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PROPAGATION RELATIONS FOR SOLUTIONS OF SOME HIGHER ORDER CAUCHY PROBLEMS

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Abstract

The Huygens' property is exploited to study propagation relations for solutions of certain types of linear higher order Cauchy problems. Motivated by the solution properties of the abstract wave problem, addition formulas are developed for the solution operators of these problems. The application of these alternative forms of the solution operators to data leads to connecting operator relations between distinct solutions of the problems at different times. We examine this solution behaviour for both analytic and abstract Cauchy problems. A basic algorithm for constructing addition formulas for solutions of ordinary differential equations is included.

1. Introduction

The Huygens' property plays an important role in studying the behaviour of solutions of Cauchy problems. In the development of series representations of classical heat functions in terms of heat polynomials and associated functions, Rosenbloom and Widder [6] made use of this property. For the standard one space variable wave problem, the Huygens' property is reflected in propagations along the characteristics. To obtain other implications of this property, we consider the abstract wave problem

$$w''(t) = Aw(t); \quad w(0) = \varphi_1, w'(0) = \varphi_2 \tag{1.1}$$

where $A = B^2$ with B the generator of a continuous group $G_B(t)$ in a Banach space X and where $\varphi_j \in D(B^2)$, $j = 1, 2$. We can write the solution of this in the form

$$w(t) = O_1(t)\varphi_1 + O_2(t)\varphi_2 \tag{1.2}$$

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where $O_1(t)$ and $O_2(t)$ are formal solution operators defined by the relations

$$O_1(t)\varphi_1 = \cosh(tB)\varphi_1 = \{G_B(t)\varphi_1 + G_B(-t)\varphi_1\}/2$$
$$O_2(t)\varphi_2 = \int_0^t O_1(\sigma)\varphi_2\,d\sigma. \tag{1.3}$$

One can express these evaluations in terms of solutions of related heat problems by means of transmutations [2]–[4] (also, see Goldstein [5] for the development of the cosine operator and its uses for solving wave problems). Now, let $w_1(t)$ denote the solution (1.2) corresponding to $\varphi_1 = \varphi$, $\varphi_2 = 0$ and let $w_2(t)$ denote the solution (1.2) corresponding to $\varphi_1 = 0$, $\varphi_2 = \varphi$. Then $w_2'(t) = w_1(t)$. Using the addition formulas for the hyperbolic cosine and sine functions, we find $O_1(t_1 \pm t_2) = O_1(t_2)O_1(t_1) \pm AO_2(t_1)O_2(t_2)$ and $O_2(t_1 \pm t_2) = O_2(t_1)O_1(t_2) \pm O_1(t_1)O_2(t_2)$. If we now apply these to the data function φ, we obtain the formulas

$$w_1(t_1 \pm t_2) = O_1(t_2)w_1(t_1) \pm AO_2(t_2)w_2(t_1)$$
$$w_2(t_1 \pm t_2) = O_1(t_2)w_2(t_1) \pm O_2(t_2)w_1(t_1). \tag{1.4}$$

Now, take $t_1 = t_2 = t > 0$ in the first of these. If we then add and subtract the resulting relations, it follows that

$$w_1(2t) + \varphi = 2O_1(t)w_1(t) \quad \text{and} \quad w_1(2t) - \varphi = 2AO_2(t)w_2(t). \tag{1.5}$$

Similarly, using (1.4b), we can show that

$$w_2(2t) = 2O_1(t)w_2(t) = 2O_2(t)w_1(t). \tag{1.6}$$

These solution propagation relations are easily verified for the one space variable wave problem by calling upon the d'Alembert formulas. For wave problems with a larger number of space variables, (1.5) and (1.6) yield multiple integral solution relationships. By making other selections for t_1 and t_2 in (1.4), we can obtain additional relations among the functions $w_1(t)$ and $w_2(t)$. In these developments, the role of the Huygens' property is taken on by the addition formulas for the formal solution operators.

In this paper, we consider the problem of developing relations analogous to (1.4)-(1.6) for certain classes of solvable Cauchy problems of order greater than 2. One class of these of particular interest involves a partial differential equation of the form

$$u^{(n)}(x,t) = \lambda_1 P(D)u^{(n-1)}(x,t) + \lambda_2 P^2(D)u^{(n-2)}(x,t) + \cdots$$
$$+ \lambda_n P^n(D)u(x,t). \tag{1.7}$$

In this, the λ_j are real constants, $x = (x_1, \ldots, x_m)$, $D = (D_1, D_2, \ldots, D_m)$ with $D_j\varphi(x) = \partial\varphi(x)/\partial x_j$, $P(D)$ is a linear partial differential operator in the D_j with constant coefficients and $u^{(j)}(x,t) = \partial^j u(x,t)/\partial t^j$. Let $u_k(x,t)$ denote the solution of this equation corresponding to the initial conditions $u_k^{(j)}(x,0) = \varphi(x)$ if $j = k-1$ and 0 otherwise for $k = 1, 2, \cdots, n$. Entireness conditions on $\varphi(x)$ will be stated later. With (1.7), we associate the ordinary differential equation

$$y^{(n)}(t) = \lambda_1 y^{(n-1)}(t) + \lambda_2 y^{(n-2)}(t) + \cdots + \lambda_n y(t). \tag{1.8}$$

Let $y_k(t)$ denote the solution of this corresponding to the conditions $y_k^{(j)}(0) = 1$ if $j = k-1$ and 0 otherwise. Let $O_k(t) = y_k(tP(D))$. Then the function $u_k(x,t)$ can be expressed as $u_k(x,t) = O_k(t)\varphi(x)$ and can be determined by using quasi inner products if $\varphi(x)$ has suitable growth. To develop propagation relations among the $u_k(x,t)$, we require addition formulas for the solution operators $O_k(t)$ and, hence, addition formulas for the $y_k(t)$.

In Section 2 to follow, we outline the steps leading to the simple algorithm for deriving the addition formulas for the solution functions $y_k(t)$ of (1.8) satisfying the conditions $y_k^{(j)}(0) = 1$ for $j = k-1$ and 0 otherwise, $j = 0, 1, \cdots, n-1$. Associated with each y_k is an $n \times n$ "addition formula" matrix. An addition formula for the expression $y_k(t_1 - t_2)$ is also easy to write down if the function $y_k(t)$ is strictly even or odd in t. If $y_k(t)$ is neither even nor odd, an expression for $y_k(t_1 - t_2)$ can be obtained in terms of the even and odd parts of y_k but it is inconvenient to use in connection with Cauchy problems involving equations such as (1.7). The results of Section 2 will then be applied in Sections 3 and 4 to develop propagation relations among the solution functions $u_k(x,t)$ associated with special equations of the form (1.7). Finally, in Section 5, we consider a third order abstract Cauchy problem.

2. The addition formula algorithm

For $k = 1, 2, \cdots, n$, let $y_k(t)$ denote a solution of (1.8) such that $y_k^{(j)}(0) = 1$ for $j = k-1$ and 0 otherwise, $j = 0, 1, \cdots, n-1$. Let $y(t)$ be any solution of (1.8). Using linear independence, the function $y(t+\tau)$ can be expressed as a linear combination of the $y_k(t)$. But since $y(t+\tau)$ is symmetric in t and τ, it can also be expressed as a linear combination of the $y_k(\tau)$. From this it follows that

$$y(t+\tau) = Y(t) \cdot M \cdot Y^{\mathrm{T}}(\tau) \tag{2.1}$$

in which $Y(t)$ is the row vector $(y_1(t), y_2(t), \cdots, y_n(t))$, $Y^{\mathrm{T}}(t)$ is the transpose of $Y(t)$, and M is an $n \times n$ matrix of constants. We need to determine the entries m_{ij} of this matrix M. But, by symmetry,

$$y^{(\ell)}(t+\tau) = \frac{\partial^p}{\partial t^p}\left(\frac{\partial^q}{\partial \tau^q} y(t+\tau)\right) \tag{2.2}$$

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for $p+q=\ell$ with $p\geq 0$, $q\geq 0$. This in turn implies that

$$y^{(\ell)}(t+\tau)=(Y(t))^{(p)}\cdot M\cdot(Y^{\mathrm{T}}(\tau))^{(q)}. \tag{2.3}$$

At $t=0$, $\tau=0$, and using the definitions of the $y_k(t)$, we find

$$m_{pq}=y^{(p+q)}(0) \quad \text{for} \quad 0\leq p+q\leq 2n-2. \tag{2.4}$$

ALGORITHM 2.1. *Let $y(t)$ be any solution of the differential equation (1.8). Then the function $y(t+\tau)$ satisfies (2.1) with*

$$M=\begin{bmatrix} y(0) & y'(0) & \cdots & \cdots & y^{(n-1)}(0) \\ y'(0) & y''(0) & \cdots & \cdots & y^{(n)}(0) \\ y''(0) & y'''(0) & \cdots & \cdots & y^{(n+1)}(0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y^{(n-1)}(0) & y^{(n)}(0) & \cdots & \cdots & y^{(2n-2)}(0) \end{bmatrix}. \tag{2.5}$$

It should be noted that the function $y(t+\tau)$ defined by (2.1) does indeed satisfy (1.8) with t replaced by $t+\tau$.

Thus, we see that obtaining the addition formulas for the basic solutions $y_k(t)$ of (1.8) reduces to computing the derivative entries in the matrix (2.5) and this can be accomplished by invoking the initial conditions on the $y_k(t)$ and repeatedly differentiating and evaluating terms in (1.8). Let us construct the matrix M for the solution function $y_1(t)$ of (1.8). Now the initial conditions give $y_1(0)=1$ and $y_1^{(k)}(0)=0$ for $k=1,2,\cdots,n-1$. From (1.8), we have $y_1^{(n)}(0)=\lambda_n$. If we repeatedly differentiate the two sides of (1.1) $n-2$ times, it is easy to show that

$$y_1^{(n+j)}(0)=\lambda_1 y_1^{(n+j-1)}(0)+\cdots+\lambda_j y_1^{(n)}(0), \quad j=1,2,\cdots,n-2. \tag{2.6}$$

It readily follows from these relations that

$$y_1^{(n+1)}(0)=\lambda_1\lambda_n, \quad y_1^{(n+2)}(0)=(\lambda_1^2+\lambda_2)\lambda_n, \quad y_1^{(n+3)}(0)=(\lambda_1^3+2\lambda_1\lambda_2+\lambda_3)\lambda_N,$$

and, in general,

$$y_1^{(n+j)}(0)=\lambda_n\cdot P_j(\lambda_1,\lambda_2,\ldots,\lambda_j)$$

where $P_j(\lambda_1,\lambda_2,\ldots,\lambda_j)$ is a generating polynomial for all partitions of j. Each term in this polynomial is the product of a numerical coefficient by powers of the λ_i. The sum of the products of the exponents of the λ_i by the i is equal to j and the numerical coefficient represents the number of different possible rearrangements of the λ_i in that term. This polynomial includes all such terms. We can now fill in the entries of (2.5) to obtain the matrix M_1. The reader can construct analogous combinatoric type formulas for the entries in the matrices M corresponding to the addition formulas for the functions $y_2(t), \cdots, y_n(t)$. In the applications, special choices of the λ_j are taken in (1.7) in order that the addition matrices have relatively simple forms.

The formulas for the corresponding $y_i(t-\tau)$ can have more complicated structures depending upon whether (1.8) contains only even order derivative terms or a mixture of odd and even order derivative terms. Now we can use the decomposition $Y^{\mathsf{T}}(t) = Y_E^{\mathsf{T}}(t) + Y_O^{\mathsf{T}}(t)$ of $Y^{\mathsf{T}}(t)$ into even and odd parts to obtain $Y^{\mathsf{T}}(-t) = Y_E^{\mathsf{T}}(t) - Y_O^{\mathsf{T}}(t) = Y^{\mathsf{T}}(t) - 2Y_O^{\mathsf{T}}(t)$ and, hence,

$$\begin{aligned} y_i(t-\tau) &= Y(t)MY_E^{\mathsf{T}}(\tau) - Y(t)MY_O^{\mathsf{T}}(\tau) \\ &= Y(t)MY^{\mathsf{T}}(\tau) - 2Y(t)MY_O^{\mathsf{T}}(\tau). \end{aligned} \tag{2.7}$$

One can determine the evenness or oddness of the $y_j(t)$ functions from (1.8) and the initial conditions that they satisfy.

3. Cauchy problems I

In this section, we take n to be even and consider obtaining propagation relations among the solutions of the set of Cauchy problems

$$\begin{aligned} &u_j^{(n)}(x,t) = P^n(D)u(x,t), \\ &u_j^{(k)}(x,0) = \varphi(x) \text{ if } k = j-1 \text{ and 0 otherwise, for } j = 1,2,\ldots,n. \end{aligned} \tag{3.1}$$

Associated with these are the ordinary differential equation problems

$$\begin{aligned} &y_j^{(n)}(t) = y_j(t), \\ &y_j^{(k)}(0) = 1 \text{ if } k = j-1 \text{ and 0 otherwise, for } j = 1,2,\ldots,n. \end{aligned} \tag{3.2}$$

By employing Algorithm 2.1, one can show, with effort, that the addition matrices are given by $M_j = M_1C^{j-1}$, $j = 1,2,\ldots,n$ where

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{3.3}$$

Thus, we have the addition formulas

$$y_j(t+\tau) = Y(t)M_1C^{j-1}Y^{\mathsf{T}}(\tau), \quad j = 1,2,\ldots,n. \tag{3.4}$$

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Upon completing the multiplication of the matrices and vectors in this, we obtain the following scalar forms for these addition formulas:

$$y_i(t+\tau) = \sum_{j=1}^{i} y_j(t)y_{i+1-j}(\tau) + \sum_{j=i+1}^{n} y_j(t)y_{n+i+1-j}(\tau), \quad i = 1, 2, \dots, n. \quad (3.5)$$

Double "angle" formulas for the $y_j(2t)$ follow from these by replacing τ in these by t.

We next obtain formulas for the $y_i(t-\tau)$. From the initial conditions on the $y_i(t)$, it is easy to see that the $y_i(t)$ with odd subscripts are even functions and the $y_i(t)$ with even subscripts are odd functions. Thus, $y_i(-t) = (-1)^{i-1}y_i(t)$, $i = 1, 2, \dots, n$. From this it follows that the formulas for the $y_i(t-\tau)$ become

$$y_i(t-\tau) = \sum_{j=1}^{i}(-1)^{i-j} y_j(t)y_{i+1-j}(\tau) + \sum_{j=i+1}^{n}(-1)^{n+i-j} y_j(t)y_{n+i+1-j}(\tau). \quad (3.6)$$

When $i = 1$ and $\tau = t$, this becomes

$$1 = y_1(0) = y_1^2(t) + \sum_{j=2}^{n}(-1)^{n+1-j} y_j(t)y_{n+2-j}(t) \quad (3.7)$$

and for $i = 3, 5, \dots, n-1$,

$$0 = y_i(0) = \sum_{j=1}^{i}(-1)^{i-j} y_j(t)y_{i+1-j}(t) + \sum_{j=i+1}^{n}(-1)^{n+1-j} y_j(t)y_{n+i+1-j}(t) \quad (3.8)$$

(for i even, these relations are trivial since the terms in their last members cancel pairwise). If we think of the $y_i(t)$ as the ith coordinate of a point $(y_1, y_2, \dots, y_n)$ in n space, then the vector $Y(t)$ describes a manifold determined by the quadratic relations in (3.7) and (3.8). (When $n = 2$, this is the hyperbola $y_1^2(t) - y_2^2(t) = 1$.)

Observe that we have deduced the above relations without knowing the explicit formulas of the functions $y_i(t)$. We leave it to the reader, using series methods, to show that

$$y_i(t) = t^{i-1} \sum_{k=0}^{\infty} \frac{t^{nk}}{(nk+i-1)!}. \quad (3.9)$$

These can also be expressed as

$$y_j(t) = t^{j-1}{}_0F_{n-1}(-; \frac{j}{n}, \frac{j+1}{n}, \dots, \frac{n-1}{n}, \frac{n+1}{n}, \dots, \frac{n+j-1}{n}; \frac{t^n}{n^n}).$$

The double angle formulas obtained from (3.4) and (3.7) and (3.8) then yield identities among these ${}_0F_{n-1}$ hypergeometric functions.

With the availability of these $y_j(t)$ and their addition formulas, we can now return to the Cauchy problems (3.1). As was noted in Section 1, the solutions of these are given formally by

$$u_j(x,t) = O_j(t)\varphi \tag{3.10}$$

where $O_j(t) = y_j(tP(D))$. In order that this be well defined, let $\varphi(z) = \varphi(z_1, \cdots, z_n)$ be entire of growth < 1 in each z_j, i.e. $|\varphi(z_1, \cdots, z_n)| \le M \exp\left(\sum_{j=1}^n \tau_j |z_j|^{\rho_j}\right)$ where $\rho_j < 1$ for all j. Using the quasi inner product approach of [1], we can rewrite (3.10) in the form

$$u_j(x,t) = \int_0^\infty e^{-\sigma}\left\{y_j(\underline{t}) \circ H(x,\underline{\sigma})\right\} d\sigma \tag{3.11}$$

and these u_j are defined for all real value vectors x and real t. We note that $\partial u_j(x,t)/\partial t = u_{j-1}(x,t)$ for $j = 2, 3, \ldots, n$. Upon replacing the addition formulas (3.5) by the corresponding solution operator formulas for (3.1) and applying them to $\varphi(x)$, we obtain:

THEOREM 3.1. *For all real t and τ, the solution propagation relations for the problems (3.1) are given by the formulas*

$$\begin{aligned} u_i(x,t+\tau) = \sum_{j=1}^{i} O_{i+1-j}(\tau) u_j(x,t) + \sum_{j=i+1}^{n} O_{n+i+1-j}(\tau) u_j(x,t),& \\ i = 1, 2, \ldots, n.& \end{aligned} \tag{3.12}$$

The corresponding formulas for the $u_j(t-\tau)$ follow by replacing τ in these by $-\tau$ and then using the fact that $O_j(-\tau) = (-1)^{j-1} O_j(\tau)$. Making use of this fact for the formula in (3.12) corresponding to $i = 1$, we also obtain:

THEOREM 3.2. *Double "angle" relations satisfied by a solution $u_1(x,t)$ of (3.1) are defined by the formulas*

$$\begin{aligned} u_1(x,2t) + \varphi(x) &= 2\left\{O_1(t)u_1(x,t) + \sum_{k=1}^{(n-2)/2} O_{n-2k+1}(t) u_{2k+1}(x,t)\right\}, \\ u_1(x,2t) - \varphi(x) &= 2\sum_{k=1}^{n/2} O_{n-2k+2}(t) u_{2k}(x,t). \end{aligned} \tag{3.13}$$

These are higher order versions of the formulas of type (1.5). One can construct formulas analogous to these for the functions $u_2(x,t), \ldots, u_n(x,t)$. For example, suppose that i is odd with $1 < i \le n-1$. Since $u_i(x, t-t) = 0$, we can add the second of the formulas in (3.13) with $\tau = t$ and $\tau = -t$ to obtain:

COROLLARY 3.3.

$$u_i(x,2t) = 2\sum_{k=0}^{(i-1)/2} O_{i-2k}(t)u_{2k+1}(x,t) + 2\sum_{k=(i+1)/2}^{(n-2)/2} O_{n+i-2k}(t)u_{2k}(x,t).$$

We leave the reader to construct similar formulas for the other $u_j(x,t)$.

4. Cauchy problems II

In order to treat cases of (1.7) with a second term in the right hand member, let us consider the set of Cauchy problems

$$\begin{aligned} &u_j^{(4)}(x,t) = \lambda P^2(D)u_j^{(2)}(x,t) + \mu P^4(D)u_j(x,t),\\ &u_j^{(k)}(x,0) = \varphi(x) \text{ if } k = j-1 \text{ and } 0 \text{ otherwise, for } j = 1,2,3,4. \end{aligned} \tag{4.1}$$

In these, x is a single space variable and λ and μ are positive parameters. Associated with this is the set of initial value problems in ordinary differential equations

$$\begin{aligned} &y_j^{(4)}(t) = \lambda y_j^{(2)}(t) + \mu y_j(t),\\ &y_j^{(k)}(0) = 1 \text{ if } k = j-1 \text{ and } 0 \text{ otherwise.} \end{aligned} \tag{4.2}$$

Now the characteristic roots of the equation in this are given by $r_2 = -r_1$, $r_1 = \sqrt{\left(\lambda + \sqrt{\lambda^2+4\mu}\right)/2}$, $r_3 = i\sqrt{\left(\sqrt{\lambda^2+4\mu} - \lambda\right)/2}$ and $r_4 = -r_3$. Moreover, applying Algorithm 2.1, we can show that the addition matrices associated with the problems (4.2) are given by

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \mu\\ 0 & 0 & \mu & 0\\ 0 & \mu & 0 & \lambda\mu \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \mu\\ 0 & 0 & \mu & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 1 & 0 & \lambda\\ 1 & 0 & \lambda & 0\\ 0 & \lambda & 0 & \lambda^2+\mu \end{bmatrix} \quad \text{and} \quad M_4 = \begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & \lambda\\ 1 & 0 & \lambda & 0 \end{bmatrix}. \tag{4.3}$$

Now the functions $y_1(t)$ and $y_3(t)$ are even while $y_2(t)$ and $y_4(t)$ are odd. Then it follows from this and the M_j matrices that

$$\begin{aligned}
y_1(t \pm \tau) &= y_1(t)y_1(\tau) \pm \mu(y_4(t)y_2(\tau) + y_2(t)y_4(\tau)) \\
&\quad + \mu y_3(t)y_3(\tau) \pm \lambda\mu y_4(t)y_4(\tau), \\
y_2(t \pm \tau) &= [\pm y_1(t)y_2(\tau) + y_1(\tau)y_2(t)] + \mu[\pm y_3(t)y_4(\tau) + y_3(\tau)y_4(t)], \\
y_3(t \pm \tau) &= [y_1(t)y_3(\tau) + y_1(\tau)y_3(t)] \pm y_2(t)y_2(\tau) \pm \lambda[y_2(t)y_4(\tau) + y_2(\tau)y_4(t)] \\
&\quad + \lambda y_3(t)y_3(\tau) \pm (\lambda^2 + \mu)y_4(t)y_4(\tau), \\
y_4(t \pm \tau) &= [\pm y_1(t)y_4(\tau) + y_4(t)y_1(\tau)] + [y_2(t)y_3(\tau) \pm y_2(\tau)y_3(t)] \\
&\quad + \lambda[\pm y_3(t)y_4(\tau) + y_3(\tau)y_4(t)]. \qquad (4.4)
\end{aligned}$$

Then the "double angle" formulas for these are

$$\begin{aligned}
y_1(2t) &= y_1^2(t) + 2\mu y_2(t)y_4(t) + \mu y_3^2(t) + \lambda\mu y_4^2(t), \\
y_2(2t) &= 2y_1(t)y_2(t) + 2\mu y_3(t)y_4(t), \\
y_3(2t) &= 2y_1(t)y_3(t) + y_2^2(t) + 2\lambda y_2(t)y_4(t) + \lambda y_3^2(x) + (\lambda^2 + \mu)y_4^2(t), \\
y_4(2t) &= 2y_1(t)y_4(t) + 2y_2(t)y_3(t) + 2\lambda y_3(t)y_4(t). \qquad (4.5)
\end{aligned}$$

Finally, taking $\tau = t$ in the formulas (4.4) corresponding to the minus signs, we obtain the pair of identities

$$\begin{gathered}
y_1^2(t) - 2\mu y_2(t)y_4(t) + \mu y_3^2(t) - \lambda\mu y_4^2(t) = 1, \\
2y_1(t)y_3(t) - 2\lambda y_2(t)y_4(t) - y_2^2(t) + \lambda y_3^2(t) - (\lambda^2 + \mu)y_4^2(t) = 0. \qquad (4.6)
\end{gathered}$$

Once again, the solutions of the problems (4.1) can be expressed in the form $u_j(x,t) = O_j(t)\varphi(x) = y_j(tP(D))\varphi(x)$ for $j = 1,2,3,4$ provided that $u_j(x,t)$ exists. Now the $y_j(t)$ functions can be expressed as linear combinations of the $\exp(r_j t)$ and, hence, the operator $O_j(t)$ can be expressed as a linear combination of the exponential operators $\exp(r_j tP(D))$. If $\varphi(z)$ is entire of growth < 1, then the functions $\exp(r_j tP(D))\varphi(x)$ can be computed as solutions of a generalized initial value heat problem [1]. Under these conditions, the $u_j(x,t)$ exist and are defined for all real x and t.

If we replace t in the relations (4.5) by $tP(D)$, we obtain identities among the operators $O_j(2t)$ and the operators $O_j(t)$. If we apply the resulting identities to the function $\varphi(x)$, we obtain the solution propagation relations. The first two of these are expressed by:

THEOREM 4.1. *Let $u_j(x,t)$ denote the solutions of (4.1) for $j = 1,2,3,4$. Then we have the double "angle" relations*

$$\begin{aligned}
u_1(x,2t) &= O_1(t)u_1(x,t) + 2\mu O_4(t)u_2(x,t) + \mu O_3(t)u_3(x,t) + \lambda\mu O_4(xt), \\
u_2(x,2t) &= 2O_1(t)u_2(x,t) + 2\mu O_3(t)u_4(x,t). \qquad (4.7)
\end{aligned}$$

Note that we can replace $O_4(t)u_2(x,t)$ in the first of these by $O_2(t)u_4(x,t)$. Similarly, in the second of these, $O_1(t)u_2(x,t) = O_2(t)u_1(x,t)$ and $O_3(t)u_4(x,t) = O_4(t)u_3(x,t)$. Finally, let us note that the first identity in (4.6) translates into the solution propagation relation

$$O_1(t)u_1(x,t) - 2\mu O_4(t)u_2(x,t) + \mu O_3(t)u_3(x,t) - \lambda\mu O_4(t)u_4(x,t) = \varphi(x). \tag{4.8}$$

5. Third order abstract problems

We finally consider the abstract equation

$$u'''(t) = Bu''(t) + B^2u'(t) - B^3u(t), \tag{5.1}$$

where B generates a continuous group in a Banach space X, subject to the usual initial conditions

$$u_i^{(j)}(0) = 0, \quad j \neq i-1, \text{ and } u_i^{(i-1)}(0) = \varphi, \quad i,j = 1,2,3, \ \varphi \in D(B^4). \tag{5.2}$$

The ordinary differential equation problems associated with (5.1) and (5.2) are given by

$$\begin{gathered} y'''(t) - \lambda y''(t) - \lambda^2 y'(t) + \lambda^3 y(t) = 0, \\ y_i^{(j)}(0) = 0, \quad j \neq i-1, \text{ and } y_i^{(i-1)}(0) = 1, \quad i,j = 1,2,3, \end{gathered} \tag{5.3}$$

and their solutions have the explicit forms

$$\begin{aligned} y_1(t,\lambda) &= \lambda\{\sinh(\lambda t)/\lambda\}/2 + \cosh(\lambda t) - \lambda e^{\lambda t}/2, \\ y_2(t,\lambda) &= \sinh(\lambda t)/\lambda, \\ y_3(t,\lambda) &= \frac{1}{2}\int_0^t \left(\sinh(\lambda\sigma)/\lambda + \sigma e^{\lambda\sigma}\right) d\sigma. \end{aligned} \tag{5.4}$$

Moreover, if we apply Algorithm 2.1 to (5.3), we obtain the addition formulas

$$\begin{aligned} y_1(t+\tau,\lambda) &= y_1(t,\lambda)y_1(\tau,\lambda) - \lambda^3\left[y_2(t,\lambda)y_3(\tau,\lambda) + y_3(t,\lambda)y_2(\tau,\lambda)\right] \\ &\quad - \lambda^4 y_3(t,\lambda)y_3(\tau,\lambda), \\ y_2(t+\tau,\lambda) &= \left[y_1(t,\lambda)y_2(\tau,\lambda) + y_2(t,\lambda)y_1(\tau,\lambda)\right] \\ &\quad + \lambda^2\left[y_2(t,\lambda)y_3(\tau,\lambda) + y_3(t,\lambda)y_2(\tau,\lambda)\right], \\ y_3(t+\tau,\lambda) &= \left[y_1(t,\lambda)y_3(\tau,\lambda) + y_3(t,\lambda)y_1(\tau,\lambda)\right] + y_2(t,\lambda)y_2(\tau,\lambda) \\ &\quad + \left[y_2(t,\lambda)y_3(\tau,\lambda) + y_3(t,\lambda)y_2(\tau,\lambda)\right] + 2\lambda^2 y_3(t,\lambda)y_3(\tau,\lambda). \end{aligned} \tag{5.5}$$

Formulas (5.4) and (5.5) will now be interpreted for the problems (5.1), (5.2). Let the operators $O_i(t)$ be defined by $O_i(t) \cdot \varphi = y_i(t, B) \cdot \varphi, i = 1, 2, 3$ and let $G_B(t)$ denote the group of operators noted in the introduction. Then we have

$$\begin{aligned} u_1(t) &= O_1(t) \cdot \varphi = \frac{1}{2} B w_2(t) + w_1(t) - \frac{1}{2} B G_B(t) \cdot \varphi, \\ u_2(t) &= O_2(t) \cdot \varphi = w_2(t), \\ u_3(t) &= O_3(t) \cdot \varphi = \frac{1}{2} \int_0^t (w_2(\sigma) + \sigma G_B(\sigma) \cdot \varphi) d\sigma, \end{aligned} \tag{5.6}$$

where the functions $w_1(t)$ and $w_2(t)$ are defined as in the example in the introduction. Choosing $\tau = t$ and $\lambda = B$ in (5.5), we deduce:

THEOREM 5.1. *The double "angle" relations satisfied by the solutions of (5.1), (5.2) are given by the formulas*

$$\begin{aligned} u_1(2t) &= O_1(t) \cdot u_1(t) - 2B^3 O_3(t) \cdot u_2(t) - B^4 O_3(t) \cdot u_3(t), \\ u_2(2t) &= 2O_2(t) \cdot u_1(t) + 2B^2 O_3(t) \cdot u_2(t), \\ u_3(2t) &= 2O_3(t) \cdot u_1(t) + O_2(t) \cdot u_2(t) + 2O_3(t) \cdot u_2(t) \\ &\quad + 2B^2 O_3(t) \cdot u_3(t). \end{aligned}$$

Since n is odd for this problem, there is no convenient way to compute the evaluations $O_i(t - \tau)\varphi, \ i = 1, 2, 3$.

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