SINGULAR SETS OF SOME KLEINIAN GROUPS

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Introduction

In our paper [1] we showed that there exist Schottky groups whose singular sets have positive 1-dimensional measure. Since the example was very complicated, it is natural to seek for simpler examples. Further the problem how about the singular sets of more general groups occurs.

In §§1-3 we investigate the measures of the singular sets of some Kleinian groups and the convergence problem of the (-2)-dimensional Poincaré theta-series. The main result is that there exist Kleinian groups whose fundamental domains are bounded by five mutually disjoint circles and whose singular sets have positive 1-dimensional measure. But it seems still open whether the singular sets of the Kleinian groups whose fundamental domains are bounded by four mutually disjoint circles can have positive 1-dimensional measure or not. In §4, as applications of the preceding chapters, similar problems about Schottky subgroups formed from Kleinian groups by inversion method are treated.

§1. Kleinian groups whose fundamental domains are bounded by N mutually disjoint circles

1. Consider the properly discontinuous groups G of the linear transformations whose fundamental domain B_0 is bounded by N mutually disjoint circles $\{K_i\}_{i=1}^{N}$. Then there exist two different kinds of generators. A generator S_{i_0} of the first kind transforms the outside of a boundary circle K_{i_0} onto the inside of a boundary circle K'_{i_0} different from K_{i_0} and a generator S_{j_0} of the second kind transforms the outside of K_{j_0} onto the inside of K_{j_0} itself. The former is the hyperbolic or loxodromic transformation and the latter is the elliptic trans formation with period 2.

Take any generator of G

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$$T_i(z) = \frac{a_i z + b_i}{c_i z + d_i}, \qquad (a_i d_i - b_i c_i = 1).$$

If $T_i(z)$ is hyperbolic or loxodromic, then $a_i + d_i$ is either real and $|a_i + d_i| > 2$ or complex and non zero. If $T_i(z)$ is the elliptic transformation with period 2, then $T_i^2(z)$ is the identical transformation, so $a_i + d_i$ is zero.

2. Now we start from B_0 and form a properly discontinuous group of linear transformations with the fundamental domain B_0.

Take $2p$ $(N \geqq 2p)$ boundary circles $\{H_i, H_i'\}_{i=1}^p$ from $\{K_i\}_{i=1}^N$. Let S_i be a hyperbolic or loxodromic generator which transforms the outside of H_i onto the inside of H_i'. We denote by S_i^{-1} the inverse transformation of S_i. Then $\{S_i\}_{i=1}^p$ generate a Schottky group G_1, which is a subgroup of G and whose fundamental domain $B_1 \supset B_0$ is bounded by $\{H_i, H_i'\}_{i=1}^p$. Let $\{T_j\}_{j=1}^q$ be the elliptic transformations with period 2 corresponding to the remaining boundary circles $\{K_j\}_{j=1}^q$, where $N - 2p = q$. Then $\{T_j\}_{j=1}^q$ generate a properly discontinuous group G_2 whose fundamental domain $B_2 \supset B_0$ is the outside of the boundary circles $\{K_j\}_{j=1}^q$. By combining two groups G_1 and G_2, a new group $G_1 \cdot G_2$, which is generated by $\{S_i\}_{i=1}^p$ and $\{T_j\}_{j=1}^q$, is obtained and is called a Kleinian group. It is easily seen that the fundamental domain of G coincides with $B_0 = B_1 \cap B_2$ and G is properly discontinuous. In the special case of $N = 2p$, $G = G_1$ is a Schottky group, and if N is odd, there exists necessarily at least one elliptic transformation with period 2 and G is a Kleinian group. If $p = 0$ especially, G is generated by the only elliptic transformations $\{T_j\}_{j=1}^N$ with period 2. To the domain B_0 with N boundary circles, there exist

$$N! \sum_{p=1}^{[N/2]} \frac{1}{2^p \cdot p!(N-2p)!}$$

Kleinian groups in all, according to determination of generators, where $\left[\frac{N}{2}\right]$ denotes the maximal integer not exceeding $\frac{N}{2}$.

3. Let G be a Kleinian group generated by combining two groups G_1 and G_2 as the above, where G_1 is a Schottky group with fundamental domain B_1 generated by p generators and their inverses and G_2 with fundamental domain B_2 is also generated by $q = N - 2p$ elliptic transformations with period 2 and of course the fundamental domain of G is $B_0 = B_1 \cap B_2$.

We denote by ST the transformation obtained by composition of transformations S and T contained in G, that is,

$$ST(z) = S(T(z)).$$

We put $SS = S^2$ and $S^\lambda = S \cdot S^{\lambda-1}$ inductively for any integer $\lambda (>1)$. For a negative integer λ, S^λ denotes $(S^{-1})^{|\lambda|}$. Then any element S of G has the form

$$(1) \qquad \begin{aligned} S &= S_{(\lambda_k)} T_{j_k} \cdots S_{(\nu_1)} T_{j_1} S_{(\nu_0)}, \text{ viz.,} \\ S(z) &= S_{(\nu_k)}(T_{j_k}(\cdots (T_{j_1}(S_{(\nu_0)}(z))\cdots)), \end{aligned}$$

where ν_i $(i=0, \ldots, k)$ are integers and $S_{(\nu_i)}$ denotes the $|\nu_i|$ product of generators of G_1 or their inverses and T_{j_i} $(T_{j_i}^2 = \text{identity})$ denotes the generator of G_2. We call the sum

$$m = \sum_{i=0}^{k} |\nu_i| + k$$

the grade of S.

The image $S(B_0)$ of the fundamental domain B_0 by S $(\in G)$ with grade m $(\neq 0)$ is bounded by N circles $S(H_i)$, $S(H_i')$ and $S(K_j)$, $(i=1, \ldots, p, j=1, \ldots, q, N = 2p+q)$, the one $C^{(m-1)}$ of which is contained in the boundary of the image of B_0 under some T $(\in G)$ with grade $m-1$. For simplicity, we say that the outer boundary circle $C^{(m-1)}$ of $S(B_0)$ is a circle of grade m. Circles $\{H_i, H_i'\}_{i=1}^{p} \cup \{K_j\}_{j=1}^{q}$, which bound B_0, are of grade 1. The number of circles of grade m is obviously equal to $N(N-1)^{m-1}$.

Denote by D_m the $N(N-1)^{m-1}$-ply connected domain bounded by the whole circles of grade m. Evidently $\{D_m\}$ $(m=1, 2, \ldots)$ is a monotone increasing sequence of domains. The complementary set D_m^c of D_m with respect to the extended z-plane consists of $N(N-1)^{m-1}$ mutually disjoint closed discs. The set $E = \bigcap_{m=1}^{\infty} D_m^c$ is perfect and nowhere dense. We call E the singular set of G. The group G is properly discontinuous in the complementary set of E. It is well-known that, in the special case when G is a Schottky group with $p \geqq 2$, the logarithmic capacity of E is positive (See Myrberg [4]) and that the 2-dimensional measure of E is equal to zero (See Sario [6]). Applying Myrberg's method to such a Kleinian group, it is seen that the logarithmic capacity of E of this group is positive in the case of $N \geqq 3$. In our paper [1], we proved, using the example of the fundamental domain of such a group bounded by 36 circles, that there exist Schottky groups whose singular sets have the

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positive 1-dimensional measure.

4. Let $H(z)$ be a rational function none of whose poles is contained in the singular set E of the Kleinian group G. Denote by $z_j=(a_jz+b_j)/(c_jz+d_j)$ $(j=0, 1, \ldots)$ all the elements of G. The identical transformation of G is denoted by z_0.

Consider the series

$$\Theta_\nu(z)=\sum_{j=0}^{\infty}H(z_j)(c_jz+d_j)^{-\nu}, \tag{2}$$

where ν is a positive integer. This is a so-called $(-\nu)$-dimensional Poincaré theta-series.

Let D be the complementary domain of the set E and D' be a relatively closed subdomain of D. Since the point $-d_j/c_j$ $(j\neq 0)$ is the image of infinity by the inverse transformation z_j^{-1} of z_j $(j\neq 0)$ and since G is properly discontinuous in D, there are only finitely many points $-d_j/c_j$ $(j\neq 0)$ in D'. Denote by D'' a non-empty subdomain of D' obtained by deleting suitable neighborhoods of points $-d_j/c_j$ and ∞.

Let $e_i (i=1, \ldots, k)$ be poles of $H(z)$ in D and let U_i be neighborhoods of e_i such that $|H(z)|=M_1$ on the boundary of U_i. By using the proper discontinuity of G and by taking M_1 sufficiently large, we may assume that $D^*=D''-\bigcup_{S\in G}\bigcup_{i=1}^{k}S(U_i)$ are not empty.

Then we have the following propositions (cf. [1]).

Proposition 1. *The $(-\nu)$-dimensional Poincaré theta-series*

$$\Theta_\nu(z)=\sum_{j=0}^{\infty}H(z_j)(c_jz+d_j)^{-\nu},\ z_j=\frac{a_jz+b_j}{c_jz+d_j}\in G$$

converges absolutely and uniformly in D^* if and only if the series

$$\sum_{j=0}^{\infty}(c_jz+d_j)^{-\nu}$$

converges absolutely and uniformly in D''.

We put

$$P_\nu(z)=\sum_{j=0}^{\infty}|c_jz+d_j|^{-\nu}$$

where ν is a positive number. We call $P_\nu(z)$ the $(-\nu)$-dimensional P-series.

Petersson [5] showed that if G is a Fuchsian group, $P_\nu(z)$ converges for $\nu>2$, and that if G is a Fuchsian group of the first kind, $P_\nu(z)$ diverges for $\nu<2$. We have the following

PROPOSITION 2. *The series $P_\nu(z)$ converges uniformly in D'' if and only if the series*

$$\sum_{j=1}^{\infty}|c_j|^{-\nu}, \qquad (\nu>0) \tag{3}$$

converges.

Let

$$S^{(m)}: z' = S^{(m)}(z) = \frac{az+b}{cz+d}, \qquad (ad-bc=1)$$

be a transformation of grade m ($\geqq 1$) in G. Then the radius r_0 of a circle C of grade m by $S^{(m)}(z)$ is given by

$$2\pi r_0 = \int_H \left|\frac{dS^{(m)}(z)}{dz}\right| |dz| = \int_H \frac{|dz|}{|cz+d|^2},$$

where H is a suitable one in $\{H_i, H_i'\}_{i=1}^{p} \cup \{K_j\}_{j=1}^{q}$ which $S^{(m)}$ carries to C. Hence, we have

$$2\pi r_0 = \frac{1}{|c|^2}\int_H \frac{|dz|}{|z+(d/c)|^2}.$$

Again we note that the point $-d/c$ is outside of B_0. If we put

$$\Delta = \max_{z\in H}|z+(d/c)| \text{ and } \delta = \min_{z\in H}|z+(d/c)|,$$

then

$$\frac{r}{\Delta^2}\cdot\frac{1}{|c|^2} \leqq r_0 \leqq \frac{r}{\delta^2}\cdot\frac{1}{|c|^2}, \tag{4}$$

where r is the radius of H.

Such inequality holds for all circles of grade m. Hence we have the following.

PROPOSITION 3. *The series $\sum_{j=1}^{\infty}|c_j|^{-\nu}$ converges if and only if $\sum_{m=1}^{\infty} l_m^{(\nu)}$ converges, where $l_m^{(\nu)}$ is the sum of terms $(r^{(m)})^{\nu/2}$ obtained for radii $r^{(m)}$ of circles of grade m.*

Combining Propositions 1, 2 and 3, we have the following important Proposition 4, which we shall use later.

PROPOSITION 4. *Let ν be a positive integer. The following four propositions are equivalent to each other*: (i) *The* $(-\nu)$*-dimensional Poincaré theta-series* $\Theta_\nu(z)$ *converges absolutely and uniformly in* D^*. (ii) *The* $(-\nu)$*-dimensional P-series* $P_\nu(z)$ *converges uniformly in* D''. (iii) *The series* $\sum_{j=1}^{\infty}|c_j|^{-\nu}$ *converges.* (vi) *The series* $\sum_{m=1}^{\infty} l_m^{(\nu)}$ *converges.*

It is evident that, if $\lim_{m\to\infty} l_m^{(\nu)} = 0$, then the singular set of G is of $\left(\frac{\nu}{2}\right)$-dimensional measure zero. Hence, from Proposition 4, we get

PROPOSITION 5. *If any one of the conditions* (i), (ii), (iii) *and* (iv) *in Proposition* 4 *is valid, then the singular set of G is of* $\left(\frac{\nu}{2}\right)$*-dimensional measure zero.*

5. We shall state the concept of isometric circles of linear transformations due to Ford [2] and some important properties of them.

For a linear transformation of the form

$$T(z) = \frac{az+b}{cz+d}, \quad ad-bc=1, \ c \neq 0,$$

the circle $I : |cz+d| = 1$ is called the isometric circle of the transformation. The radius of I equals $1/|c|$.

(I) By a transformation lengths and areas inside its isometric circle are increased in magnitude and lengths and areas outside the isometric circle are decreased in magnitude. A transformation carries its isometric circle into the isometric circle of the inverse transformation. The radii of the isometric circles of a transformation and its inverse are equal.

Let G be a properly discontinuous group of linear transformations. We suppose that, if an element of G transforms the point at infinity into itself, then the element is the identity of G. Consider two arbitrary transormations of G

$$T : T(z) = \frac{az+b}{cz+d}, \quad ad-bc=1, \ c \neq 0,$$

and

$$S : S(z) = \frac{\alpha z+\beta}{\gamma z+\delta}, \quad \alpha\delta-\beta\gamma=1, \ \gamma \neq 0.$$

For a moment we assume that $S \neq T^{-1}$. The isometric circle of $ST = S(T(z))$ is the circle

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$$|(\gamma a+\delta c)z+\gamma b+\delta d|=1.$$

Denote by I_S, I'_S, I_T, I'_T and I_{ST} isometric circles of S, S^{-1}, T, T^{-1} and ST, respectively. Let g_S, g'_S, g_T, g'_T and g_{ST} be their centers, and let R_S, R_T and R_{ST} be radii of I_S, I_T and I_{ST}.

As to these values, the relation

$$R_{ST}=\frac{1}{|\gamma a+\delta c|}=\frac{R_S\cdot R_T}{|g'_T-g_S|} \tag{5}$$

holds.

As to the location of isometric circles, we have from (I) the following.

(II) If I_S and I'_T are exterior to each other, then I_{ST} is contained in I_T. If I_S and I'_T are tangent externally, then I_{ST} lies in I_T and is tangent internally.

If the grade of the transformation in G is m, its isometric circle is called an isometric circle of grade m. The number of the isometric circles with grade m is obviously equal to $N(N-1)^{m-1}$.

From the definition of isometric circles follows:

Proposition 6. *The convergence of the series* $\sum_{j=1}^{\infty}|c_j|^{-\nu}$ *is equivalent to the convergence of the sum of* ν*-th powers of radii of isometric circles for all the elements of* G.

§ 2. Measure of the singular sets of Kleinian groups

6. Given a set ε of points in the z-plane and a positive number δ, we denote by $I(\delta, \varepsilon)$ a family of a countable number of closed discs U of diameter $l_U\leqq\delta$ such that every point of ε is an interior point of at least one U.

We call the quantity

$$\Lambda^{\eta}\varepsilon=\lim_{\delta\to0}\left[\inf_{\{I(\delta,\varepsilon)\}}\sum_{U\in I(\delta,\varepsilon)}l_U^{\eta}\right]$$

the η-dimensional measure of ε.

7. Denoting by $r_j^{(m)}$ and by $r_i^{(m+1)}$ $(i=1,\ldots,N-1)$ the radius of the outer boundary circle $C_j^{(m)}$, that is, a circle of grade m and the radii of $N-1$ inner boundary circles $C_i^{(m+1)}$ $(i=1,\ldots,N-1)$ of the image B_m of the fundamental domain B_0 by a transformation $S^{(m)}$ $(\in G)$ with grade m, we have the following (See [1].),

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PROPOSITION 7. *There exist positve constants K_0 (<1) and k_0 depending only on B_0 such that*

(6) $$k_0 r_j^{(m)} \leqq r_i^{(m+1)} \leqq K_0 r_j^{(m)}. \qquad (i=1, \ldots, N-1).$$

8. In [1] we obtained the important criterion which determines that the singular set E of a Schottky group G has the positive η-dimensional measure. Since the method will be needed in the following, we state it here again.

Denote by F_{n_0} the family of all closed discs bounded by circles of grade n $(\geqq n_0)$. It is easy to see that F_{n_0} is a covering of the singular set of our Kleinian group G and that the diameter of any disc of F_{n_0} is less than a given δ (>0) for sufficiently large n_0. This fact is verified by Proposition 7.

Consider a family $I(\delta, E)$ of coverings of E stated in No. 6. Since E is compact, the set E is covered by a finite number of discs $\mathfrak{C}_1, \ldots, \mathfrak{C}_k$ of a covering of E in $I(\delta, E)$. Take an arbitrary $\mathfrak{C}_i$ among these k circles and let l_i $(\leqq \delta/2)$ be the radius of $\mathfrak{C}_i$.

Let δ be sufficiently small. For a $\mathfrak{C}_i$ fixed, we can find closed discs ${}^iC^{(m_1)}, \ldots, {}^iC^{(m_{N(i)})}$ in $\bigcup_{n=1}^{\infty} F_n$ satisfying the following conditions:

(i) The radius ${}^ir^{(m_j)}$ of ${}^iC^{(m_j)}$ $(1 \leqq j \leqq N(i))$ of grade m_j is larger than l_i;

(ii) There exist at least one circle of grade m_j+1 lying inside the boundary of ${}^iC^{(m_j)}$, meeting $\mathfrak{C}_i$ and of radius ${}^ir^{(m_j+1)}$ not greater than l_i;

(iii) $\bigcup_{j=1}^{N(i)} {}^iC^{(m_j)} \supset \mathfrak{C}_i \cap E$.

It is easy to see that there exist a constant κ independent of i such that $N(i) \leqq \kappa$. We can prove $\kappa = 5$ by some geometrical consideration.

By the inequality (6) of Proposition 7,

$$k_0 {}^ir^{(m_j)} \leqq {}^ir^{(m_j+1)} \leqq l_i < {}^ir^{(m_j)}.$$

Construct such discs $\{{}^iC^{(m_j)}\}$ for every $\mathfrak{C}_i$ $(i=1, \ldots, k)$. Then it is obvious that $\bigcup_{i=1}^{k} \bigcup_{j=1}^{N(i)} {}^iC^{(m_j)} \supset E$ and

$$\sum_{i=1}^{k} \sum_{j=1}^{N(i)} ({}^ir^{(m_j)})^\eta \leqq \kappa k_0^{-\eta} \sum_{i=1}^{k} l_i^\eta.$$

Thus we have

PROPOSITION 8. *Let $F_{n_0}^{\delta/k_0}$ be a covering of E constructed by discs in F_{n_0} whose radii are not greater than $\delta/2\,k_0$ and let r_C be the radius of a disc C in $F_{n_0}^{\delta/k_0}$. Then it holds*

(7) $$L^{\eta}E=\liminf_{\delta\to 0\ \{F_{n_0}^{\delta/k_0}\}}\sum_{C\in F_{n_0}^{\delta/k_0}}(2r_C)^{\eta}\leqq\kappa\left(\frac{k_0}{2}\right)^{-\eta}\Lambda^{\eta}E,$$

By Proposition 8, we can prove

PROPOSITION 9. *Given a Kleinian group defined in* § 1, *if*

(8) $$\sum_{T_i}(R_{S^{(m)}})^{\mu}\geqq(R_{S^{(m-1)}})^{\mu},\quad(0<\mu<4,\ S^{(m)}=S^{(m-1)}T_i)$$

for radius $R_{S^{(m-1)}}$ *of any isometric circle* $I_{S^{(m-1)}}$ *of grade* $m-1$ *and radii* $R_{S^{(m)}}$ *of* $N-1$ *isometric circles* $I_{S^{(m)}}$ *of grade* m, *then the* $\left(\frac{\mu}{2}\right)$*-dimensional measure of the singular set* E *of* G *is positive.*

Proof. Take a covering $F_{n_0}^{\delta/k_0}$ of E constructed by a finite number of close discs $D^1_{S^{(m_1)}},\ldots,D^1_{S^{(m_Q)}}$, which are bounded by circles

(9) $$C^1_{S^{(m_1)}},\ldots,C^1_{S^{(m_Q)}}$$

respectively, where $C^1_{S^{(m_j)}}$ $(1\leqq j\leqq Q)$ is a circle of grade m_j, that is, an outer boundary circle of the image $S^{(m_j)}(B_0)$.

Denote by $r_{S^{(m_j)}}$ the radius of a circle $C^1_{S^{(m_j)}}$. Then, from (4)

(10) $$\sum_{j=1}^{Q}(r_{S^{(m_j)}})^{\mu/2}\geqq k(G)\sum_{j=1}^{Q}(R_{S^{(m_j)}})^{\mu},$$

where $R_{S^{(m_j)}}$ is the radius of the isometric circle $I_{S^{(m_j)}}$.

From the construction of $F_{n_0}^{\delta/k_0}$, there exist in (9) some systems $\{W^1_{m_k^*}\}$, each of which consists of $N-1$ boundary circles with the following properties:

(i) $N-1$ circles of $W^1_{m_k^*}$ have same grade number m_k^*, while the grade of circles of different systems are not necessarily equal,

(ii) $N-1$ circles of each system $W^1_{m_k^*}$ are bounded by a circle of grade m_k^*-1.

Let $C^1_{S_i^{(m_k^*)}}$ $(i=1,2,\ldots,N-1)$ and $C_{S^{(m_k^*-1)}}$ be circles of a system $W^1_{m_k^*}$ and a circle surrounding them respectively, where $S_i^{(m_k^*)}=S^{(m_k^*-1)}T_i$ $(i=1,2,\ldots,N-1$; T_i; a generator or its inverse). By the assumption (8), it holds, for each system,

$$\sum_{i=1}^{N-1}(R_{S_i^{(m_k^*)}})^{\mu}\geqq R^{\eta}_{S^{(m_k^*-1)}}.$$

After replacing $N-1$ circles of each system $W^1_{m_k^*}$ by a circle surrounding them, we have also a new covering of E consisting of closed discs $D^2_{S(m_1)}, \dots, D^2_{S(m_{Q'})}$, which are bounded by circles

$$(11) \qquad C^2_{S(m_1)}, \dots, C^2_{S(m_{Q'})}, \qquad (Q' < Q).$$

Then there exist in (11) some systems $\{W^2_{m_l*}\}$ which satisfy the above condition (i) and (ii) and hence, for each system of $\{W^2_{m_l*}\}$, it holds also

$$\sum_{i=1}^{N-1} (R_{S_i(m_l*)})^\mu \geqq R^\mu_{S(m_l*-1)}.$$

Repeating this procedure, we obtain the following

$$(12) \qquad \sum_{j=1}^{Q} (R_{S(m_j)})^\mu \geqq \sum_{S(m_0)} (R_{S(m_0)})^\mu,$$

where $m_0 = \min_{1 \leqq j \leqq Q} m_j$ and the summation in the right hand side is taken over all transformations in G with grade m_0. By a similar argument, we see

$$(13) \qquad \sum_{S(m_0)} (R_{S(m_0)})^\mu \geqq \sum_{S(1)} (R_{S(1)})^\mu,$$

where $\sum_{S(1)}$ denotes the sum with respect to all generators and their inverses. Here the quantity in the right hand side is a positive constant. Thus, for any covering $F^{\delta/k_0}_{n_0}$ of E, we have from (10), (12) and (13)

$$\sum_{j=1}^{Q} (r_{S(m_j)})^{\mu/2} \geqq k(G) \sum_{S(1)} (R_{S(1)})^\mu > 0.$$

Putting $\eta = \frac{\mu}{2}$ in (7), we can prove our proposition from the above inequality and Proposition 8.

Remark. In [1], we obtained the following result: given a Schottky group G, if

$$(14) \qquad \sum_{T_i} (R_{S(m)})^\mu \geqq R^\mu_{S(m-1)}, \quad (0 < \mu < 4, \ S^{(m)} = T_i S^{(m-1)}),$$

then the $\left(\frac{\mu}{2}\right)$-dimensional measure of the singular set E of G is positive.

This Theorem is valid for a Kleinian group defined in No. 2. But the process of proving Theorem contained the obscurity with respect to the relation between a covering consisting of the images of B_0 by transformations of G and radii of their isometric circles.

From the property (II) in No. 5, it is seen that $R_{S(m-1)}$ and $R_{S(m)}$ ($S^{(m)} =$

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$T_iS^{(m-1)}$) are the radius of the isometric circle $I_{S^{(m-1)}}$ and the radii of $N-1$ isometric circles $I_{S^{(m)}}$ contained in $I_{S^{(m-1)}}$. By the property (I) in No. 5, (14) is equivalent to the condition

$$\sum_{T_i^{-1}}(R_{S^{-(m)}})\geqq R^{\mu}_{S^{-(m-1)}}), \quad (S^{-(m)}=S^{-(m-1)}T_i^{-1})$$

and this is also equivalent to (8).

§ 3. Computing functions of a Kleinian group

9. Let us consider a transformation

$$S^{(m)}=S^{(m-1)}T_1=S^{(m-2)}T_2T_1, \qquad (T_2\neq T_1^{-1})$$

of a Kleinian group G, where T_1 and T_2 are generators or their inverses. Let $R_{S^{(m)}}$ and $R_{S^{(m-1)}}$ be the radii of the isometric circles of $S^{(m)}$ and $S^{(m-1)}$. Then we have from (5)

$$\left(\frac{R_{S^{(m)}}}{R_{S^{(m-1)}}}\right)^{\mu}=\frac{R^{\mu}_{T_1}}{|g'_{T_1}-g_{S^{(m-1)}}|^{\mu}}, \qquad (0<\mu<4).$$

We consider the following function

(15) $$f^{(\mu)}_{T_2}(z)=\sum_{T_1^{-1}}\frac{R^{\mu}_{T_1}}{|g_{T_1^{-1}}-z|^{\mu}}, \qquad (T_2\neq T_1^{-1})$$

for the boundary circle H_{i_1} of B_0, which is one of B_0 mapped onto H'_{i_1} by T_2, where z varies on the closed disc bounded by H_{i_1}, and the notation $\sum_{T_1^{-1}}$ denotes the summation with respect to the $N-1$ generators or their inverses except T_2. It T_2 is an elliptic transformation with period 2, then $H_{i_1}=H'_{i_1}$.

It is obvious that

$$f^{(\mu)}_{T_2}(g_{S^{(m-1)}})=\sum_{T_1^{-1}}\frac{R^{\mu}_{T_1}}{|g_{T_1^{-1}}-g_{S^{(m-1)}}|^{\mu}}. \qquad (T_2\neq T_1^{-1}),$$

where $g_{T_1^{-1}}$ does not belong to the closed disc bounded by a boundary circle H_{i_1} and $g_{S^{(m-1)}}$ is surrounded by H_{i_1}, since $g_{S^{(m-1)}}=S^{-(m-1)}(\infty)$, ($S^{-(m-1)}=T_2^{-1}S^{-(m-2)}$, $\infty\in B_0$) and $S^{-(m-1)}(B_0)$ is contained in the domain bounded by H_{i_1}. We call $f^{(\mu)}_{T_2}(z)$ the μ-dimensional computing function of T_2 and there exist N computing functions $f^{(\mu)}_{T_\nu}(z)$, ($\nu=1,\ldots,N$) in all, since the first element T_2 of $S^{(m-1)}$ is any generator or its inverse of G. Such functions $\{f^{(\mu)}_{T_\nu}(z)\}$ ($\nu=1,\ldots,N$) are called the μ-dimensional computing functions of a Kleinian group

G.

10. Take a generator T_1 and its μ-dimensional computing function

$$f_{T_1}^{(\mu)}(z) = \sum_{T_\nu \neq T_1} \frac{R_{T_\nu}^{\mu}}{|g_{T_\nu} - z|^{\mu}},$$

where the notation $\sum_{T_\nu \neq T_1}$ denotes the summation with respect to the $N-1$ generators or their inverses T_ν except T_1. Then $f_{T_1}^{(\mu)}(z)$ is defined in the closed disc D_{T_1} : $|z - a_{t_1}| \leqq r_{t_1}$ bounded by H_{t_1} which is a boundary circle of B_0 mapped onto H'_{t_1} by T_1. Since any center g_{T_ν} of the isometric circles of generators or their inverses T_ν $(T_\nu \neq T_1)$ is a pole of T_ν, it is in the outside of D_{T_1} and hence the denominator of each term of $f_{T_1}^{(\mu)}(z)$ does not vanish. From this fact $f_{T_1}^{(\mu)}(z)$ is uniformly continuous in D_{T_1}. Then we can choose δ depending only on any small ε, so that it holds $|f_{T_1}^{(\mu)}(z) - f_{T_1}^{(\mu)}(z')| < \varepsilon$ for z and z' satisfying $|z - z'| < \delta$ in D_{T_1}. Denote by E_1 the subset of E contained in D_{T_1}. Since, from Proposition 7, any radius $r^{(m)}$ of circles of grade m is equal or less than $K_0^{m-1} r^{(1)} (K_0 < 1)$, which tends to zero for $m \to \infty$, there exists a grade number m_0 depending only on δ so that for any $S^{(m)} = S^{(m-1)} T_1$, $m \geqq m_0$, there is $z_0 \in E_1$ such that $g_{S^{(m)}} \in D_\delta(z_0)$, where $D_\delta(z_0)$ denotes the disc with center z_0 and with radius δ. Hence it can be seen that

$$(16) \qquad |f_{T_1}^{(\mu)}(z_0) - f_{T_1}^{(\mu)}(g_{S^{(m)}})| < \varepsilon, \qquad (\forall m \geqq m_0,\ S^{(m)} = S^{(m_0)} S^{(m')}).$$

Suppose that it holds

$$(17) \qquad f_{T_1}^{(\mu)}(z) > \lambda_1, \qquad \text{for any } z \in E_1$$

Then we have from (16) and (17)

$$|f_{T_1}^{(\mu)}(g_{S^{(m)}})| \geqq |f_{T_1}^{(\mu)}(z_0)| - |f_{T_1}^{(\mu)}(g_{S^{(m)}}) - f_{T_1}^{(\mu)}(z_0)| > \lambda_1 - \varepsilon.$$

Now we have the following

THEOREM 1. *Let G be a Kleinian group whose fundamental domain is bounded by N boundary circles as in* No. 2. *If it holds that*

$$(18) \qquad f_{T_1}^{(\mu)}(z) > \lambda_i > 1, \qquad (i = 1, \ldots, N)$$

on the singlar subset E_i of E contained in the boundary circle H_i $(i = 1, \ldots, N)$ of B_0 respectively, then the singular set E of G has the positive $\left(\frac{\mu}{2}\right)$-dimensional measure.

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Proof. For any i, take ε so small that it may hold $\lambda_i - \varepsilon > 1$ $(i = 1, \ldots, N)$. Then we can determine the grade number m_0 such that the inequalities

$$f_{T_i}^{(\mu)}(g_{S^{(m)}}) > \lambda_i - \varepsilon > 1, \qquad (m \geqq m_0\ ;\ i = 1, \ldots, N)$$

hold. Hence we have the following inequalities

$$\sum_{T_1} (R_{S^{(m)}})^{\mu} \geqq R^{\mu}_{S^{(m-1)}}, \qquad (S^{(m)} = S^{(m-1)}T_1 = S^{(m-2)}T_2T_1,\ T_2^{-1} \neq T_1),$$

for radius $R_{S^{(m-1)}}$ of any isometric circle $I_{S^{(m-1)}}$ of grade $m-1$ and radii $R_{S^{(m)}}$ of the $N-1$ isometric circles $I_{S^{(m)}}$ of grade m. Thus, by Proposition 9, we get the theorem. q.e.d.

Noting Proposition 4, we get the following

Corollary. *If the condition* (18) *is satisfied for the computing functions of the Kleinian group, then the* $(-\mu)$*-dimensional P-series* $P_\mu(z)$ *does not converge in* D''.

§ 4. Examples of Kleinian groups whose singular sets have positive 1-dimensional measure

11. P. J. Myrberg [3] treated also the convergence problem of the (-2)-dimensional Poincaré theta-series $\Theta_2(z)$ with respect to Schottky groups and Kleinian groups and gave the examples in which $\Theta_2(z)$ does not converge. But in his paper it was not treated from the view point of the measure of the singular sets of such groups.
In this chapter, by using the condition (for $\mu = 2$) of Theorem 1, we shall give, more systematically, the examples in which the singular sets of Kleinian groups have positive 1-dimensional measure and $\Theta_2(z)$ does not converge in D^*. Further we shall try to make the number N of the boundary circles as small as possible.

As the preliminary to give examples, we shall show how to construct a transformation T which maps the outside of a circle H onto the inside of another circle H', where H and H' have equal radii, though in general we can set up infinitely many such transformations.

Denote two circles by

$$H : |z - q| = r, \ H' : |z - q'| = r.$$

It T is restricted by the conditions: $q' = T(\infty)$ and $q = T^{-1}(\infty)$, it is easily seen that T has the following form

$$z' = T(z) = \frac{q'z - (qq' + r^2 e^{i\theta})}{z - q}, \tag{19}$$

where θ is any real number and the isometric circles I_T and I'_T are H and H', respectively.

12. Example The case of $N = 5$.

Consider the four circles H_j $(j = 1, 2, 3, 4)$ with centers $a_j = \sqrt{2}\, e^{i(2j-1)\pi/4}$ $(i^2 = -1)$ and equal radii $R = 1 - \varepsilon$, respectively. If we let these four circles correspond to two hyperbolic transformations S_1 and S_2 by (19) such that the outside of H_1 is mapped onto the inside of H_3 by S_1 and the outside of H_2 is mapped onto the inside of H_4 by S_2, we obtain a Fuchsian group G_1 of the second kind with the fixed circle $|z| = 1 + \varepsilon_1$. Next we describe a circle H_5 with center at the origin and radius $r = \sqrt{2} - 1$ and let it correspond to an elliptic transformation S_5 with period 2.

Combining Fuchsian group G_1 with the group G_2 generated by only S_5, we obtain a Kleinian group G, that is, a combination group $G_1 \cdot G_2$, whose fundamental domain B_0 is connected and bounded by five circles H_j $(j = 1, 2, 3, 4, 5)$.

For convenience of calculation, we may consider the limit case $\varepsilon = 0$. In this case B_0 is no more connected. Then the fixed circle of G_1 is $|z| = 1$. Denote by D_j $(j = 1, 2, 3, 4, 5)$ the closed discs bounded by H_j $(j = 1, 2, 3, 4, 5)$ and U the closed unit disc. The singular set E of G is contained in the domain $\bigcup_{j=1}^{5} \{D_j \cap U\}$. It can be seen from (21) that the generating transformations S_j $(j = 1, 2)$ and S_5 have the following forms:

$$\tag{20} \begin{aligned} S_j &= \frac{-\sqrt{2}\, z + e^{i(2j-1)\pi/4}}{e^{-i(2j-1)\pi/4} z - \sqrt{2}}, \quad S_j^{-1} = S_{j+2}, \qquad (j = 1, 2) \\ S_5 &= \frac{r^2 e^{i\theta}}{z}. \qquad (\theta;\ {}^{\forall}\text{real number},\ r = \sqrt{2} - 1). \end{aligned}$$

By virtue of symmetricity of the figure, it is sufficient to calculate the values of the computing functions $f_{S_5}^{(2)}(z)$ in D_5 and $f_{S_1}^{(2)}(z)$ in $U \cap D_1$. Since the centers and radii of the isometric circles I_{S_j}, I'_{S_j} $(j = 1, 2)$ and I_{S_5} are easily known from (20), the values of $f_{S_1}^{(2)}(z)$ and $f_{S_5}^{(2)}(z)$ in the above restricted domains can be calculated as follows.

(I) Case of $f_{S_5}^{(2)}(z)$.

It holds that in D_5

$$f_{S_5}^{(2)}(z) = \sum_{j=1}^{4} \frac{1}{|z-g_j|^2},$$

where g_j coincides with the center a_j of H_j. Since $|z-g_j| \leqq 2\sqrt{2}-1$ in D_5, we have

$$f_{S_5}^{(2)}(z) > \frac{4}{(2\sqrt{2}-1)^2} = \frac{4(9+4\sqrt{2})}{49} > 1.$$

(II) Case of $f_{S_1}^{(2)}(z)$.

It holds that in $U \cap D_1$

$$f_{S_1}^{(2)}(z) = \sum_{j=2}^{4} \frac{1}{|z-g_j|^2} + \frac{(\sqrt{2}-1)^2}{|z|^2}.$$

Considering

$$\frac{1}{|z-g_2|^2} + \frac{1}{|z-g_4|^2},$$

we see easily that it attains the minimum at $c_1 = (1+i)/\sqrt{2}$ in $U \cap U_1$. Since $|c_1 - g_4|^2 = (\sqrt{2})^2 + 1 = 3$, the above sum is equal or greater than 2/3. Since $1/|z-g_3|^2$ and $(\sqrt{2}-1)^2/|z|^2$ attain also the minimum at c_1, it holds

$$f_{S_1}^{(2)}(z) \geqq \frac{2}{3} + 2(\sqrt{2}-1)^2 > 1.$$

So taking ε sufficiently small, we conclude that in this example all functions $f_{S_j}^{(2)}(z)$ $(j=1, 2, 3, 4, 5)$ are greater than 1 in the singular set contained in D_j respectively. Hence for the case of $\mu=2$ the condition of Theorem 1 is satisfied and the 1-dimensional measure of the singular set of such a Kleinian group is positive. Thus the (-2)-dimensional Poincaré theta-series $\Theta_2(z)$ does not converge in D^* for such group.

Remark. We can easily see that for sufficiently small δ the $(1+\delta)$-dimensional measure of the singular set of the above Kleinian group is positive. Even in the case of $N=5$, there exist Kleinian groups whose singular sets do not belong to Painlevé null sets.

By Example we obtain the following theorem.

Theorem 2. *Under Kleinian groups whose fundamental domains are bounded by mutually disjoint N $(\geqq 5)$ circles, there exist ones with respect to which the*

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(-2)*-dimensional Poincaré theta-series* $\Theta_2(z)$ *does not converge in* D^* *and whose singular sets have positve* 1*-dimensional measure.*

13. Let us consider an application of Theorems 1 and 2 to Schottky groups.

Consider the totality G^* formed by the elements of even grade of a Kleinian group G. We can easily prove that G^* is a Schottky group independent of the sort of the generators.

Take any generator S_{i_0} of G. Then any element $S_{i_2}\cdot S_{i_1}$ of grade 2 of G is represented by the form

$$S_{i_2}S_{i_0}^{-1}\cdot S_{i_0}S_{i_1}.$$

Therefore any element of G^* is generated by

$$T_i = S_{i_0}S_i, \qquad T_i^{-1} = (S_i)^{-1}S_{i_0}^{-1} = (S_{i_0}\cdot S_i)^{-1}, \tag{21}$$

where S_i runs in $N-1$ generators except $S_{i_0}^{-1}$. We see that G^* is a subgroup of G generated by $2N-2$ generators and their inverses (21). Since S_{i_0} is any generator, there are N ways about the determination of the generators of G^*. Though the generators of G may contain an elliptic transformation with period 2, the generators of G^* are all hyperbolic or loxodromic transformations. Because, since the generator T_i maps the boundary circle K_i onto a circle of grade 2 in the boundary circle $K_{i_0}^{-1}$ $(\neq K_i)$, onto which S_{i_0} maps the boundary circle K_{i_0}, so the circles K_i and $T_i(K_i)$ are mutually disjoint and hence T_i is a hyperbolic or a loxodromic transformation. It is easily seen that the fundamental domain of G^* is $B_0 + S_{i_0}(B_0)$. We call such a method, which forms Schottky subgroup from Kleinian group, the inversion method.

14. With respect to a Schottky subgroup G^* of G given by inversion method, we have the following

THEOREM 3. *If the condition* (18) *of Theorem* 1 *is valid for a Kleinian group* G, *then the singular set of a Schottky subgroup* G^* *of* G *has the positive* $\left(\frac{\mu}{2}\right)$*-dimensional measure, and the* $(-\mu)$*-dimensional P-series* $P_\mu(z)$ *with respect to* G^* *does not converge in* D''.

Proof. From the assumption of Theorem, it is clear that the singular set E of G has positive $\left(\frac{\mu}{2}\right)$-dimensional measure. Since it can be seen that the singular set E^* of G^* coincides with E from the definition of the singular set,

E^* has also positive $\left(\frac{\mu}{2}\right)$-dimensional measure. q.e.d.

15. In Example, take a generator S_5 and form a system of generators and their inverses:

$$T_i = S_5 S_i, \quad T_i^{-1} = S_i^{-1} S_5^{-1} \qquad (i = 1, 2, 3, 4). \tag{22}$$

Then (22) generate a Schottky subgroup G^* of the Kleinian group G. Then from Theorems 2 and 3, we obtain the following

THEOREM 4. *Let G be a Kleinian group whose fundamental domain is bounded by N circles. Under Schottky subgroups given by inversion method from G, if $N = 5$, there exist Schottky groups whose fundamental domains are bounded by* 8 *boundary circles and whose singular sets have positive* 1-*dimensional measure.*

16. PROBLEM. The Schottky's condition [7] implies that $l_m^{(2)}$ tends to zero for $m \to \infty$ in Proposition 3. Hence the 1-dimensional measure of the singular set of any Kleinian group with fundamental domain B_0 bounded by three mutually disjoint circles is always zero, since B_0 satisfies the Schottky's condition.

Then the 1-dimensional measure of the singular set of a Schottky subgroup with fundamental domain bounded by four circles, which is given from the above Kleinian group by inversion method, is also zero. But it remains to be proved whether there exist or not Kleinian groups with fundamental domains bounded by 4 circles whose singular sets have positive 1-dimensional measure.

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