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Cesàro Operators on the Hardy Spaces of the Half-Plane

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Abstract. In this article we study the Cesàro operator

$$\mathcal{C}(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\,d\zeta,$$

and its companion operator $\mathcal{T}$ on Hardy spaces of the upper half plane. We identify $\mathcal{C}$ and $\mathcal{T}$ as resolvents for appropriate semigroups of composition operators and we find the norm and the spectrum in each case. The relation of $\mathcal{C}$ and $\mathcal{T}$ with the corresponding Cesàro operators on Lebesgue spaces $L^p(\mathbb{R})$ of the boundary line is also discussed.

1 Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ denote the upper half of the complex plane. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{U})$ is the space of analytic functions $f\colon \mathbb{U} \to \mathbb{C}$ for which

$$\|f\|_{H^p(\mathbb{U})} = \sup_{y>0}\Big(\int_{-\infty}^{\infty} |f(x+iy)|^p\,dx\Big)^{\frac{1}{p}} < \infty.$$

For $p = \infty$ we denote by $H^\infty(\mathbb{U})$ the space of all bounded analytic functions on $\mathbb{U}$ with the supremum norm.

The spaces $H^p(\mathbb{U})$, $1 \le p \le \infty$, are Banach spaces and for $p = 2$, $H^2(\mathbb{U})$ is a Hilbert space. For the rest of the paper we use the notation $\|f\|_p$ for $\|f\|_{H^p(\mathbb{U})}$.

Let $1 \le p < \infty$ and suppose $f \in H^p(\mathbb{U})$. Then f satisfies the growth condition

$$|f(z)|^p \le \frac{C\|f\|_p^p}{\operatorname{Im}(z)}, \quad z \in \mathbb{U}, \tag{1.1}$$

where C is a constant. Furthermore, the limit $\lim_{y\to 0} f(x+iy)$ exists for almost every x in $\mathbb{R}$, and we may define the boundary function on $\mathbb{R}$, denoted by f^*, as

$$f^*(x) = \lim_{y\to 0} f(x+iy).$$

This function is p-integrable and

$$\|f\|_p^p = \int_{-\infty}^{\infty} |f^*(x)|^p\,dx.$$

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Thus $H^p(\mathbb{U})$ can be viewed as a subspace of $L^p(\mathbb{R})$. For more details on Hardy spaces $H^p(\mathbb{U})$ see [2, 4, 5].

For $f \in L^p(\mathbb{R})$ the well-known Cesàro transformation is defined by

$$\mathtt{C}(f)(x) = \frac{1}{x}\int_0^x f(u)\,du$$

for $x \in \mathbb{R}$, with an appropriate convention if $x = 0$. This defines a bounded operator on $L^p(\mathbb{R})$ for $p > 1$ ([1]). In particular if $f^* \in L^p(\mathbb{R})$ is the boundary function of $f \in H^p(\mathbb{U})$, the question arises whether the transformed function $\mathtt{C}(f^*)$ is also the boundary function of an $f \in H^p(\mathbb{U})$.

In this article we consider the half-plane version of the Cesàro operator, which is formally defined by

$$\mathcal{C}(f)(z) = \frac{1}{z}\int_0^z f(\zeta)d\zeta, \quad f \in H^p(\mathbb{U}).$$

It turns out that this formula defines an analytic function on $\mathbb{U}$ for each $f \in H^p(\mathbb{U})$ and the resulting operator is bounded on $H^p(\mathbb{U})$, $p > 1$. In addition the companion operator

$$\mathcal{T}(f)(z) = \int_z^\infty \frac{f(\zeta)}{\zeta}d\zeta, \quad f \in H^p(\mathbb{U})$$

is also shown to be bounded on $H^p(\mathbb{U})$ for $p \geq 1$. In fact it will be seen later that these two operators are adjoints of each other. We find the norm and the spectrum of $\mathcal{C}, \mathcal{T}$ and we show that for the boundary functions we have $\mathcal{C}(f)^* = \mathtt{C}(f^*)$. The whole discussion is based on the observation that $\mathcal{C}$ and $\mathcal{T}$ can be obtained as resolvent operators for appropriate strongly continuous semigroups of simple composition operators on $H^p(\mathbb{U})$. Cesàro operators of the unit disc were studied in [7].

2 Related Semigroups, Strong Continuity

For each $t \in \mathbb{R}$ consider the following analytic self maps of $\mathbb{U}$:

$$\phi_t(z) = e^{-t}z, \quad z \in \mathbb{U},$$

and for $1 \leq p < \infty$, the corresponding weighted composition operators on $H^p(\mathbb{U})$:

$$T_t(f)(z) = e^{-\frac{t}{p}} f(\phi_t(z)), \quad f \in H^p(\mathbb{U}).$$

For $f \in H^p(\mathbb{U})$ we have

$$\begin{aligned}\|T_t(f)\|_p^p &= e^{-t}\sup_{y>0}\Big(\int_{-\infty}^{+\infty} |f(e^{-t}x + ie^{-t}y)|^p\,dx\Big)\\ &= \sup_{v>0}\Big(\int_{-\infty}^{+\infty} |f(u+iv)|^p\,du\Big) = \|f\|_p^p.\end{aligned}$$

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Thus each T_t is an isometry on $H^p(\mathbb{U})$. Furthermore, it is easy to see that the family $\{T_t\}$ satisfies $T_t T_s = T_{t+s}$ for each $t, s \in \mathbb{R}$ and $T_0 = I$ the identity operator, so $\{T_t\}$ is a group of isometries. We will use this group or the positive and negative semigroups $\{T_t,\ t \geq 0\}$ and $\{T_{-t},\ t \geq 0\}$ in our study of the operators $\mathcal{C}$ and $\mathcal{T}$.

Proposition 2.1 *For $1 \leq p < \infty$, the group $\{T_t\}$ is strongly continuous on $H^p(\mathbb{U})$. The infinitesimal generator Γ of $\{T_t\}$ is given by $\Gamma(f)(z) = -zf'(z) - \frac{1}{p}f(z)$, and its domain is $D(\Gamma) = \{f \in H^p(\mathbb{U}) : zf'(z) \in H^p(\mathbb{U})\}$.*

Proof For the strong continuity we need to show $\lim_{t\to 0} \|T_t(f) - f\|_p = 0$ for every $f \in H^p(\mathbb{U})$. To do this, observe first that the set $\mathcal{A}^p(\mathbb{U})$ containing all functions in $H^p(\mathbb{U})$ that are continuous up to the boundary (*i.e.*, functions in $H^p(\mathbb{U})$ that are continuous on $\overline{\mathbb{U}}$) is dense in $H^p(\mathbb{U})$. Thus for $f \in H^p(\mathbb{U})$ and arbitrary $\epsilon > 0$ we can find $g \in \mathcal{A}^p(\mathbb{U})$ such that $\|f - g\|_p < \epsilon$. Then

$$\begin{aligned}\|T_t(f) - f\|_p &\leq \|T_t(f) - T_t(g)\|_p + \|T_t(g) - g\|_p + \|g - f\|_p\\ &= 2\|f - g\|_p + \|T_t(g) - g\|_p\\ &\leq 2\epsilon + \|T_t(g) - g\|_p,\end{aligned}$$

and thus it suffices to show that $\|T_t(g) - g\|_p \to 0$ as $t \to 0$ for $g \in \mathcal{A}^p(\mathbb{U})$. But for such functions g it is clear that $T_t(g)(x) \to g(x)$ for all $x \in \mathbb{R}$ and furthermore, trivially, $\|T_t(g)\|_p \to \|g\|_p$ because T_t are isometries. We can then apply [2, Lemma 1, p. 21] to conclude $\|T_t(g) - g\|_p \to 0$ and the assertion about strong continuity follows.

By definition the domain $D(\Gamma)$ of Γ consists of all $f \in H^p(\mathbb{U})$ for which the limit $\lim_{t\to 0}(e^{-(t/p)} f \circ \phi_t - f)/t$ exists in $H^p(\mathbb{U})$ and

$$\Gamma(f) = \lim_{t\to 0} \frac{e^{-(t/p)} f \circ \phi_t - f}{t}, \quad f \in D(\Gamma).$$

The growth estimate (1.1) shows that convergence in the norm of $H^p(\mathbb{U})$ implies in particular pointwise convergence, therefore for each $f \in D(\Gamma)$,

$$\begin{aligned}\Gamma(f)(z) &= \lim_{t\to 0} \frac{e^{-\frac{t}{p}} f(e^{-t}z) - f(z)}{t} = \frac{\partial}{\partial t}\big(e^{-\frac{t}{p}} f(e^{-t}z)\big)\Big|_{t=0}\\ &= -zf'(z) - \frac{1}{p} f(z).\end{aligned}$$

This shows that $D(\Gamma) \subseteq \{f \in H^p(\mathbb{U}) : zf'(z) \in H^p(\mathbb{U})\}$.

Conversely, let $f \in H^p(\mathbb{U})$ such that $zf'(z) \in H^p(\mathbb{U})$. Then for $z \in \mathbb{U}$, we can write

$$\begin{aligned}T_t(f)(z) - f(z) &= \int_0^t \frac{\partial}{\partial s}\big(e^{-\frac{s}{p}} f(\phi_s(z))\big)\, ds\\ &= \int_0^t -e^{-\frac{s}{p}} \phi_s(z) f'(\phi_s(z)) - \frac{1}{p} e^{-\frac{s}{p}} f(\phi_s(z))\, ds\\ &= \int_0^t T_s(F)(z)\, ds,\end{aligned}$$

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where $F(z) = -zf'(z) - \frac{1}{p}f(z)$ is a function in $H^p(\mathbb{U})$. Thus

$$\lim_{t\to 0}\frac{T_t(f)-f}{t} = \lim_{t\to 0}\frac{1}{t}\int_0^t T_s(F)\,ds.$$

From the general theory of strongly continuous (semi)groups, for $F \in H^p(\mathbb{U})$ the latter limit exists in $H^p(\mathbb{U})$ and is equal to F. Thus

$$\lim_{t\to 0}\frac{T_t(f)-f}{t} = F.$$

This says that $D(\Gamma) \supseteq \{f \in H^p(\mathbb{U}) : zf'(z) \in H^p(\mathbb{U})\}$, completing the proof. ■

Remark 2.2 An immediate corollary of the previous proposition is that the two semigroups $\{T_t,\ t \geq 0\}$ and $\{T_{-t},\ t \geq 0\}$ are strongly continuous on $H^p(\mathbb{U})$ and their infinitesimal generators are Γ and $-\Gamma$, respectively.

Lemma 2.3 *Let $0 < p < \infty$ and $\lambda \in \mathbb{C}$. Then*

(i) $e_\lambda(z) = z^\lambda \notin H^p(\mathbb{U})$,
(ii) $h_\lambda(z) = (i+z)^\lambda \in H^p(\mathbb{U})$ *if and only if* $\mathrm{Re}(\lambda) < -\frac{1}{p}$.

Proof This can be proved by direct calculation of the norms. We give an alternative argument involving the following well-known characterization of membership in the Hardy space of the half-plane. For a function f analytic on $\mathbb{U}$,

$$f \in H^p(\mathbb{U}) \quad \text{if and only if} \quad \psi'(z)^{1/p} f(\psi(z)) \in H^p(\mathbb{D}),$$

where $\psi(z) = i\frac{1+z}{1-z}$, a conformal map from the unit disc $\mathbb{D} = \{|z| < 1\}$ onto $\mathbb{U}$, and $H^p(\mathbb{D})$ is the usual Hardy space of the disc. We find

$$\psi'(z)^{1/p} h_\lambda(\psi(z)) = \frac{c_1}{(1-z)^{\frac{2}{p}+\lambda}}, \tag{2.1}$$

$$\psi'(z)^{1/p} e_\lambda(\psi(z)) = c_2\frac{(1+z)^\lambda}{(1-z)^{\frac{2}{p}+\lambda}}, \tag{2.2}$$

where c_1, c_2 are nonzero constants.

Next recall that if ν is a complex number, $(1-z)^\nu \in H^p(\mathbb{D})$ if and only if $\mathrm{Re}(\nu) > -1/p$ (this follows for example from [2, p. 13, Ex. 1]). Applying this to (2.1) we obtain $h_\lambda \in H^p(\mathbb{U})$ if and only if $\mathrm{Re}(-\frac{2}{p} - \lambda) > -1/p$ and this gives the desired conclusion.

In the case of e_λ, the right-hand side of (2.2) belongs to $H^p(\mathbb{D})$ if and only if both terms $(1+z)^\lambda$ and $(1-z)^{-\frac{2}{p}-\lambda}$ belong to $H^p(\mathbb{D})$, because each of the two terms is analytic and nonzero at the point where the other term has a singularity. Thus for e_λ to belong to $H^p(\mathbb{U})$, both conditions $\mathrm{Re}(\lambda) > -1/p$ and $\mathrm{Re}(-\frac{2}{p} - \lambda) > -1/p$ must be satisfied, which is impossible. ■

For an operator A denote by $\sigma_\pi(A)$ the set of eigenvalues of A, by $\sigma(A)$ the spectrum of A, and by $\rho(A)$ the resolvent set of A on $H^p(\mathbb{U})$.

Proposition 2.4 *Let $1 \le p < \infty$ and consider $\{T_t\}$ acting on $H^p(\mathbb{U})$. Then*

(i) $\sigma_\pi(\Gamma)$ *is empty;*
(ii) $\sigma(\Gamma) = i\mathbb{R}$.

In particular Γ is an unbounded operator.

Proof (i) Let γ be an eigenvalue of Γ and let f be a corresponding eigenvector. The eigenvalue equation $\Gamma(f) = \gamma f$ is equivalent to the differential equation

$$zf'(z) + \left(\gamma + \frac{1}{p}\right) f(z) = 0.$$

The nonzero analytic solutions of this equation on $\mathbb{U}$ have the form $f(z) = cz^{-(\gamma+\frac{1}{p})}$ with $c \neq 0$, which by Lemma 2.3 are not in $H^p(\mathbb{U})$. It follows that $\sigma_\pi(\Gamma) = \varnothing$.

(ii) Because each T_t is an invertible isometry its spectrum satisfies $\sigma(T_t) \subseteq \partial\mathbb{D}$. The spectral mapping theorem for strongly continuous groups [6, Theorem 2.3] says that $e^{t\sigma(\Gamma)} \subseteq \sigma(T_t)$. Thus if $w \in \sigma(\Gamma)$, then $e^{tw} \in \partial\mathbb{D}$, so that $\sigma(\Gamma) \subseteq i\mathbb{R}$. We will show that in fact $\sigma(\Gamma) = i\mathbb{R}$.

Let $\mu \in i\mathbb{R}$ and assume that $\mu \in \rho(\Gamma)$. Let $\lambda = \mu + \frac{1}{p}$ and consider the function

$$f_\lambda(z) = i\lambda(i+z)^{-\lambda-1}.$$

Since $\mathrm{Re}(-\lambda - 1) = -1 - 1/p < -1/p$, this function is in $H^p(\mathbb{U})$. Since $\mu \in \rho(\Gamma)$, the operator $R_\mu = (\mu - \Gamma)^{-1} : H^p(\mathbb{U}) \to H^p(\mathbb{U})$ is bounded. The image function $g = R_\mu(f_\lambda)$ satisfies the equation $(\mu - \Gamma)(g) = f_\lambda$, or equivalently,

$$\left(\mu + \frac{1}{p}\right) g(z) + zg'(z) = f_\lambda(z), \quad z \in \mathbb{U}.$$

Thus g solves the differential equation $\lambda g(z) + zg'(z) = i\lambda(i+z)^{-\lambda-1}$, $z \in \mathbb{U}$. It is easy to check that the general solution of this equation in $\mathbb{U}$ is

$$G(z) = cz^{-\lambda} + (i+z)^{-\lambda}, \quad c \text{ a constant.}$$

Using the notation of Lemma 2.3 we find

$$(\psi'(z))^{1/p} G(\psi(z)) = \frac{c_1 + cc_2(1+z)^{-\lambda}}{(1-z)^{\frac{2}{p}-\lambda}},$$

where c_1, c_2 are nonzero constants. This last expression represents an analytic function on the unit disc, which, however, by an argument similar to that in the proof of Lemma 2.3, is not in the Hardy space $H^p(\mathbb{D})$ of the unit disc for any value of c, because $\mathrm{Re}(\lambda) = 1/p$. Thus $G(z)$ is not in $H^p(\mathbb{U})$ for any c, and this is a contradiction. It follows that $\sigma(\Gamma) = i\mathbb{R}$, and this completes the proof. ■

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3 The Cesàro Operators

It follows from the above that when $1 < p < \infty$, the point $\lambda_p = 1 - 1/p$ is in the resolvent set of the generator Γ. The resolvent operator $R(\lambda_p, \Gamma)$ is therefore bounded. Let $f \in H^p(\mathbb{U})$ and let $g = R(\lambda_p, \Gamma)(f)$. It follows that $(\lambda_p - \Gamma)(g) = f$ or equivalently $(1-1/p)g(z)+zg'(z)+(1/p)g(z) = f(z)$. Thus g satisfies the differential equation $g(z) + zg'(z) = f(z)$, $z \in \mathbb{U}$. Fix a point w on the imaginary axis. Then

$$zg(z) = \int_w^z f(\zeta)\, d\zeta + c, \quad z \in \mathbb{U},$$

with c a constant. Now let $z = iy \to 0$ along the imaginary axis. Since $g \in H^p(\mathbb{U})$ with $p > 1$, the estimate (1.1) implies that $zg(z) \to 0$. Therefore,

$$c = -\int_w^0 f(\zeta)\, d\zeta.$$

(The existence of this integral is also a consequence of (1.1).) It follows that the integral of f on the segment $[0, z]$ exists and we have

$$g(z) = R(\lambda_p, \Gamma)(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\, d\zeta.$$

Theorem 3.1 *Let $1 < p < \infty$ and let $\mathcal{C}$ be the operator defined by*

$$\mathcal{C}(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\, d\zeta, \quad f \in H^p(\mathbb{U}).$$

Then $\mathcal{C}\colon H^p(\mathbb{U}) \to H^p(\mathbb{U})$ is bounded. Furthermore,

$$\|\mathcal{C}\| = \frac{p}{p-1} \quad \textit{and} \quad \sigma(\mathcal{C}) = \Big\{ w \in \mathbb{C} : \Big| w - \frac{p}{2(p-1)} \Big| = \frac{p}{2(p-1)} \Big\}.$$

Proof As found above, for $1 < p < \infty$,

$$\mathcal{C} = R(\lambda_p, \Gamma), \quad \lambda_p = 1 - 1/p \in \rho(\Gamma).$$

The spectral mapping theorem [3, Lemma VII.9.2] gives

$$\begin{aligned}\sigma(\mathcal{C}) &= \Big\{ \frac{1}{\lambda_p - z} : z \in \sigma(\Gamma) \Big\} \cup \{0\} = \Big\{ \frac{1}{1 - 1/p - ir} : r \in \mathbb{R} \Big\} \cup \{0\} \\ &= \Big\{ w \in \mathbb{C} : \Big| w - \frac{p}{2(p-1)} \Big| = \frac{p}{2(p-1)} \Big\},\end{aligned}$$

giving the spectrum of $\mathcal{C}$ on $H^p(\mathbb{U})$.

Since the spectral radius of $\mathcal{C}$ is equal to $p/(p-1)$, it follows that

$$\|\mathcal{C}\| \geq \frac{p}{p-1}.$$

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On the other hand, we can apply the Hille–Yosida–Phillips theorem to the group $\{T_t\}$ of isometries [3, Corollary VIII.1.14] to obtain

$$\|\mathcal{C}\| = \|R(\lambda_p, \Gamma)\| \le \frac{1}{1-\frac{1}{p}} = \frac{p}{p-1},$$

and the proof is complete. ■

Remark 3.2 It follows from Lemma 2.3 that $h(z) = (i+z)^{-2}$ belongs to $H^1(\mathbb{U})$. The transformed function $-i(i+z)^{-1} = \frac{1}{z}\int_0^z h(\zeta)\,d\zeta$ is analytic on $\mathbb{U}$ but does not belong to $H^1(\mathbb{U})$. Thus $\mathcal{C}$ does not take $H^1(\mathbb{U})$ to $H^1(\mathbb{U})$.

We now consider the negative part $\{T_t : t \le 0\}$ of the group $\{T_t\}$ and we rename it $\{S_t\}$. That is, for $f \in H^p(\mathbb{U})$,

$$S_t(f)(z) = e^{\frac{t}{p}} f(\phi_{-t}(z)) = e^{\frac{t}{p}} f(e^t z), \quad t \ge 0.$$

It is clear that $\{S_t\}$ is strongly continuous on $H^p(\mathbb{U})$ and that its generator is $\Delta = -\Gamma$. It follows from Proposition 2.4 that

$$\sigma(\Delta) = i\mathbb{R}, \quad \sigma_\pi(\Delta) = \varnothing.$$

Let $\mu_p = \frac{1}{p} \in \rho(\Delta)$, and consider the bounded resolvent operator $R(\mu_p, \Delta)$. Let $f \in H^p(\mathbb{U})$ and let $g = R(\mu_p, \Delta)(f)$. Then $\mu_p g - \Delta(g) = f$ or equivalently

$$zg'(z) = -f(z), \quad z \in \mathbb{U}.$$

Fix a point $a \in \mathbb{U}$, then we have

$$g(z) = -\int_a^z \frac{f(\zeta)}{\zeta}\,d\zeta + c$$

with c a constant. Now let $z \to \infty$ within a half-plane $\mathrm{Im}(z) \ge \delta > 0$. Then $g(z) \to 0$ [2, Corollary 1, p. 191]; thus the following limit exists and gives the value of c:

$$c = \lim_{\substack{w\to\infty \\ \mathrm{Im}(w)\ge\delta}} \int_a^w \frac{f(\zeta)}{\zeta}\,d\zeta.$$

We therefore find

$$g(z) = \int_z^a \frac{f(\zeta)}{\zeta}\,d\zeta + \lim_{\substack{w\to\infty \\ \mathrm{Im}(w)\ge\delta}} \int_a^w \frac{f(\zeta)}{\zeta}\,d\zeta = \lim_{\substack{w\to\infty \\ \mathrm{Im}(w)\ge\delta}} \int_z^w \frac{f(\zeta)}{\zeta}\,d\zeta.$$

We now show that the above limit exists unrestrictedly when $w \to \infty$ within the half-plane $\mathbb{U}$. Indeed for $\delta > 0$ we have,

$$\int_z^w \frac{f(\zeta)}{\zeta}\,d\zeta = \int_z^{w+i\delta} \frac{f(\zeta)}{\zeta}\,d\zeta + \int_{w+i\delta}^w \frac{f(\zeta)}{\zeta}\,d\zeta = J_w + I_w.$$

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With the change of variable $\zeta = w + is\delta$, $0 \le s \le 1$, we have

$$\begin{aligned}
|I_w| &= \Big| -i\delta \int_0^1 \frac{f(w+is\delta)}{w+is\delta}\, ds \Big| \le \delta \int_0^1 \frac{|f(w+is\delta)|}{|w+is\delta|}\, ds \\
&\le \frac{\delta}{|w|} \int_0^1 |f(w+is\delta)|\, ds \\
&= \frac{1}{|w|} \int_y^{y+\delta} |f(x+iu)|\, du,
\end{aligned}$$

where $w = x + iy$ and $u = y + s\delta$. For $p = 1$ the Fejér–Riesz inequality for the half-plane [2, Exercise 6, p. 198] implies through the last inequality

$$|I_w| \le \frac{1}{|w|} \int_0^\infty |f(x+iu)|\, du \le \frac{1}{2|w|} \|f\|_1 \to 0, \quad \text{as } w \to \infty;$$

while for $p > 1$, the growth estimate (1.1) gives

$$\begin{aligned}
|I_w| &\le C_p \Big(\frac{1}{|w|} \int_y^{y+\delta} u^{-\frac{1}{p}}\, du \Big) \|f\|_p \\
&\le C_p' \frac{(y+\delta)^{1/q}}{|w|} \|f\|_p \qquad\qquad (1/q = -1/p + 1) \\
&\le C_p' \frac{(|w|+\delta)^{1/q}}{|w|} \|f\|_p \to 0 \qquad \text{as } w \to \infty.
\end{aligned}$$

Thus in all cases $I_w \to 0$ and we have

$$\lim_{w\to\infty} \int_z^w \frac{f(\zeta)}{\zeta}\, d\zeta = \lim_{w\to\infty} J_w + \lim_{w\to\infty} I_w = g(z),$$

i.e., the unrestricted limit exists. For this reason we can write

$$g(z) = \int_z^\infty \frac{f(\zeta)}{\zeta}\, d\zeta.$$

Theorem 3.3 *Let $1 \le p < \infty$ and let $\mathcal{T}$ be the operator defined by*

$$\mathcal{T}(f)(z) = \int_z^\infty \frac{f(\zeta)}{\zeta}\, d\zeta, \quad f \in H^p(\mathbb{U}).$$

Then $\mathcal{T}\colon H^p(\mathbb{U}) \to H^p(\mathbb{U})$ is bounded. Furthermore,

$$\|\mathcal{T}\| = p, \quad \textit{and} \quad \sigma(\mathcal{T}) = \Big\{ w \in \mathbb{C} : \Big| w - \frac{p}{2} \Big| = \frac{p}{2} \Big\}.$$

Proof We have found above that

$$\mathcal{T} = R(\mu_p, \Delta), \quad \mu_p = 1/p \in \rho(\Delta).$$

Using the spectral mapping theorem as in the proof of Theorem 3.1, we find the spectrum of $\mathcal{T}$ on $H^p(\mathbb{U})$ to be

$$\sigma(\mathcal{T}) = \left\{ w \in \mathbb{C} : \left| w - \frac{p}{2} \right| = \frac{p}{2} \right\},$$

from which we read that the spectral radius is equal to p, therefore $\|\mathcal{T}\| \geq p$. An application of the Hille–Yosida–Phillips theorem as in the proof of Theorem 3.1 gives $\|\mathcal{T}\| = \|R(\mu_p, \Delta)\| \leq \frac{1}{1/p} = p$, and this completes the proof. ∎

Suppose now $1 < p < \infty$ and let q be the conjugate index, $1/p + 1/q = 1$. Recall the duality $(H^p(\mathbb{U}))^* = H^q(\mathbb{U})$ which is realized through the pairing

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f^*(x) \overline{g^*(x)}\, dx.$$

For the semigroups $T_t(f)(z) = e^{-t/p} f(e^{-t}z)$ and $S_t(g)(z) = e^{t/q} g(e^t z)$ acting on $H^p(\mathbb{U})$ and $H^q(\mathbb{U})$, respectively, we find

$$\langle T_t(f), g \rangle = \int_{-\infty}^{\infty} e^{-\frac{t}{p}} f^*(e^{-t}x) \overline{g^*(x)}\, dx = \int_{-\infty}^{\infty} f^*(x) \overline{e^{\frac{t}{q}} g^*(e^t x)}\, dx = \langle f, S_t(g) \rangle.$$

Thus $\{T_t\}$ and $\{S_t\}$ are adjoints of each other. From the general theory of operator semigroups, this relation of being adjoint on reflexive spaces is inherited by the infinitesimal generators and subsequently by the resolvent operators [6, Corollaries 10.2, 10.6]. It follows that $\mathcal{C}$ and $\mathcal{T}$ are adjoints of each other on the reflexive Hardy spaces of the half-plane.

4 Boundary Correspondence

We now examine the boundary correspondence between $\mathcal{C}$ and its real line version $\mathtt{C}$, as well as between $\mathcal{T}$ and the corresponding real line operator $\mathtt{T}$ defined on $L^p(\mathbb{R})$ by

$$\mathtt{T}(f)(x) = \begin{cases} \displaystyle\int_x^{\infty} \frac{f(u)}{u}\, du, & x > 0, \\ \displaystyle -\int_{-\infty}^{x} \frac{f(u)}{u}\, du, & x < 0, \end{cases}$$

while $\mathtt{T}(f)(0)$ can be chosen arbitrarily. It is well known (and easy to prove) that $\mathtt{T}$ is bounded on $L^p(\mathbb{R})$ for $p \geq 1$.

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Theorem 4.1 *Consider the operators $\mathcal{C}$, $\mathcal{T}$ on the Hardy spaces $H^p(\mathbb{U})$ and the operators $\mathtt{C}$, $\mathtt{T}$ on the spaces $L^p(\mathbb{R})$. Then the following hold:*

(i) *For $1 < p < \infty$ and $f \in H^p(\mathbb{U})$, $\mathcal{C}(f)^* = \mathtt{C}(f^*)$.*
(ii) *For $1 \le p < \infty$ and $f \in H^p(\mathbb{U})$, $\mathcal{T}(f)^* = \mathtt{T}(f^*)$.*

Proof (i) For $f \in H^p(\mathbb{U})$ and $\delta > 0$, consider the function

$$f_\delta(z) = f(z + i\delta), \quad \operatorname{Im}(z) > -\delta.$$

It is clear that $f_\delta \in H^p(\mathbb{U})$ and $f_\delta^*(x) = f_\delta(x)$, and we have

$$\begin{aligned}\|\mathcal{C}(f)^* - \mathtt{C}(f^*)\|_{L^p(\mathbb{R})} &\le \|\mathcal{C}(f)^* - \mathcal{C}(f_\delta)^*\|_{L^p(\mathbb{R})} + \|\mathcal{C}(f_\delta)^* - \mathtt{C}(f_\delta^*)\|_{L^p(\mathbb{R})} \\ &\quad + \|\mathtt{C}(f_\delta^*) - \mathtt{C}(f^*)\|_{L^p(\mathbb{R})} \\ &\le (\|\mathcal{C}\| + \|\mathtt{C}\|)\|f_\delta - f\|_{H^p(\mathbb{U})} + \|\mathcal{C}(f_\delta)^* - \mathtt{C}(f_\delta^*)\|_{L^p(\mathbb{R})}.\end{aligned}$$

We can make $\|f_\delta - f\|_{H^p(\mathbb{U})}$ as small as we wish by choosing δ close enough to 0. Thus in order to prove that $\mathcal{C}(f)^*(x) = \mathtt{C}(f^*)(x)$ a.e. on $\mathbb{R}$, it suffices to show $\mathcal{C}(f_\delta)^*(x) = \mathtt{C}(f_\delta^*)(x)$, *i.e.*,

$$\mathcal{C}(f_\delta)^*(x) = \frac{1}{x}\int_0^x f_\delta^*(u)\,du$$

for almost all x. Now for $z = x + iy \in \mathbb{U}$, since $f_\delta(z)$ is analytic on $\{\operatorname{Im}(z) > -\delta\}$, its integral on the segment $[0, z]$ can be obtained by integrating over the path $[0, x] \cup [x, z]$, so we have

$$\mathcal{C}(f_\delta)(z) = \frac{1}{z}\int_0^z f_\delta(\zeta)\,d\zeta = \frac{1}{z}\int_{[0,\,x]} f_\delta(\zeta)\,d\zeta + \frac{1}{z}\int_{[x,\,z]} f_\delta(\zeta)\,d\zeta.$$

If $x \ne 0$, then clearly the limit of the first integral as $y \to 0$ is

$$\lim_{y\to 0}\frac{1}{z}\int_{[0,\,x]} f_\delta(\zeta)\,d\zeta = \frac{1}{x}\int_0^x f_\delta(u)\,du.$$

The limit of the second integral vanishes. Indeed since $f \in H^p(\mathbb{U})$, f is bounded over every half-plane $\{z : \operatorname{Im}(z) \ge \delta\}$ and we find

$$\sup_{\zeta\in[x,\,z]} |f_\delta(\zeta)| \le \sup_{\operatorname{Im}(z)\ge\delta} |f(z)| = M < \infty.$$

Therefore,

$$\Big|\frac{1}{z}\int_{[x,\,z]} f_\delta(\zeta)\,d\zeta\Big| \le \frac{1}{|x|}\Big(\sup_{\zeta\in[x,\,z]} |f_\delta(\zeta)|\Big)\, y \le \frac{M}{|x|} y \to 0$$

as $y \to 0$. It follows that

$$\mathcal{C}(f_\delta)^*(x) = \lim_{y\to 0}\mathcal{C}(f_\delta)(z) = \frac{1}{x}\int_0^x f_\delta(u)\,du = \frac{1}{x}\int_0^x f_\delta^*(u)\,du$$

and the proof of (i) is complete.

(ii) We argue as in part (i) and use the same notation. If $f \in H^p(\mathbb{U})$, $1 \le p < \infty$, let f_δ be defined as in part (i). Using the triangle inequality as in part (i) we see that it suffices to show that for almost all x, $\mathcal{T}(f_\delta)^*(x) = \mathtt{T}(f_\delta^*)(x)$. Let $z = x + iy \in \mathbb{U}$ with $x > 0$ (the case $x < 0$ is similar). Choose $s > 0$, and consider the path $[z, x] \cup [x, x+s] \cup [x+s, z+s(1+i)]$ as an alternative path of integration to obtain

$$\begin{aligned}\mathcal{T}(f_\delta)(z) &= \lim_{s\to\infty} \int_z^{z+s(1+i)} \frac{f_\delta(\zeta)}{\zeta}\, d\zeta \\ &= \int_z^x \frac{f_\delta(\zeta)}{\zeta}\, d\zeta + \lim_{s\to\infty} \Big(\int_x^{x+s} \frac{f_\delta(\zeta)}{\zeta}\, d\zeta + \int_{x+s}^{z+s(1+i)} \frac{f_\delta(\zeta)}{\zeta}\, d\zeta \Big).\end{aligned}$$

For the first integral inside the limit it is clear that

$$\lim_{s\to\infty} \int_x^{x+s} \frac{f_\delta(\zeta)}{\zeta}\, d\zeta = \int_x^{+\infty} \frac{f_\delta(u)}{u}\, du.$$

Write $I(s) = \int_{x+s}^{z+s(1+i)} \frac{f_\delta(\zeta)}{\zeta}\, d\zeta$, the second integral inside the limit, then

$$\begin{aligned}|I(s)| &= \Big| \int_0^{y+s} \frac{f_\delta(x+s+it)}{x+s+it} i\, dt \Big| \le \int_0^{y+s} \frac{|f_\delta(x+s+it)|}{|x+s+it|}\, dt \\ &\le \frac{1}{x+s} \int_0^{y+s} |f_\delta(x+s+it)|\, dt.\end{aligned}$$

If $p = 1$, then the Fejér–Riesz inequality for the upper half-plane gives

$$\int_0^{y+s} |f_\delta(x+s+it)|\, dt \le \int_0^{\infty} |f_\delta(x+s+it)|\, dt \le \frac{1}{2} \|f_\delta\|_1 \le \frac{1}{2} \|f\|_1.$$

Thus $I(s) \le \frac{1}{2(x+s)} \|f\|_1 \to 0$ as $s \to \infty$. If $p > 1$, then using (1.1) we obtain

$$\begin{aligned}\int_0^{y+s} |f_\delta(x+s+it)|\, dt &\le C_p \|f_\delta\|_p \int_0^{y+s} \frac{1}{t^{1/p}}\, dt \\ &\le C_p' \|f\|_p (y+s)^{-\frac{1}{p}+1},\end{aligned}$$

which implies $I(s) \le C_p' \|f\|_p \frac{(y+s)^{-\frac{1}{p}+1}}{x+s} \to 0$ as $s \to \infty$. We have shown

$$\mathcal{T}(f_\delta)(z) = \int_z^x \frac{f_\delta(\zeta)}{\zeta}\, d\zeta + \int_x^{+\infty} \frac{f_\delta(u)}{u}\, du$$

for each $z = x + iy \in \mathbb{U}$. Taking the limit of the above as $y \to 0$, we find

$$\mathcal{T}(f_\delta)^*(x) = \int_x^{+\infty} \frac{f_\delta(u)}{u}\, du = \mathtt{T}(f_\delta^*)(x)$$

and the proof is complete. ∎

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As a consequence we have the following corollary.

Corollary 4.2 *For $1 \leq p < \infty$ let $\mathcal{H}_p$ be the closed subspace of $L^p(\mathbb{R})$ consisting of all boundary functions f^* of $f \in H^p(\mathbb{U})$. Then $\mathtt{C}(\mathcal{H}_p) \subset \mathcal{H}_p$ for $1 < p < \infty$, and $\mathtt{T}(\mathcal{H}_p) \subset \mathcal{H}_p$ for $1 \leq p < \infty$.*

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