1^4 + \alpha_{04}q_2^4 + \alpha_{22}q_1^2q_2^2\}$$

and the out of phase motion is governed by the Hamiltonian

$$(10.2)\qquad H(0, p_2, 0, q_2) = \frac{p_2^2}{2m} + m\alpha_{02}q_2^2 + m\alpha_{04}q_2^4$$

The motion resulting from this Hamiltonian can be expressed in terms of Jacobian elliptic functions as follows:

$$(10.3a)\qquad \frac{q}{A} = cn\left[K(k)\ (1 - \frac{4t}{T})\right] \qquad \alpha_{04} > 0$$

$$(10.3b)\qquad \frac{q}{A} = sn\left[K(k)\ \frac{4t}{T}\right] \qquad \alpha_{04} < 0$$

where

$$(10.4a)\qquad k^2 = \frac{\alpha_{04}A^2}{\{2\alpha_{04}A^2 + \alpha_{02}\}} \qquad \alpha_{04} > 0$$

$$(10.4b)\qquad k^2 = -\frac{\alpha_{04}A^2}{\{\alpha_{04}A^2 + \alpha_{02}\}} \qquad \alpha_{04} < 0$$

Here $K(k)$ is the complete Elliptic Function $\int_0^{\pi/2}(1 - k^2\sin^2\theta)^{-1/2}d\theta$, A is the amplitude of the motion and the period T is given by

$$T \equiv \frac{2\pi}{\omega} = \left(\frac{8}{\pm\ \alpha_{04}}\right)^{\frac{1}{2}}\frac{k}{A}K(k).$$

(The upper or lower sign is taken according as α_{04} is greater than or less than zero respectively.)

This solution can be expanded in terms of a Fourier series, of which the first two significant terms are

$$(10.5)\qquad q_2(t) = A\{(1+\varepsilon)\sin\omega t + \varepsilon\sin 3\omega t\}$$

where $\varepsilon = k^2/16$. If we make the substitutions

$$(10.6)\qquad Y = 2\alpha_{04}\left(\frac{A^2}{\omega^2}\right) \qquad X \equiv \frac{2\alpha_{02}}{\omega^2}$$

we arrive at the amplitude-frequency relations in a dimensionless form

$$(\pm Y)^{1/2} = \frac{2k}{\pi} K(k) \tag{10.7}$$

where k takes the values given by (10.4a) and (10.4b), namely

$$k^2 = \frac{1}{2 + X/Y} \qquad \alpha_{04} > 0 \tag{10.8a}$$

$$k^2 = \frac{-1}{1 + X/Y} \qquad \alpha_{04} < 0 \tag{10.8b}$$

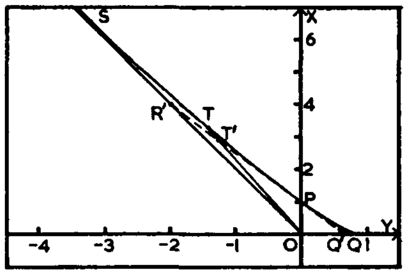

DIAGRAM 8

The amplitude-frequency relation for the motion governed by the Hamiltonian

$$H(0, p_2, 0, q_2) = \frac{p_2^2}{2m} + m\alpha_{02} q_2^2 + m\alpha_{04} q_2^4$$

Here $X = 2\alpha_{04} \dfrac{A^2}{\omega^2}$ and $Y = \dfrac{2\alpha_{02}}{\omega^2}$.

The broken line $R'PQ'$ is the Duffing Approximation (10.9) and the full line SPQ the exact relation, obtained from (10.7) and (10.8).

If $\alpha_{04} > 0$, the representative point lies on the section PQ, the point P representing zero amplitude and Q the limit of infinite amplitude. The numerical values are $OQ' = 2/3$ and $OQ = 0.6966$. Thus the Duffing approximation is a close one, even for high amplitudes.

If $\alpha_{04} < 0$, the representative point lies on the section of the curve to the left of P. In this case $V(0, q_2)$ has an upper limit V_{max} and the amplitude has an upper limit A_{max}. For the motion to be possible, the representative point lies to the right of the straight line OR'. The true curve approaches this line asymptotically whereas the Duffing approximation (broken line) intersects it. For a particular amplitude, the representative point T, and its approximation in the Duffing equation, T', are collinear with the origin. The point R' corresponds to $A = A_{max}$ in the Duffing Approximation.

For small amplitudes (10.7) and (10.8) can be replaced by the well known Duffing approximation [6]

$$X + \tfrac{3}{2}Y = 1 \tag{10.9}$$

The amplitude frequency relations (10.7) and (10.8) are plotted in diagram 8 together with the approximation (10.9).

Substitution of $q_2(t)$ given by (10.5) into the Hill Equation (3.3b) gives the equation

$$(10.10) \qquad \frac{d^2\xi_r}{dt^2} + [a + bA^2\{(1+\varepsilon)\sin\omega t + \varepsilon \sin 3\omega t\}^2]\xi_r = 0$$

The period of the term in curly brackets is π/ω and so the period of $\xi_r(t)$ is π/ω or $2\pi/\omega$ as is the case in the Mathieu Equation considered in sections 4 and 8. In fact (10.10) differs from the Mathieu equation only by terms in ε, and as the period is the same, we may expect the stability to differ from that of diagram 2 by terms of order ε only. This is what in fact happens. Using a trial function $\psi = 1$, there results the stability line

$$(10.11) \qquad \frac{a}{\omega^2} \cong -\tfrac{1}{2}\frac{bA^2}{\omega^2}(1 + 2\varepsilon + 2\varepsilon^2)$$

and using $\psi = 1 + \mu \cos 2\omega t$, we find, after applying the variational technique

$$(10.12) \qquad \begin{aligned} \frac{a}{\omega^2} = &-\tfrac{1}{2}\frac{bA^2}{\omega^2}(1 + \tfrac{3}{2}\varepsilon + \tfrac{3}{2}\varepsilon^2) + 2 \\ &- 2\sqrt{1 + \frac{bA^2}{\omega^2}\frac{\varepsilon + \varepsilon^2}{4} + \frac{b^2A^4}{\omega^4}\left(\tfrac{1}{32} - \tfrac{5}{54}\varepsilon^2 + \tfrac{1}{32}\varepsilon^3 + \tfrac{1}{64}\varepsilon^4\right)} \end{aligned}$$

Not only is the stability diagram different, but also a slightly different technique is required for its use. The examples discussed in Sections 4, 8 and 9 all had the property that the amplitude A was independent of the frequency ω. In the more general case this is no longer true. The exact relation between these quantities is indicated in equations (10.7) and (10.8) and an approximate one in equation (10.9). These are illustrated on diagram 8. However, for any particular mechanical system, the representative point can cover only part of this curve. Which part this is depends only on the sign of α_{04}. (We recall that we are restricting ourselves to the case in which $\alpha_{02} > 0$). If α_{04} is positive, inspection shows that the representative point lies on the section PQ. If α_{04} is negative, this point lies on the section to the left of P. In this case the amplitude cannot increase indefinitely since $V(0, q_2)$ has a maximum at $q_2 = \sqrt{-\alpha_{02}/2\alpha_{04}}$. We call this value of q_2, $A_{\max}$, and it is clear that if q_2 ever attains a value greater than $A_{\max}$, then q_2 continues to increase indefinitely. The restriction

$$(10.13) \qquad A^2 < -\frac{\alpha_{02}}{2\alpha_{04}} \equiv A^2_{\max}$$

can be written, using (10.6),

$$(10.14) \qquad X > -2Y$$

The amplitude frequency relations given by (10.7) and (10.8) obey this inequality, and are actually asymptotic to the line

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$$X = -2Y \tag{10.15}$$

as is illustrated in Diagram 8. The Duffing approximation does not conform to this restriction and breaks down for amplitudes near to $A_{\max}$. This is discussed in the caption of Diagram 8. It is clear from inspection of this diagram that, with this one exception, the Duffing approximation is very good.

Now using the defining relations of a and b, equations (4.7) and the definition of X and Y, equation (10.6), we see that

$$Y = \frac{\alpha_{04}}{\alpha_{22}}\left(\frac{bA^2}{\omega^2}\right) \quad \text{and} \quad X = \frac{\alpha_{02}}{\alpha_{20}}\left(\frac{a}{\omega^2}\right) \tag{10.16}$$

Thus if we make a trivial change of scale, the whole of diagram 8 can be transferred bodily onto a stability diagram; the former X and Y axes are now a/ω^2 and bA^2/ω^2 axes. The situation is illustrated in diagram 9. The use

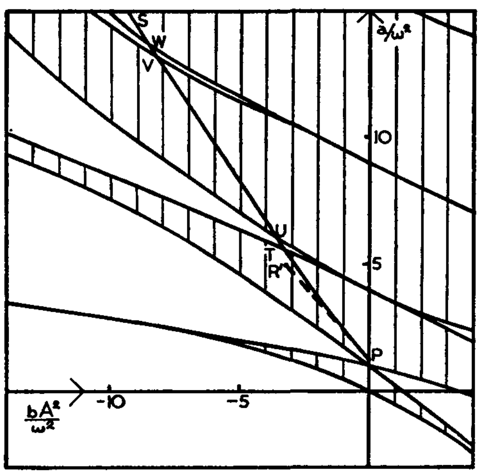

DIAGRAM 9

The Stability Diagram for the equation

$$\frac{d^2\xi}{dt^2} + [a + b\{\varphi(t)\}^2\xi] = 0$$

where $\varphi(t)$ is the solution of amplitude A of the motion governed by the Hamiltonian

$$H(p, q) = \frac{p^2}{2m} + m\alpha_{02}q^2 + m\alpha_{04}q^4; \quad \alpha_{04} < 0$$

The representative point lies on the line PS. When the amplitude is small, it is near P and as the amplitude is increased it moves away from P, approaching infinity as A approaches $A_{\max}$. The line PR is the Duffing approximation (10.9). Use of this Approximation can lead to qualitatively wrong results about the stability. In the case illustrated in the diagram, increasing the amplitude leads to a change in stability as the representative point reaches T, U, V, W, and so on. Use of the Duffing Approximation implies that the representative point moves directly to R as A is increased to $A_{\max}$ and there is apparently no change in stability at all.

of this diagram differs from that of diagram 2, for example, in that the representative point is confined to the curve PS instead of to the horizontal line QK of diagram 2. Reference to diagram 9 shows that error about the stability of the system can easily occur unless the amplitude frequency relation is known accurately.

APPENDIX I

The minimum of $g(\mu)$ (Equation 8.4)

We derive here an expression for the minimum of $g(\mu)$ where $g(\mu)$ is given by

(A1) $$g(\mu) = (a_0 + a_1\mu + a_2\mu^2)^{-1}(b_0 + b_1\mu + b_2\mu^2)$$

This can be written

(A2) $$(a_0g(\mu) - b_0) + (a_1g(\mu) - b_1)\mu + (a_2\, g(\mu) - b_2)\mu^2 = 0$$

We suppose that when $\mu = \mu_0$, $g(\mu)$ takes an extremum value or

(A3) $$g'(\mu_0) = 0$$

Multiplying (A2) by μ, differentiating with respect to μ and putting $\mu = \mu_0$, leads to, in view of (A3),

$$(a_0g(\mu_0) - b_0) + 2(a_1g(\mu_0) - b_1)\mu_0 + 3(a_2g(\mu_0) - b_2)\mu_0{}^2 = 0$$

and subtracting this from the expression obtained from (A3) by multiplying by 3 and putting $\mu = \mu_0$ gives

(A4) $$2(a_0g(\mu_0) - b_0) + (a_1g(\mu_0) - b_1)\mu_0 = 0$$

We can find another independent expression for μ_0 by differentiating (A2) and putting $\mu = \mu_0$. This is

(A5) $$(a_1g(\mu_0) - b_1) + (a_2g(\mu_0) - b_2)\mu_0 = 0$$

Eliminating μ_0 from (A4) and (A5) gives the quadratic relation for $g(\mu_0)$

A(6) $$\begin{vmatrix} 2(a_0g(\mu_0) - b_0), & (a_1g(\mu_0) - b_1) \\ (a_1g(\mu_0) - b_1), & 2(a_2g(\mu_0) - b_2) \end{vmatrix} = 0$$

Expanding this determinant gives the following expression for the two extremum values of $g(\mu)$

$$\frac{-(-a_1b_1 + 2a_0b_2 + 2a_2b_0) \pm \sqrt{(-a_1b_1 + 2a_0b_2 + 2a_2b_0)^2 - (a_1^2 - 4a_2a_0)(b_1^2 - 4b_2b_0)}}{(a_1^2 - 4a_2a_0)}$$

If the variational method is used with two parameters λ and μ it is possible, using the same procedure, to obtain a 4×4 determinant of the same type as that in (A6), which gives a quartic for $g(\mu_0, \lambda_0)$. Similarly an N-parameter trial function gives an equation of degree 2^N for $q_{\min}$. Because it is difficult to deal with equations like this it is usually more convenient to use the iteration technique of the second part of section 7.

APPENDIX II

Notation for eigenvalues of Mathieu Equation

References mentioned in the text discuss this equation in different forms. For the convenience of the reader these are listed here

Whittaker (Ref. 12) $\dfrac{d^2\xi}{dz^2} + (a + 16q \sin 2z)\xi = 0$

Stoker (Ref. 10) $\dfrac{d^2u}{dz^2} + (\delta + \varepsilon \cos z)\, u = 0$

This paper $\dfrac{d^2\xi}{dt^2} + (a + bA^2 \sin^2 \omega t)\xi = 0$

McLachlan [6] uses the same notation as Whittaker [12] but with $-2q$ replacing $16q$. (This is the same as that used by Mathieu). The relations between the various parameters is as follows

$$\begin{aligned} a - 16q &= 4(\delta - \varepsilon) = a/\omega^2 \\ 32q &= 8\varepsilon = bA^2/\omega^2 \end{aligned}$$

In the notation of Stoker [Reference 10], the equations of stability lines listed in Table 2 are

$$\begin{aligned} \delta &= \tfrac{1}{2} - \tfrac{1}{2}(1 + 2\varepsilon^2)^{1/2} \\ \delta &= \tfrac{1}{4}\varepsilon + \tfrac{5}{4} - (1 - \tfrac{1}{2}\varepsilon + \tfrac{5}{16}\varepsilon^2)^{1/2} \\ \delta &= -\tfrac{1}{4}\varepsilon + \tfrac{5}{4} - (1 + \tfrac{1}{2}\varepsilon + \tfrac{5}{16}\varepsilon^2)^{1/2} \\ \delta &= \tfrac{5}{2} - \tfrac{3}{2}(1 + \tfrac{1}{9}\varepsilon^2)^{1/2} \end{aligned}$$

References

[1] J. M. Blatt and J. N. Lyness: The Practical Use of Variation Methods in Non-Linear Mechanics; to be published in J. Aust. Math. Soc.
[2] R. Courant and D. Hilbert: Methods of Mathematical Physics, Vol. I, Interscience, 1953.
[3] S. Goldstein: Camb. Phil. Soc., 23, 303 (1927).
[4] E. L. Ince: Ordinary Differential Equations; Longmans Green and Co., 1927.
[5] J. N. Lyness: A Property of an Iteration Process,
[6] N. W. McLachlan: Theory and Application of Mathieu Functions, Oxford, 1947.
[7] H. Poincaré: Les Méthodes Nouvelles de la Mécanique Céleste; Tome 1, Gauthier Villars et Fils, Paris, 1892.

[8] R. M. Rosenberg and C. P. Atkinson, Journal of Appl. Mech., 3, 377 (1959).
[9] J. Schwinger: Lecture notes on Nuclear Physics (unpublished).
[10] J. J. Stoker: Nonlinear Vibrations, Interscience, N.Y., 1950.
[11] N. Svartholm: Ph. D. Thesis, Lund, 1945.
[12] E. T. Whittaker and G. N. Watson: Modern Analysis, C.U.P., 1927.
[13] E. T. Whittaker: Analytical Dynamics, C.U.P., 1937.

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