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EMBEDDINGS INTO EFFICIENT GROUPS

by JENS HARLANDER

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A finite presentation F/N of a group G is called efficient if $d_F(N) = d(H_2(G)) + d(F) - r(H_1(G))$. A finitely presented group is called efficient if it admits an efficient presentation. We show that a finitely presented group embeds into an efficient group.

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1. Background

If A is a G-group, then $d_G(A)$ denotes the minimal number of G-group generators of A. For example the normal subgroup N of a group F is an F-group via conjugation and $d_F(N)$ is the minimal number of elements that generate N as a normal subgroup. If G acts trivially on A we omit the subscript and simply write $d(A)$ for the minimal number of generators for the group A.

Given a finite presentation $\mathcal{P} = \langle X|R\rangle$ of a group G, let F be the free group on X and $N = N(R)$ be the normal closure of R in F. Then $F/N = G$. We also refer to F/N as a presentation for G. Now

$$(*) \qquad N/[F,N] = H_2(G) \oplus Z^{d(F)-r(H_1(G))},$$

where $r(H_1(G))$ is the torsion free rank of the finitely generated abelian group $H_1(G)$. To see this, consider the exact 5-term sequence

$$H_2(F) \to H_2(G) \to N/[F,N] \to F/[F,F] \to G/[G,G] \to 0$$

associated with the extension $N \to F \to G$ (see Brown [**9**, page 47]). Since F is free, $H_2(F) = 0$ and the result follows. In particular we have

$$d(N/[F,N]) = d(H_2(G)) + d(F) - r(H_1(G)).$$

For more details and additional references see Beyl, Tappe [**5**, page 18]. The presentation $\mathcal{P} = \langle X|R\rangle$ is called *efficient* if

$$|R| = d_F(N) = d(N/[F,N]).$$

The group G is called efficient if it admits an efficient presentation. Examples of efficient groups are finitely generated abelian groups (Epstein [**13**]), fundamental groups of closed 3-manifolds [**13**] and also finite groups with balanced presentations. Such finite groups have trivial Schur-multiplier. Whether finite groups with trivial Schur-multiplier are efficient (i.e., admit balanced presentations in this case) was answered negatively by Swan [**26**]. He gave examples of non-efficient metabelian groups with trivial H_2. Finite metacyclic groups are efficient. This was shown by Wamsley [**27**] and Beyl [**4**]. Infinite metacyclic groups however need not be efficient, a result due to Baik and Pride [**2**] (see also Baik [**1**]). The first examples of torsion-free non-efficient groups were found by Lustig [**21**]. For more references on the subject of efficiency see Baik, Pride [**3**], Beyl, Rosenberger [**6**], Campbell, Robertson, Williams [**10, 11**], Johnson, Robertson [**18**], Kenne [**20**] and Robertson, Thomas, Wotherspoon [**24**].

Suppose $\langle X|R\rangle$ is a finite presentation for a group H. Assume that u and w are words in $X^{\pm 1}$ and let G be the quotient of H presented by $\langle X|R, w\rangle$. Suppose the following conditions are satisfied:

1. $[u, w]$ represents the trivial element of H;
2. u represents an element of infinite order of G;
3. The presentation $\langle X|R, w\rangle$ is efficient.

The group G can be used to embed a given group into an efficient group by an iterated amalgamated product. Before we state our main result we introduce more notation. Let $S(K, G, l)$ be the fundamental group of a graph of groups supported by a graph with vertices $v, v_1, \ldots, v_l$ and oriented edges $e_1, \ldots, e_l$, where e_i starts at v and ends at v_i. The group at v is K, all other vertex groups are G (as above) and the edge groups are infinite cyclic. Edge maps are given by choosing elements of infinite order in K and the other vertex groups.

Theorem. *Suppose that K is a finitely presented group that admits a generating set consisting of elements of infinite order. Suppose furthermore that, in case both $H_2(K)$ and $H_2(G)$ have torsion, the first torsion-numbers of these abelian groups are not relatively prime. Then there exists an integer l such that $S(K, G, l)$ is efficient.*

Note that the condition on the torsion numbers ensures that $d(H_2(K) \oplus H_2(G)) = d(H_2(K)) + d(H_2(G))$.

There is considerable flexibility in choosing G. For example we can take $H = \langle a, b|a^n = b^n\rangle$, $u = ab$ and $w = a^n$. In that case we get $G = \langle a, b|a^n = b^n, a^n\rangle = Z_n * Z_n$, the free product of two cyclic groups of order n. Or we could take $H = \langle a, b, c|[a, b], [a, c]\rangle$, $u = a$ and $w = [b, c]$. Here we obtain $G = \langle a, b, c|[a, b], [a, c], [b, c]\rangle = Z \oplus Z \oplus Z$. Note that in both cases $H_2(G)$ is torsion-free. Before we prove the Theorem we point out some consequences.

Corollary 1. *Let K be a finitely presented group and let $d(K) = k$. Let F_k be the free*

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group of rank k. Then there exists an integer l such that $S(K * F_k, Z_n * Z_n, l)$ is efficient. In particular a finitely presented group can be embedded into a finitely presented efficient group.

Proof. Let $y_1, \dots, y_k$ be a set of generators for K and let $a_1, \dots, a_k$ be a basis for F_k. Then $y_1 a_1, \dots, y_k a_k, a_1, \dots, a_k$ is a generating set of $K * F_k$ consisting of elements of infinite order. Now apply the Theorem. □

We remark that the author showed in [15] that a finite group can be embedded into a finite efficient group. In fact, if K is finite, then $K \times \prod_{i=1}^{l} Z_p$ is efficient for l big enough and p a prime.

Corollary 2. *Let K be a finitely presented group of finite cohomological dimension k. If $k \neq 2$, then K can be embedded into an efficient group of cohomological dimension k. If $k = 2$, then K can be embedded into an efficient group of virtual cohomological dimension 2.*

Proof. If $k = 1$ then, by Stallings' Theorem [25], K is free and thus itself efficient. So suppose $k \geq 2$. Since K is torsion-free, it admits a generating set consisting of elements of infinite order. We can apply the Theorem to see that $\tilde{G} = S(K, G, l)$ is efficient for big enough l and an appropriately chosen group G. If $k = 2$ take $G = Z_n * Z_n$. The virtual cohomological dimension of both Z and $Z_n * Z_n$ is one and hence $vcd(\tilde{G}) = vcd(K) = 2$ (see Bieri [7, page 83]). If $k \geq 3$ take $G = Z \oplus Z \oplus Z$. Since $cd(G) = 3$ it follows that $cd(\tilde{G}) = cd(K) = k$. □

Whether a group of cohomological dimension 2 can be embedded into an efficient group of cohomological dimension 2 is related to the question whether a group of cohomological dimension 2 has geometric dimension 2. A discussion of these matters can be found in Section 4 of this paper.

2. The main lemma

The proof of the main theorem in this article is based on an idea of Wolfgang Metzler. He realized (see [23]) that wedging on standard 2-complexes of $Z_2 \times Z_4$ to a given 2-complex allows one to bypass the commutator question, a serious obstruction encountered when attempting to generalize results from higher dimensions into dimension 2. Hog-Angeloni and Metzler have successfully applied this trick to various situations (see Metzler [23] and also Metzler, Hog-Angeloni [16]). We will present a generalized $Z_2 \times Z_4$ trick, which is tailored to our situation. Suppose $\mathcal{P}_H = \langle X | R \rangle$ is a finite presentation for the group H and u and w are words in $X^{\pm 1}$ so that the commutator $[u, w]$ represents the trivial element of H. Let G be the quotient of H represented by $\mathcal{P}_G = \langle X | R, w \rangle$. Let n be the order of the element of G represented by u (the order can be infinite). Next assume that K is another group admitting a finite presentation $\mathcal{P}_K = \langle Y|S, [f, t]\rangle$, where t is a consequence of $S, [f, t]$ in $\mathcal{P}_K$, that is $t \in N(S, [f, t])$, and f represents an element of order n in K. We can form the free product with amalgamation $\tilde{G} = K *_{Z_n} G$ with presentation $\mathcal{P} = \langle X, Y|R, w, S, [f, t], u = f\rangle$. The key observation here is that the normal closure of the relations in $\mathcal{P}$ is generated by $|R| + 1 + |S| + 1$ elements, which is one less than expected. Indeed $\{R, S, w = t, u = f\}$ is a generating set for that normal closure. Just observe that

$$[f, t] = [u, w] = 1$$

modulo the relations $f = u, t = w$ and R. Since we assumed that $t = 1$ modulo S and $[f, t]$, this shows that $w = 1$ modulo $R, S, f = u$ and $t = w$. If we iterate the above process we obtain the following.

Lemma. *Suppose K is a group admitting a presentation $\mathcal{P}_K = \langle Y|S, [f_i, t_i]\rangle$, $1 \le i \le l$, where each t_i is contained in $N(S, [f_1, t_1], \ldots, [f_l, t_l])$, and each f_i represents an element of infinite order. Let*

$$\mathcal{P} = \langle X_i, Y|S, [f_i, t_i], R_i, w_i, u_i = f_i\rangle,$$

$1 \le i \le l$, $\langle X_i|R_i, w_i\rangle$ presenting G, u_i representing an element of infinite order in G, be the standard presentation for the amalgamated product $S(K, G, l)$. Then the normal closure of the relations in $\mathcal{P}$ is generated by $|S| + (|R| + 2)l$ elements.

A free product version of the above Proposition with $Z_2 \times Z_4$ factors is implicit in **[23]**, dealing with commutators of relators, that is with elements of $[N, N]$ rather than $[F, N]$.

3. Proof of the theorem

Let F/N be a finite presentation for the group K, where F is a free group with basis Y and each element y of Y represents an element of infinite order in K. Let $m = d(N/[F, N])$. We can find elements $s_1, \ldots, s_m$ of N so that $s_1[F, N], \ldots, s_m[F, N]$ generates $N/[F, N]$. Since N is the normal closure of finitely many elements, we can find elements $f_i \in F, t_i \in N$, $1 \le i \le l$, so that $N = N(s_1, \ldots, s_m, [f_i, t_i])$. Thus

$$\mathcal{P}_K = \langle Y|s_1, \ldots, s_m, [f_i, t_i]\rangle,$$

$1 \le i \le l$, presents K. Note that because $\{[y^{\pm 1}, r]\,|\,y \in Y, r \in N\}$ generates $[F, N]$ we may assume that each f_i is equal to some $y^{\pm 1}$, in particular that each f_i has infinite order in K. Let $\mathcal{P}_G = \langle X|R, w\rangle$ be an efficient presentation of a group G as in the previous section. Then we have a word u in $X^{\pm 1}$ representing an element of infinite order in G and $[u, w] = 1$ modulo R. Let

$$\mathcal{P} = \langle X_i, Y | S, [f_i, t_i], R_i, w_i, f_i = u_i \rangle,$$

$1 \le i \le l$, be the standard presentation for the amalgamated product $S(K, G, l)$ as in the Lemma. Let $\tilde{F}$ be the free group on the generators in $\mathcal{P}$ and let $\tilde{N}$ be the normal closure of the relations in $\mathcal{P}$. Furthermore let F_i be the free group on X_i and let N_i be the normal closure of R_i and w_i in $F(X_i)$. So F_i/N_i presents the vertex group G at v_i in the above amalgamated product. We know from the Lemma that $d_F(\tilde{N}) \le m + (|R| + 2)l$. We claim that $d(\tilde{N}/[\tilde{F}, \tilde{N}]) = m + (|R| + 2)l$ and thus that $\tilde{F}/\tilde{N}$ is efficient. Before we show this, let us make some general remarks. Suppose F_i/N_i is a finite presentation for $G_i, i = 1, 2$, and that C is a finitely generated subgroup of both G_1 and G_2. Let F/N be a presentation for the amalgamated product $G = G_1 *_C G_2$, obtained from the presentations F_i/N_i and a fixed finite generating set for C. Then we have an exact sequence (see Hannerbauer [**14**])

$$0 \to (ZG \otimes_{G_1} N_1/[N_1, N_1]) \oplus (ZG \otimes_{G_2} N_2/[N_2, N_2]) \to N/[N, N] \to ZG \otimes_C IC \to 0.$$

If we apply $Z \otimes_G -$ we obtain the exact sequence

$$H_2(C) \to N_1/[F_1, N_1] \oplus N_2/[F_2, N_2] \to N/[F, N] \to H_1(C) \to 0.$$

In case C is infinite cyclic, $H_2(C) = 0$ and $H_1(C) = Z$ and we obtain

$$N/[F, N] = N_1/[F_1, N_1] \oplus N_2/[F_2, N_2] \oplus Z.$$

If we apply this result to our presentation $\tilde{F}/\tilde{N}$ of $S(K, G, l)$ we get

$$\tilde{N}/[\tilde{F}, \tilde{N}] = N/[F, N] \oplus \bigoplus_{i=1}^{l} N_i/[F_i, N_i] \oplus Z^l.$$

This follows from the above discussion and induction on l since

$$S(K, G, l) = S(K, G, l-1) *_C G,$$

with C infinite cyclic. Since $F_i/N_i = \langle X_i | R_i, w_i \rangle$ is an efficient presentation for G, we have $d(N_i/[F_i, N_i]) = |R| + 1$. Since the first torsion-numbers of $H_2(G)$ and $H_2(K)$ are not relatively prime (in case both $H_2(K)$ and $H_2(G)$ contain torsion), we have $d(H_2(K) \oplus H_2(G)) = d(H_2(K)) + d(H_2(G))$. Since $N/[F, N]$ is the direct sum of $H_2(K)$ and a free abelian group and each $N_i/[F_i, N_i]$ is the direct sum of $H_2(G)$ and a free abelian group (see equation (*) on the first page), we have

$$d(\tilde{N}/[\tilde{F}, \tilde{N}]) = d(N/[F, N]) + \sum_{i=1}^{l} d(N_i/[F_i, N_i]) + l.$$

Hence $d(\tilde{N}/[\tilde{F}, \tilde{N}]) = m + (|R| + 2)l$ as claimed. □

4. Groups of dimension 2

A group G has cohomological dimension 2 if the trivial G-module Z admits a projective resolution of length 2. The geometric dimension of G is 2 if there exists a 2-dimensional $K(G, 1)$-complex. A group G is of type FL if Z admits a resolution of finite length consisting of finitely generated free ZG-modules. A presentation $\mathcal{P}$ is aspherical if the associated 2-complex $K(\mathcal{P})$ modelled on $\mathcal{P}$ is aspherical (that is, it has trivial second homotopy group). Note that in that case $K(\mathcal{P})$ is a $K(G, 1)$-complex and thus G and all it's subgroups have geometric dimension 2. These definitions can be found in [**9**].

Efficient groups of cohomological dimension 2 are of interest in connection with the longstanding open question whether cohomological dimension 2 implies geometric dimension 2. The next proposition shows that subgroups of an efficient group of cohomological dimension 2 that is FL have geometric dimension 2. This result is due to Gutierrez and Ratcliffe [**17**] (see also Bogley [**8**, page 329]). In [**17**] it is stated for subcomplexes of aspherical complexes. Such complexes give rise to presentations which are not only efficient but satisfy the Cockcroft property (see [**12**, page 149]). For the convenience of the reader we have also included a proof.

Proposition. *Let $\mathcal{P}$ be a finite presentation of a group G of cohomological dimension 2 that is of type FL. Then $\mathcal{P}$ is efficient if and only if $\mathcal{P}$ is aspherical.*

Proof. Let $\mathcal{P} = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ be an efficient presentation for G. Let F be the free group on the generators in $\mathcal{P}$ and let N be the normal closure of the relations of $\mathcal{P}$ in F. Cohomological dimension 2 together with FL implies that the relation module $N/[N, N]$ of this presentation is finitely generated stably free. This is a consequence of Schanuel's Lemma (see [**9**, page 192]). So suppose $N/[N, N] \oplus ZG^k = ZG^l$. Replacing $\mathcal{P}$ with $\langle x_1, \ldots, x_n, y_1, \ldots, y_k | r_1, \ldots, r_m, y_1, \ldots, y_k \rangle$, we obtain an efficient presentation with free relation module of rank l. In particular $l = m + k$. Since the 2-complex associated with the modified presentation is simply homotopic to the two complex associated with $\mathcal{P}$, asphericity of the new presentation implies asphericity of $\mathcal{P}$. This discussion shows that we may assume that the relation module of $\mathcal{P}$ is free of rank m. Let us look at the partial resolution (see Lyndon, Schupp [**22**, page 100])

$$\pi_2(K(\mathcal{P})) \to ZG^m \xrightarrow{\partial_2} ZG^n \xrightarrow{\partial_1} ZG \to Z \to 0$$

associated with $\mathcal{P}$ (it arises from the cellular chain complex of the universal covering of $K(\mathcal{P})$). The image of the boundary map ∂_2 is the relation module which is free of rank m. Thus it follows from Kaplansky's Theorem (see [**19**], and also [**8**, page 328]) that ∂_2 is an isomorphism and that $\pi_2(K(\mathcal{P}))$ is trivial. Thus $\mathcal{P}$ is an aspherical presentation for G. This proves one direction. That asphericity of a presentation implies efficiency is immediate from the partial resolution associated with $\mathcal{P}$. □

The property FL was needed to ensure that every finite presentation of G has stably free relation module. It should be noted that there are no examples known of finitely presented groups of cohomological dimension 2 that are not FL. We know from Corollary 2 of Section 1 that a finitely presented group K of cohomological dimension 2 can be embedded into an efficient group $S(K, G = Z_n * Z_n, l)$, which is of virtual cohomological dimension 2. If we could replace $Z_n * Z_n$ by a group G of cohomological dimension 2 for which our method works, we could eliminate "virtual". If in addition $S(K, G, l)$ is FL, then K is actually of geometric dimension 2 by the above Proposition. But we believe that such a group G is difficult to find. For our techniques to work we would have to find an efficient presentation $\mathcal{P} = \langle X|R, w\rangle$ of a group G of cohomological dimension 2 and a word u representing an element of infinite order such that $[u, w] = 1$ modulo R. Thus we would have an identity of relations $uwu^{-1}w^{-1}\prod_{i=1}^{k} f_i r_{j_i}^{\epsilon_i} f_i^{-1} = 1$, f_i words in $X^{\pm 1}$, $\epsilon_i \in \{\pm 1\}$, $r_{j_i} \in R$, which yields a non-trivial spherical element over $\mathcal{P}$ since u is not trivial. So $\mathcal{P}$ is an efficient non-aspherical presentation of a group G of cohomological dimension 2. In view of the above Proposition, G could not be FL!

Of course the group $S(K, Z_n * Z_n, l)$ contains a torsion-free subgroup of finite index of cohomological dimension 2. We conclude by remarking that a subgroup of finite index of an efficient group need not be efficient. It was shown in **[15]** that a finite group can be embedded into a finite efficient group. Since there are non-efficient finite groups (Swan's examples for instance), finite index does not preserve efficiency.

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FB Mathematik
Universität Frankfurt
Robert-Mayer-Str. 8
60054 Frankfurt/Main
Germany