INDIVISIBILITY OF HEEGNER CYCLES OVER SHIMURA CURVES AND SELMER GROUPS

HAINING WANG (ID)

Shanghai Center for Mathematical Sciences, Fudan University,
No. 2005 Songhu Road, Shanghai, 200438, China
(wanghaining1121@outlook.com)

Abstract In this article, we show that the Abel–Jacobi images of the Heegner cycles over the Shimura curves constructed by Nekovar, Besser and the theta elements contructed by Chida–Hsieh form a bipartite Euler system in the sense of Howard. As an application of this, we deduce a converse to Gross–Zagier–Kolyvagin type theorem for higher weight modular forms generalising works of Wei Zhang and Skinner for modular forms of weight 2. That is, we show that if the rank of certain residual Selmer group is 1, then the Abel–Jacobi image of the Heegner cycle is nonzero in this residual Selmer group.

Contents

1. Introduction

In a seminal work of Bertolini–Darmon [1], the authors constructed an Euler–Kolyvagin type system using Heegner points on various Shimura curves. The cohomology classes in this system satisfy beautiful reciprocity laws that resemble the so-called Jochnowitz's congruences. More precisely, these reciprocity laws relate the theta elements of the Gross points on the Shimura sets given by certain definite quaternion algebras to the reductions of the Heegner points on the Shimura curves given by certain indefinite quaternion algebras. These theta elements encode the algebraic part of the special values of the L-functions of elliptic curves over an imaginary quadratic field, while the Heegner points provide natural classes in the Galois cohomologies of the elliptic curves over such an imaginary quadratic field. These reciprocity laws enabled the authors to construct annihilators for elements in the Selmer groups attached to these elliptic curves over an imaginary quadratic field. As an application of these constructions, the authors proved the one-sided divisibility of the anticyclotomic Iwasawa main conjectures for these elliptic curves. The method of Bertolini–Darmon is axiomatised in [14] where it is shown that the theta elements and the Heegner points (almost) form a bipartite Euler system in his sense. See also the recent work [5] for a refinement.

The present article addresses the question of constructing a bipartite Euler system for higher weight modular forms over an imaginary quadratic field. On the analytic side, the theta elements are constructed by Chida–Hsieh in [8]. On the geometric side, it is natural to consider the Heegner cycles constructed by Nekovar [25] over the classical modular curves and by Besser [2] and Iovita–Speiss [17] over the Shimura curves given by indefinite quaternion algebras. In this article, we show that these Heegner cycles and theta elements indeed form a bipartite Euler system. As an application of this, we prove a converse to the Gross–Zagier–Kolyvagin type theorem which can be seen as the Selmer rank 1 case of a generalisation of the Kolyvagin conjecture for higher weight modular forms. We follow the strategy of Wei Zhang in his proof of the original Kolyvagin conjecture for modular forms of weight 2.

There are other attempts to generalise the work of Bertolini–Darmon [1] to a higher weight case. Notably, in [9], the authors indeed proved the one-sided divisibility for the anticyclotomic Iwasawa main conjecture for higher weight modular forms. Their construction relies on a clever trick using congruences between values of weight 2 modular forms and higher weight modular forms when evaluated at Gross points. This strategy works well when the root number of the involved L-function is $+1$ but does not apply to questions when the root number is -1, which is the case of interest in the present article. We also remark that in [7], the author works directly with the Heegner cycles but still

in the case when the root number is $+1$ and he is able to prove the first reciprocity law and apply it to prove a version of the Bloch–Kato conjecture in the rank 0 case. In this article, we prove the remaining second reciprocity law, which forms the main arithmetic input to our proof of the converse to the Gross–Zagier–Kolyvagin type theorem.

1.1. Main results

To precisely describe our results, we first introduce some notations. Let $f \in S_k^{\text{new}}(N)$ be a newform of level $\Gamma_0(N)$ with even weight $k \geq 2$ and K be an imaginary quadratic field whose discriminant is given by $-D_K$ with $D_K > 0$. We assume that N and D_K are relatively prime to each other. We also assume that N admits a factorisation $N = N^+ N^-$ with N^+ only divisible by primes that are split in K and N^- only divisible by primes that are inert in K. Throughout this article we assume that the following generalised Heegner hypothesis is satisfied:

N^- is square-free and consists of even number of prime factors that are inert in K.
(Heeg)

Let l be a distinguished rational prime such that $l \nmid ND_K$ and $k < l - 1$. Let $E = \mathbf{Q}(f)$ be the Hecke field of f and we fix an embedding $\iota_l : \mathbf{Q}^{\text{ac}} \hookrightarrow \mathbf{C}_l$ such that it induces a place λ of E. Let E_λ be the completion of E at λ and $\mathcal{O} = \mathcal{O}_{E_\lambda}$ be the valuation ring of E_λ. We fix a uniformiser $\varpi \in \mathcal{O}$ and let $\mathbf{F}_\lambda$ be the residue field of $\mathcal{O}$. If $n \geq 1$, then we will write $\mathcal{O}_n = \mathcal{O}/\varpi^n$. Let $\mathbb{T} = \mathbb{T}(N^+, N^-)$ be the l-adic completion of the Hecke algebra acting faithfully on the subspace of $S_k(N)$ consisting of forms that are new at primes dividing N^-. Let $\phi_f : \mathbb{T} \to \mathcal{O}$ be the morphism corresponding to the Hecke eigensystem of f and $\phi_{f,n} : \mathbb{T} \to \mathcal{O}_n$ be the reduction of ϕ_f modulo ϖ^n. Let $I_{f,n}$ be the kernel of $\phi_{f,n}$ and $\mathfrak{m}_f$ be the unique maximal ideal containing $I_{f,n}$. We denote by

$$\rho_{f,\lambda} : G_\mathbf{Q} \to \mathrm{GL}_2(E_\lambda)$$

the λ-adic Galois representation attached to f whose residual Galois representation is denoted by $\bar{\rho}_{f,\lambda}$. In this article, we will mainly consider the twist $\rho_{f,\lambda}(\frac{2-k}{2})$, which we will denote by $\rho_{f,\lambda}^*$, whose representation space is denoted by $V_{f,\lambda}$. We fix a $G_\mathbf{Q}$-stable lattice $\mathrm{T}_{f,\lambda}$ in $V_{f,\lambda}$. The residual Galois representation of $\rho_{f,\lambda}^*$ will be denoted by $\bar{\rho}_{f,\lambda}^*$. It is well-known that the representation $\rho_{f,\lambda}^*$ appears in the cohomology of a certain Shimura curve with coefficient in the l-adic local system $\mathcal{L}_{k-2}$ corresponding to the representation $\mathrm{Sym}^{k-2}\mathrm{st} \otimes \det^{\frac{k-2}{2}}$ of GL_2. Here, st is the standard representation of GL_2. To define these Shimura curves, we will introduce certain quaternion algebras. Let B' be the indefinite quaternion algebra of discriminant N^- and $\mathcal{O}_{B',N^+}$ be an Eichler order of level N^+ contained in some maximal order $\mathcal{O}_{B'}$ of B'. These data define a Shimura curve $X = X_{N^+,N^-}^{B'}$ which is a coarse moduli space of abelian surfaces with quaternionic multiplications. We wish not to assume that $N^- > 1$, in which case X is a projective curve over $\mathbf{Q}$. In the case when $N^- = 1$, X will denote the compactification of the classical modular curve over $\mathbf{Q}$. However, we only give the constructions and proofs for the more complicated case of $N^- > 1$; the proof for the case of modular curves is almost completely similar. We will rigidify the moduli problem of X by adding

an auxiliary full level-d structure and denote the resulting fine moduli space by X_d. Let $A_d \to X_d$ be universal abelian surface and $\pi_{k,d} : W_{k,d} \to X_d$ be the Kuga–Sato variety given by the $\frac{k-2}{2}$-fold fibre product of A_d over X_d. One can construct certain projectors ϵ_d and ϵ_k that cut out the Chow motive corresponding to the space of modular forms in the Kuga–Sato variety $W_{k,d}$. Then the representation $\mathrm{T}_{f,\lambda}$ occurs in $\epsilon_d \epsilon_k \mathrm{H}^{k-1}(W_{k,d,\mathbf{Q}^{\mathrm{ac}}}, \mathcal{O}(\frac{k}{2})) = \mathrm{H}^1(X_{\mathbf{Q}^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O})(1))$. We will put the following assumption on the residue Galois representation $\bar{\rho}_{f,\lambda}$.

Assumption 1 (CR*). The residual Galois representation $\bar{\rho}_{f,\lambda}$ satisfies the following assumptions:

(1) $k < l-1$ and $|(\mathbf{F}_l^\times)^{k-1}| > 5$;

(2) $\bar{\rho}_{f,\lambda}$ is absolutely irreducible when restricted to $G_{\mathbf{Q}(\sqrt{p^*})}$ where $p^* = (-1)^{\frac{p-1}{2}} p$;

(3) If $q \mid N^-$ and $q \equiv \pm 1 \bmod l$, then $\bar{\rho}_{f,\lambda}$ is ramified at q;

(4) If $q \parallel N^+$ and $q \equiv 1 \bmod l$, then $\bar{\rho}_{f,\lambda}$ is ramified at q;

(5) The Artin conductor $N_{\bar{\rho}}$ of $\bar{\rho}_{f,\lambda}$ is prime to $N/N_{\bar{\rho}}$;

(6) There is a place $q \parallel N$ such that $\bar{\rho}_{f,\lambda}$ is ramified at q.

We remark that our assumption (CR*) is essentially the assumption (CR$^+$) in [9]. It is used to invoke results in [8] and [9]. The assumption (CR*)(6) is needed to apply the main result of [35]. In order to apply the main results of [9], we also assume the following technical assumption on f:

$$a_l(f) \not\equiv 1 \bmod l \text{ if } k = 2. \tag{PO}$$

Let K_m be the ring class field over K of level m for some integer $m \geq 1$. In Subsection 3.1, we define a certain Heegner cycle $\epsilon_d Y_{m,k} \in \epsilon_d \epsilon_k \mathrm{CH}^{\frac{k}{2}}(W_{k,d} \otimes K_m) \otimes \mathbf{Z}_l$ and an Abel–Jacobi map for some $n \geq 1$

$$\mathrm{AJ}_{k,n} : \epsilon_d \epsilon_k \mathrm{CH}^{\frac{k}{2}}(W_{k,d} \otimes K_m) \otimes \mathbf{Z}_l \to \mathrm{H}^1(K_m, \mathrm{T}_{f,n}).$$

The images of $\epsilon_d Y_{m,k}$ under the map $\mathrm{AJ}_{k,n}$ give the cohomology class

$$\kappa_n(m) := \mathrm{AJ}_{k,n}(\epsilon_d Y_{m,k}) \in \mathrm{H}^1(K_m, \mathrm{T}_{f,n})$$

and we define

$$\kappa_n := \mathrm{Cor}_{K_1/K} \kappa_n(1) \in \mathrm{H}^1(K, \mathrm{T}_{f,n}).$$

Our main result concerns the element κ_1. The element κ_1 in fact lives in the residual Selmer group

$$\mathrm{Sel}_{\mathcal{F}(N^-)}(N^+, \mathrm{T}_{f,1})$$

defined by some Selmer structure $\mathcal{F}(N^-)$ spelled out in (4.4). Our main result is the following.

Theorem 2. *Suppose (f, K) is a pair that satisfies the generalised Heegner hypothesis* (Heeg) *and* (PO). *Suppose in addition that f is ordinary at l and that $\bar{\rho}_{f,\lambda}$ satisfies*

the hypothesis (CR*). *If* $\dim_{\mathbf{F}_\lambda} \mathrm{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1}) = 1$, *then the class* κ_1 *is nonzero in* $\mathrm{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1})$.

This theorem can be viewed as a converse to the Gross–Zagier–Kolyvagin theorem for Heegner cycles. For the other direction, one can show that if κ_n is nonzero in $\mathrm{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,n})$, then the Selmer group $\mathrm{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,n})$ is of rank 1. This follows from the result of Nekovar [25] in the case when $N^- = 1$ and its extension to the case when $N^- > 1$ in [13]. In these works, they follow the original method of Kolyvagin and use the derivative classes of the Heegner cycles to construct annihilators for the Selmer groups. We can recover their results by combining the first and second reciprocity laws proved in this article. See [37] for an example of how to carry this out. We also have the Gross–Zagier formula [39] for the Heegner cycles over the classical modular curves by Shou–Wu Zhang. Suppose that height pairing is nondegenerate; then the Gross–Zagier formula and Theorem 2 would allow us to conclude that if the rank of Selmer group is 1, then the analytic rank of the L-function $L(f/K, s)$ at $s = \frac{k}{2}$ is 1. Next, we sketch the proof of Theorem 2. First we recall the notion of an n-admissible prime for f.

Definition 1.1. We say a prime p is n-admissible for f if

(1) $p \nmid Nl$;
(2) p is an inert prime in K;
(3) l does not divide $p^2 - 1$;
(4) ϖ^n divides $p^{\frac{k}{2}} + p^{\frac{k-2}{2}} - \epsilon_p a_p(f)$ with $\epsilon_p \in \{\pm 1\}$.

These primes are level-raising primes for f. This means that one can find a newform $f^{[p]} \in S_k^{\mathrm{new}}(pN)$ that is congruent to f modulo ϖ^n. Note that $f^{[p]}$ can be realised in the space of quaternionic modular forms $S_k^B(N^+, \mathcal{O})$ of weight k associated to the definite quaternion algebra B of discriminant pN^-. This is justified in the following theorem, which we call the (unramified) arithmetic level-raising theorem for the Kuga–Sato varieties. We consider the ordinary-supersingular excision exact sequence on X with coefficients in $\mathcal{L}_{k-2}$

$$0 \to \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \to \mathrm{H}^1(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ord}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \to \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O}))_{\mathfrak{m}_f} \to 0$$

where $\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}$ is the special fibre of X over $\mathbf{F}_p^{\mathrm{ac}}$ and $X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ord}}$ (respectively $X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}$) is its ordinary locus (respectively supersingular locus). The coboundary map of the above exact sequence induces the following map:

$$\Phi_n : \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O}))_{/I_{f,n}}^{G_{\mathbf{F}_{p^2}}} \to \mathrm{H}^1(\mathbf{F}_{p^2}, \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}}).$$

Theorem 3 (Unramified level raising). *Let p be an n-admissible prime for f. We assume that the residual Galois representation $\bar{\rho}_{f,\lambda}$ satisfies* (CR*). *Then the following hold true:*

(1) *There exists a morphism $\phi_{f,n}^{[p]} : \mathbb{T}^{[p]} \to \mathcal{O}_n$ which agrees with $\phi_{f,n} : \mathbb{T} \to \mathcal{O}_n$ on all of the Hecke operators away from p and sends U_p to $\epsilon_p p^{\frac{k-2}{2}}$.*

(2) *Let $I_{f,n}^{[p]}$ be the kernel of the morphism $\phi_{f,n}^{[p]}$. We have a canonical isomorphism*

$$\Phi_n : \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O}))_{/I_{f,n}}^{G_{\mathbf{F}_{p^2}}} \xrightarrow{\cong} \mathrm{H}^1(\mathbf{F}_{p^2}, \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}})$$

which can be identified with an isomorphism

$$\Phi_n : S_k^B(N^+, \mathcal{O})_{/I_{f,n}^{[p]}} \xrightarrow{\cong} \mathrm{H}^1(\mathbf{F}_{p^2}, \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}}).$$

One can define a theta element $\Theta(f_{\pi'}^{[p]})$ associated to the Jacquet–Langlands transfer $f_{\pi'}^{[p]}$ of $f^{[p]}$ following Chida–Hsieh [8] that encodes the square root of the algebraic part of the L-value $L(f^{[p]}/K, \frac{k}{2})$. Note that the global root number of the L-function $L(f^{[p]}/K, s)$ at $s = \frac{k}{2}$ is $+1$. We have the following reciprocity formula relating the Heegner cycle class κ_n to the theta element $\Theta(f_{\pi'}^{[p]})$.

Theorem 4 (Second reciprocity law). *Let p be an n-admissible prime for f and assume that $\bar{\rho}_{f,\lambda}$ satisfies assumption $(\mathrm{CR}^\star)$. Let $f_n^{[p]}$ be a generator of $S_k^B(N^+, \mathcal{O})[I_{f,n}^{[p]}]$; then we have the following relation between the class κ_n and the theta element $\Theta(f_{\pi'}^{[p]})$:*

$$\langle \mathrm{loc}_p(\kappa_n), f_n^{[p]} \rangle_B = u \cdot \Theta(f_{\pi'}^{[p]}) \bmod \varpi^n$$

for some unit $u \in \mathcal{O}_n$.

Returning to the sketch of the proof of Theorem 2, we choose a 1-admissible prime p for f and consider the residual Selmer group $\mathrm{Sel}_{\mathcal{F}(pN^-)}(K, \mathrm{T}_{f,1})$ associated to $f^{[p]}$. The assumption that the residual Selmer group $\mathrm{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1})$ is of dimension 1 implies that the dimension of $\mathrm{Sel}_{\mathcal{F}(pN^-)}(K, \mathrm{T}_{f,1})$ drops to 0. As a consequence of the anticyclotomic Iwasawa main conjectures for $f^{[p]}$ over K proved in [35] and [9], we show that the algebraic part of the special value $L(f^{[p]}/K, \frac{k}{2})$ is indivisible by ϖ and thus $\Theta(f_{\pi'}^{[p]})$ is indivisible by ϖ. Here we will rely on the recent work of Kim–Ota [26] to compare the canonical period $\Omega_{f^{[p]}}^{\mathrm{can}}$ and another period $\Omega_{f^{[p]}, pN^-}$ that show up in the specialisation formula relating $\Theta(f_{\pi'}^{[p]})$ to $L(f^{[p]}/K, \frac{k}{2})$. Finally, the second reciprocity law implies that $\mathrm{loc}_p(\kappa_1)$ is indivisible and therefore κ_1 is nonzero.

We finish this introduction with a few remarks on the related works. First of all, in [40], the author proves the Kolyvagin conjecture without assuming the rank of the Selmer group is 1. It is reasonable to expect that one can formulate and prove an analogue of the Kolyvagin conjecture for Heegner cycles using the results of the present article as the first step of an induction process. In this article, the derived classes of the Heegner cycles are completely untouched. As pointed out by Francesc Castella, our results should also shed light on Perrin–Riou's main conjecture for generalised Heegner cycles formulated by [23, Conjecture 5.1]. Compare the proof of [5, Proposition 3.7] towards Perrin–Riou's original main conjecture for Heegner points. In an unpublished work of Castella and Skinner, the authors carried out a similar program for the big Heegner point in the sense of Howard, and it should be interesting to compare their results with the results in this article.

1.2. Notations and conventions

We will use common notations and conventions in algebraic number theory and algebraic geometry. The cohomologies in this article will be understood as the étale cohomologies. For a field K, we denote by K^{ac} the separable closure of K and let $G_K := \mathrm{Gal}(K^{\mathrm{ac}}/K)$ be the absolute Galois group of K. We let $\mathbf{A}$ be the ring of adèles over $\mathbf{Q}$ and $\mathbf{A}^{(\infty)}$ be the subring of finite adèles. For a prime p, $\mathbf{A}^{(\infty,p)}$ denotes the prime-to-p part of $\mathbf{A}^{(\infty)}$.

Let F be a local field with ring of integers $\mathcal{O}_F$ and residue field k. We let I_F be the inertia subgroup of G_F. Suppose M is a G_F-module. Then the finite part $\mathrm{H}^1_{\mathrm{fin}}(F,\mathrm{M})$ of $\mathrm{H}^1(F,\mathrm{M})$ is defined to be $\mathrm{H}^1(k,\mathrm{M}^{I_F})$ and the singular quotient $\mathrm{H}^1_{\mathrm{sing}}(F,\mathrm{M})$ of $\mathrm{H}^1(F,\mathrm{M})$ is defined to be the quotient of $\mathrm{H}^1(F,\mathrm{M})$ by the image of $\mathrm{H}^1_{\mathrm{fin}}(F,\mathrm{M})$.

We provide a list of quaternion algebras appearing in this article. Recall that N^- is square-free with even number of prime divisors and p,p' are n-admissible primes.

- B' is the indefinite quaternion algebra of discriminant N^-.
- B is the definite quaternion algebra of discriminant pN^-.
- B'' is the indefinite quaternion algebra of discriminant $pp'N^-$

2. Arithmetic level raising on Kuga–Sato varieties

2.1. Shimura curves and local system

Let N be a positive integer with a factorisation $N = N^+N^-$ with N^+ and N^- coprime to each other. We assume that N^- is square-free and is a product of *even* number of primes. Let B' be the indefinite quaternion algebra over $\mathbf{Q}$ with discriminant N^-. Let $\mathcal{O}_{B'}$ be a maximal order of B' and let $\mathcal{O}_{B',N^+}$ be the Eichler order of level N^+ in $\mathcal{O}_{B'}$. We define G' to be the algebraic group over $\mathbf{Q}$ given by $B'^\times$ and let K'_{N^+} be the open compact subgroup of $G'(\mathbf{A}^{(\infty)})$ defined by $\widehat{\mathcal{O}}^\times_{B',N^+}$. Let $X = X^{B'}_{N^+,N^-}$ be the Shimura curve over $\mathbf{Q}$ with level $K' = K'_{N^+}$. The complex points of this curve are given by the following double coset:

$$X(\mathbf{C}) = G'(\mathbf{Q})\backslash \mathcal{H}^\pm \times G'(\mathbf{A}^{(\infty)})/K'.$$

We consider the functor $\mathfrak{X}$ on schemes over $\mathbf{Z}[1/N]$ which gives the following moduli problem. Let S be a test scheme over $\mathbf{Z}[1/N]$; then $\mathfrak{X}(S)$ classifies the triple (A,ι,C) up to isomorphism where

(1) A is an S-abelian scheme of relative dimension 2;
(2) $\iota : \mathcal{O}_{B'} \hookrightarrow \mathrm{End}_S(A)$ is an embedding;
(3) C is an $\mathcal{O}_{B'}$-stable locally cyclic subgroup of $A[N^+]$ of order $(N^+)^2$.

It is well-known this moduli problem is coarsely representable by a projective scheme $\mathfrak{X}$ over $\mathbf{Z}[1/N]$ of relative dimension 1. Let l be a distinguished rational prime such that $l \nmid N$ and $k < l-1$. We let $\Lambda = \mathbf{Z}/l^n$ for some $n \geq 1$ or a finite extension of $\mathbf{Z}_l$. Then we define $\mathcal{L}'_{k-2}(\Lambda)$ to be the local system given by the composite map

$$\pi_1^{\mathrm{alg}}(X) \to K' \to \mathcal{O}^\times_{B',l} \cong \mathrm{GL}_2(\mathbf{Z}_l) \to \mathrm{GL}_{k+1}(\mathbf{Z}_l). \tag{2.1}$$

To rigidify the moduli problem $\mathfrak{X}$, we choose an auxiliary square-free integer $d \geq 5$ that is prime to Nl and add a full level-d structure to the above moduli problem; that is, we add the following data to the above moduli problem: let

$$\nu_d : (\mathcal{O}_{B'}/d)_S \to A[d] \tag{2.2}$$

be an isomorphism of $\mathcal{O}_{B'}$-stable group schemes. By forgetting the data ν_d, we have a natural map $c_d : \mathfrak{X}_d \to \mathfrak{X}$ which is Galois with covering group $G_d := (\mathcal{O}_{B'}/d)^\times/\{\pm 1\}$. Then this new moduli problem is representable by a projective scheme $\mathfrak{X}_d$ over $\mathbf{Z}[1/Nd]$ of relative dimension 1. We will set

$$K'_d = \{g = (g_v)_v \in K' : g_v \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod v \text{ for all } v \mid d\}.$$

We will denote by X_d the base change of $\mathfrak{X}_d$ to $\mathbf{Q}$. Then the $\mathbf{C}$-point of X_d is given by

$$X_d(\mathbf{C}) = G'(\mathbf{Q})\backslash \mathcal{H}^\pm \times G'(\mathbf{A}^\infty)/K'_d.$$

Let $\pi_d : \mathcal{A}_d \to \mathfrak{X}_d$ be the universal abelian surface. The sheaf $R^1\pi_{d*}\Lambda$ over $\mathfrak{X}_d$ is equipped with an action of $\mathcal{O}_{B'_l} = \mathrm{M}_2(\mathbf{Z}_l)$. We then define the following local system on $\mathfrak{X}_d$:

$$\mathcal{L}'_{k-2}(\Lambda) := \mathrm{Sym}^{k-2} e \cdot R^1\pi_{d*}\Lambda, \quad \mathcal{L}_{k-2}(\Lambda) := \mathrm{Sym}^{k-2} e \cdot R^1\pi_{d*}\Lambda\left(\frac{k-2}{2}\right)$$

where e is the idempotent given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathrm{M}_2(\mathbf{Z}_l)$. There is another construction of this local system in [2] whose cohomology is tied more closely to the Kuga–Sato variety that we will introduce below. We briefly review this construction. Define

$$\mathfrak{L}'_2(\Lambda) = \cap_{b \in B'} \ker[R^2\pi_{d*}\Lambda \xrightarrow{b - \mathrm{Nrd}(b)} R^2\pi_{d*}\Lambda]. \tag{2.3}$$

We define the weight $k-2$ local system by

$$\mathfrak{L}'_{k-2}(\Lambda) := \ker[\mathrm{Sym}^m \mathfrak{L}'_2(\Lambda) \xrightarrow{\Delta_m} \mathrm{Sym}^{m-2}\mathfrak{L}'_2(\Lambda)(-2)]$$

where $m = \frac{k-2}{2}$ and Δ_m is the Laplace defined by

$$\Delta_m(x_1, \ldots, x_m) = \sum_{1 \leq i, j \leq m} (x_i, x_j) x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_m$$

where $(\ ,\ )$ is the nondegenerate pairing

$$(\ ,\ ) : \mathfrak{L}'_2(\Lambda) \times \mathfrak{L}'_2(\Lambda) \to \Lambda(-2)$$

induced by the Poincare duality

$$(\ ,\ ) : R^2\pi_{d*}\Lambda \times R^2\pi_{d*}\Lambda \to \Lambda(-2).$$

Lemma 2.1. *We have an isomorphism*

$$\mathfrak{L}'_{k-2}(\Lambda) \cong \mathcal{L}'_{k-2}(\Lambda).$$

Proof. This result is contained in the proof of [2, Theorem 5.8]. We briefly outline the construction. First of all, there is an isomorphism between the rank 3 local system

$$\mathcal{L}'_2(\Lambda) \cong \mathrm{Sym}^2 e \cdot R^1 \pi_{d*} \Lambda.$$

Then it is an easy exercise to show that $\mathrm{Sym}^{k-2} e \cdot R^1 \pi_{d*} \Lambda$ is the kernel of the map

$$\mathrm{Sym}^m \mathrm{Sym}^2 e \cdot R^1 \pi_{d*} \Lambda \xrightarrow{\Delta_m} \mathrm{Sym}^{m-2} \mathrm{Sym}^2 e \cdot R^1 \pi_{d*} \Lambda$$

with $m = \frac{k-2}{2}$. The result follows from this. $\qquad\square$

Let $k \geq 2$ and let $\pi_{k,d} : \mathcal{W}_{k,d} \to \mathfrak{X}_d$ be the *Kuga–Sato variety of weight k* over $\mathfrak{X}_d$. This is defined by the $\frac{k-2}{2}$-fold fibre product of $\mathcal{A}_d$ over $\mathfrak{X}_d$,

$$\mathcal{W}_{k,d} := \mathcal{A}_d \times_{\mathfrak{X}_d} \cdots \times_{\mathfrak{X}_d} \mathcal{A}_d.$$

We will denote by $W_{k,d}$ the scheme $\mathcal{W}_{k,d} \otimes \mathbf{Q}$. We define ϵ_d to be the projector given by

$$\epsilon_d = \frac{1}{|G_d|} \sum_{g \in G_d} g.$$

Then the following relation clearly holds:

$$\epsilon_d \mathrm{H}^1(\mathfrak{X}_d \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}'_{k-2}(\Lambda)) \cong \mathrm{H}^1(\mathfrak{X} \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}'_{k-2}(\Lambda)).$$

The cohomology of the Kuga–Sato variety and the cohomology of the local system $\mathcal{L}'_{k-2}(\Lambda)$ are closely related.

Lemma 2.2. *There is a projector ϵ_k on $\mathcal{W}_{k,d}$ such that*

$$\epsilon_k \mathrm{H}^*(\mathcal{W}_{k,d} \otimes \mathbf{Q}^{\mathrm{ac}}, \Lambda) \cong \epsilon_k \mathrm{H}^{k-1}(\mathcal{W}_{k,d} \otimes \mathbf{Q}^{\mathrm{ac}}, \Lambda) \cong \mathrm{H}^1(\mathfrak{X}_d \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}'_{k-2}(\Lambda)).$$

Proof. This follows from the discussions in [17, (67), Lemma 10.1]. $\qquad\square$

2.2. Shimura sets and quaternionic modular forms

Let $k \geq 2$ be an even integer such that $k < l - 1$. If A is a ring, let $L_{k-2}(A) = \mathrm{Sym}^{k-2}(A^2)$ be the set of homogeneous polynomials of degree $k-2$ with coefficients in A. We present $L_{k-2}(A)$ as

$$L_{k-2}(A) = \bigoplus_{\frac{2-k}{2} \leq r \leq \frac{k-2}{2}} A \cdot \mathbf{v}_r \tag{2.4}$$

with $\mathbf{v}_r := X^{\frac{k-2}{2}-r} Y^{\frac{k-2}{2}+r}$. It gives rise to a unitary representation

$$\rho_k : \mathrm{GL}_2(A) \to \mathrm{Aut}_A(L_{k-2}(A))$$

such that $\rho_k(g)P(X,Y) = \det^{-\frac{(k-2)}{2}}(g)P((X,Y)g)$ for any $P(X,Y) \in L_k(A)$. Let A be a $\mathbf{Z}_{(l)}$-algebra; we define a pairing

$$\langle \ , \ \rangle : L_{k-2}(A) \times L_{k-2}(A) \to A$$

by the following formula:

$$\left\langle \sum_i a_i \mathbf{v}_i, \sum_j b_j \mathbf{v}_j \right\rangle_{k-2} = \sum_{\frac{2-k}{2} \le r \le \frac{k-2}{2}} a_r b_{-r} \cdot (-1)^{\frac{k-2}{2}+r} \frac{\Gamma\left(\frac{k}{2}+r\right)\Gamma\left(\frac{k}{2}-r\right)}{\Gamma(k-1)}.$$

For $P_1, P_2 \in L_{k-2}(A)$, the pairing above has the following property:

$$\langle \rho_k(g)P_1, \rho_k(g)P_2 \rangle = \langle P_1, P_2 \rangle.$$

Let $p \nmid N$ be a prime. Let B be the definite quaternion algebra of discriminant pN^-. We let G be the algebraic group over $\mathbf{Q}$ defined by $B^\times$. If $U \subset G(\mathbf{A}^{(\infty)})$ is an open compact subgroup and A is a $\mathbf{Z}_l$-algebra, we define the space $S_k^B(U, A)$ of l-adic quaternionic modular forms of weight k with values in A by

$$S_k^B(U, A) = \{h : G(\mathbf{A}) \to L_{k-2}(A) : h(agu)$$
$$= \rho_k(u_l^{-1})h(g) \text{ for } a \in B^\times \text{ and } u \in U \cdot Z(\mathbf{A}^{(\infty)})\}.$$

In the case U corresponds to an Eichler order $\mathcal{O}_{B,N^+}$ of level N^+ in a fixed maximal order $\mathcal{O}_B$, we will simply write the space $S_k^B(U, A)$ as $S_k^B(N^+, A)$. We define the *Atkin–Lehner involution* at q to be

$$\tau_q = \begin{cases} \begin{pmatrix} 0 & 1 \\ -N^+ & 0 \end{pmatrix} & \text{for } q \mid N^+; \\ J & \text{for } q \mid \infty N^-; \\ 1 & \text{for } q \nmid N. \end{cases} \tag{2.5}$$

Then we put $\tau^{N^+} = \prod_q \tau_q$ as an element in $G'(\mathbf{A})$. We will define an inner product on this space

$$\langle \ , \ \rangle_B : S_k^B(N^+, A) \times S_k^B(N^+, A) \to A \tag{2.6}$$

by the following formula:

$$\langle f_1, f_2 \rangle_B = \sum_{g \in \mathrm{Cl}(\mathrm{N}^+)} \frac{1}{|\Gamma_g|} \langle f_1(g), f_2(g\tau^{N^+}) \rangle_k \tag{2.7}$$

where $\Gamma_g = (B^\times \cap g\widehat{\mathcal{O}}_{B,N^+}^\times g^{-1} Z(\mathbf{A}^{(\infty)}))/\mathbf{Q}^\times$ and $\mathrm{Cl}(\mathrm{N}^+)$ is a set of representatives of the finite set

$$B^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}}_{B,N^+}^\times \widehat{\mathbf{Q}}^\times.$$

2.3. Reductions of Shimura curves

Let p be a prime away from N. We will consider the base change of $\mathfrak{X}_d, \mathfrak{X}$ to $\mathbf{Z}_{p^2}$ and we will denote them by the same notations. The special fibre of $\mathfrak{X}_d$ and $\mathfrak{X}$ will be denoted by $\overline{X}_d$ and $\overline{X}$. Let $x = (A, \iota, \bar{\eta}) \in \overline{X}_d(\mathbf{F}_p^{\mathrm{ac}})$ be an $\mathbf{F}_p^{\mathrm{ac}}$-point. Then the p-divisible group $A[p^\infty]$ of A can be written as $A[p^\infty] = E[p^\infty] \times E[p^\infty]$ for a p-divisible group $E[p^\infty]$ associated to an elliptic curve E and $\mathcal{O}_{B'}$ acts naturally via $\mathcal{O}_{B'} \otimes \mathbf{Z}_p = \mathrm{M}_2(\mathbf{Z}_p)$. Depending on whether $E[p^\infty]$ is *ordinary* or *supersingular*, we will accordingly call x ordinary or supersingular.

Let X_d^{ss} be the closed subscheme of $\overline{X}_d$ given by those points that are supersingular and let $X_d^{\mathrm{ord}} = \overline{X}_d - X_d^{\mathrm{ss}}$ be its complement. We will refer to X_d^{ss} as the *supersingular locus* and to X_d^{ord} as the *ordinary locus*. Let $B = B_{pN^-}$ be the definite quaternion algebra with discriminant pN^- and $\mathcal{O}_B$ be a maximal order. Note that we can naturally view $K_d^{\prime(p)}$, the prime-to-p part of K_d', as an open compact subgroup of $G(\mathbf{A}^{(\infty,p)}) = B^\times(\mathbf{A}^{(\infty,p)})$. The scheme X_d^{ss} is given by a finite set of points, and we have the following parametrisation of it.

Lemma 2.3. *We have a bijection*

$$X_d^{\mathrm{ss}} \cong B^\times(\mathbf{Q})\backslash B^\times(\mathbf{A}^{(\infty)})/K_d^{\prime(p)}\mathcal{O}_{B_p}^\times.$$

Proof. The lemma is well-known and can be proved using essentially the same method of the classical work Deuring and Serre. See [11, Lemma 9], for example. $\quad\square$

We will write

$$X_d^B = B^\times(\mathbf{Q})\backslash B^\times(\mathbf{A}^{(\infty)})/K_d^{\prime(p)}\mathcal{O}_{B_p}^\times \qquad (2.8)$$

and refer to it as the *Shimura set* associated to the definite quaternion algebra B with level $\Gamma_0(N^+) \cap \Gamma(d)$. Therefore, the above lemma can be rephrased as a natural bijection

$$X_d^{\mathrm{ss}} \cong X_d^B.$$

Let $\mathcal{O}_{B',pN^+}$ be an Eichler order of level pN^+ and let $K'(p)$ be the associated open compact subgroup in $G'(\mathbf{A}^\infty)$. Similar to (2.2), we define the open compact subgroup $K_d'(p)$ of $G'(\mathbf{A}^\infty)$ by adding a full level d-structure to $K'(p)$. We have the curve $X_d(p)$ over $\mathbf{Q}$ whose complex points are given by

$$X_d(p)(\mathbf{C}) = G'(\mathbf{Q})\backslash\mathcal{H}^\pm \times G'(\mathbf{A}^\infty)/K_d'(p).$$

We define an integral model $\mathfrak{X}_d(p)$ over $\mathbf{Z}[1/dN]$ which represents the following functor. Let S be a test scheme over $\mathbf{Z}[1/dN]$. Then $\mathfrak{X}_d(p)(S)$ classifies the tuples $(A_1, A_2, \iota_1, \iota_2, \pi_A, C, \nu_d)$ up to isomorphism where

(1) A_i for $i = 1, 2$ is an S-abelian scheme of relative dimension 2;

(2) $\iota_i: \mathcal{O}_{B'} \hookrightarrow \mathrm{End}_S(A_i)$ is an action of $\mathcal{O}_{B'}$ on $\mathrm{End}_S(A_i)$ for $i = 1, 2$;

(3) $\pi_A: A_1 \to A_2$ is an isogeny of degree p that commutes with the action of $\mathcal{O}_{B'}$;

(4) C is an $\mathcal{O}_{B'}$-stable locally cyclic subgroup of $A[N^+]$ of order N^{+2};

(5) $\nu_d: (\mathcal{O}_{B'}/d)_S \to A_1[d]$ is an isomorphism of $\mathcal{O}_{B'}$-stable group schemes.

By forgetting the data given by ν_d, we have a natural map $c_d(p): \mathfrak{X}_d(p) \to \mathfrak{X}_0(p)$ where $\mathfrak{X}_0(p)$ is the coarse moduli space representing the above functor without the data ν_d. Note that the isogeny π_A induces an isomorphism between $\gamma_d: A_1[d] \xrightarrow{\sim} A_2[d]$ and $\gamma_{N^+}: A_1[N^+] \xrightarrow{\sim} A_2[N^+]$. Again we consider the base change of $\mathfrak{X}_d(p)$ to $\mathbf{Z}_{p^2}$ and use the same symbol for this base change and denote its special fibre by $\overline{X}_d(p)$. Let $\overline{X}_0(p)$ be the image of $\overline{X}_d(p)$ under the map $c_d(p)$. We have the following descriptions of $\overline{X}_d(p)$ and $\overline{X}_0(p)$. Similarly, let $(\overline{X}, X^{\mathrm{ord}}, X^{\mathrm{ss}})$ be the image of $(\overline{X}_d, X_d^{\mathrm{ord}}, X_d^{\mathrm{ss}})$ under the map c_d. We will call $(X^{\mathrm{ord}}, X^{\mathrm{ss}})$ the ordinary and supersingular locus of $\overline{X}$.

Lemma 2.4. *The scheme $\overline{X}_d(p)$ consists of two irreducible components both isomorphic to $\overline{X}_d$ which cross transversally at the supersingular locus $X_d^{\mathrm{ss}}(p)$ of $\overline{X}_d(p)$, which in turn can be identified with the supersingular locus of $\overline{X}_d$. A similar statement holds for $\overline{X}_0(p)$.*

Proof. This is proved in [6, Theorem 4.7(v)]. □

Let $\pi_1 : \mathfrak{X}_d(p) \to \mathfrak{X}_d$ be the morphism given by sending $(A_1, A_2, \pi_A, \iota_1, \iota_2, C, \nu_d)$ to (A_1, ι_1, C, ν_d) and $\pi_2 : \mathfrak{X}_d(p) \to \mathfrak{X}_d$ be the morphism given by sending $(A_1, A_2, \pi_A, \iota_1, \iota_2, C, \nu_d)$ to

$$(A_2, \iota_2, \gamma_{N^+}(C), \gamma_d \circ \nu_d).$$

Then we can define two closed immersions $i_1 : \overline{X}_d \to \overline{X}_d(p)$ and $i_2 : \overline{X}_d \to \overline{X}_d(p)$ as in the proof of [6, Theorem 4.7(v)] such that

$$\begin{pmatrix} \pi_1 \circ i_1 & \pi_1 \circ i_2 \\ \pi_2 \circ i_1 & \pi_2 \circ i_2 \end{pmatrix} = \begin{pmatrix} \mathrm{id} & \mathrm{Frob}_p \\ S_p^{-1}\mathrm{Frob}_p & \mathrm{id} \end{pmatrix}, \tag{2.9}$$

where S_p corresponds to the central element in the spherical Hecke algebra of $\mathrm{GL}_2(\mathbf{Q}_p)$.

We will need the following important result known as the 'Ihara's lemma'. This is proved for the case of classical modular curves by Ribet [29] and Diamond–Taylor [11] for Shimura curves.

Theorem 2.5 ([11, Theorem 4]). *We have the following statements:*

(1) *The kernel of the pullback map*

$$(\pi_1^* + \pi_2^*) : \mathrm{H}^1(\mathfrak{X}_d \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathbf{F}_l)) \oplus \mathrm{H}^1(\mathfrak{X}_d \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathbf{F}_l))$$
$$\to \mathrm{H}^1(\mathfrak{X}_d(p) \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathbf{F}_l))$$

   is Eisenstein.

(2) *The cokernel of the pushforward map*

$$(\pi_{1*}, \pi_{2*}) : \mathrm{H}^1(\mathfrak{X}_d(p) \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathbf{F}_l))$$
$$\to \mathrm{H}^1(\mathfrak{X}_d \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathbf{F}_l)) \oplus \mathrm{H}^1(\mathfrak{X}_d \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathbf{F}_l))$$

   is Eisenstein.

Remark 2.6. Here the restriction on weights in the original treatment of [11] can be improved by using the recent work of [24]. However, we will rely on the freeness result on the Hecke module of quaternionic modular forms [8, Proposition 6.8], and this result is proved under the weight restriction in the Fontaine–Laffaille range.

2.4. Review of weight spectral sequence

Let K be a Henselian discrete valuation field with valuation ring $\mathcal{O}_K$ and residue field k of characteristic p. We fix a uniformiser π of $\mathcal{O}_K$. We set $S = \mathrm{Spec}(\mathcal{O}_K)$, $s = \mathrm{Spec}(k)$ and $\eta = \mathrm{Spec}(K)$. Let K^{ac} be a separable closure of K and K_{ur} the maximal unramified extension of K in K^{ac}. We denote by k^{ac} the residue field of K_{ur}. Let $I_K = \mathrm{Gal}(K^{\mathrm{ac}}/K_{\mathrm{ur}}) \subset$

$G_K = \mathrm{Gal}(K^{\mathrm{ac}}/K)$ be the inertia subgroup. Let l be a prime different from p. We set $t_l : I_K \to \mathbf{Z}_l(1)$ to be the canonical surjection given by

$$\sigma \mapsto (\sigma(\pi^{1/l^m})/\pi^{1/l^m})_m$$

for every $\sigma \in I_K$.

Let $\mathfrak{X}$ be a *strictly semi-stable scheme* over S purely of relative dimension n, which we also assume to be proper. This means that $\mathfrak{X}$ is locally of finite presentation and Zariski locally étale over

$$\mathrm{Spec}(\mathcal{O}_K[X_1, \ldots, X_n]/(X_1 \cdots X_r - \pi))$$

for some integer $1 \le r \le n$. We let X_k be the special fibre of $\mathfrak{X}$ and $X_{k^{\mathrm{ac}}}$ be its base change to k^{ac}. Let $X = \mathfrak{X}_\eta$ be the generic fibre of $\mathfrak{X}$ and $X_{K_{\mathrm{ur}}}$ be its base change to K_{ur}. We have the following natural maps $i : X_k \to \mathfrak{X}$, $j : X \to \mathfrak{X}$, $\bar{i} : X_{k^{\mathrm{ac}}} \to \mathfrak{X}_{\mathcal{O}_{K_{\mathrm{ur}}}}$ and $\bar{j} : X_{K_{\mathrm{ur}}} \to \mathfrak{X}_{\mathcal{O}_{K_{\mathrm{ur}}}}$. We have the *nearby cycle sheaf* given by

$$R^q \Psi(\Lambda) = \bar{i}^* R^q \bar{j}_* \Lambda$$

and the *nearby cycle complex* given by

$$R\Psi(\Lambda) = \bar{i}^* R\bar{j}_* \Lambda.$$

By proper base change theorem, we have $\mathrm{H}^*(X_{k^{\mathrm{ac}}}, R\Psi(\Lambda)) = \mathrm{H}^*(X_{K^{\mathrm{ac}}}, \Lambda)$. We can regard $R\Psi(\Lambda)$ as an object in the derived category $D^+(X_{k^{\mathrm{ac}}}, \Lambda[I_K])$ of sheaves of Λ-modules with continuous I_K-actions. Let $D_1, \ldots, D_m$ be the set of irreducible components of X_k. For each index set $I \subset \{1, \ldots, m\}$ of cardinality p, we set $X_{I,k} = \cap_{i \in I} D_i$. This is a smooth scheme of dimension $n - p$. For $1 \le p \le m - 1$, we define

$$X_k^{(p)} = \bigsqcup_{I \subset \{1,\ldots,m\}, \mathrm{Card}(I)=p+1} X_{I,k}$$

and let $a_p : X_k^{(p)} \to X_k$ be the natural map, so we have $a_{p*}\Lambda = \wedge^{p+1} a_{0*}\Lambda$.

Let T be an element in I_K such that $t_l(T)$ is a generator of $\mathbf{Z}_l(1)$; then T induces a nilpotent operator $T - 1$ on $R\Psi(\Lambda)$. Let $N = (T-1) \otimes \check{T}$ where $\check{T} \in \mathbf{Z}_l(-1)$ is the dual of $t_l(T)$. Then with respect to this N, we have the *monodromy filtration* $M_\bullet R\Psi(\Lambda)$ on $R\Psi(\Lambda)$ characterised by

(1) $M_n R\Psi(\Lambda) = R\Psi(\Lambda)$ and $M_{-n-1} R\Psi(\Lambda) = 0$;

(2) $N : R\Psi(\Lambda)(1) \to R\Psi(\Lambda)$ sends $M_r R\Psi(\Lambda)(1)$ into $M_{r-2} R\Psi(\Lambda)$ for $r \in \mathbf{Z}$;

(3) $N^r : Gr_r^M R\Psi(\Lambda)(r) \to Gr_{-r}^M R\Psi(\Lambda)$ is an isomorphism.

The monodromy filtration induces the *weight spectral sequence*

$$\mathrm{E}_1^{p,q} = \mathrm{H}^{p+q}(X \otimes k^{\mathrm{ac}}, Gr_{-p}^M R\Psi(\Lambda)) \Rightarrow \mathrm{H}^{p+q}(X \otimes k^{\mathrm{ac}}, R\Psi(\Lambda)) = \mathrm{H}^{p+q}(X \otimes K^{\mathrm{ac}}, \Lambda).$$

$$(2.10)$$

The E_1-term of this spectral sequence can be made explicit by

$$\mathrm{H}^{p+q}(X \otimes k^{\mathrm{ac}}, Gr_{-p}^M R\Psi(\Lambda)) = \bigoplus_{i-j=-p, i\geq 0, j\geq 0} \mathrm{H}^{p+q-(i+j)}(X_{k^{\mathrm{ac}}}^{(i+j)}, \Lambda(-i))$$

$$= \bigoplus_{i\geq \max(0,-p)} \mathrm{H}^{q-2i}(X_{k^{\mathrm{ac}}}^{(p+2i)}, \Lambda(-i)).$$

This spectral sequence is first introduced by Rapoport–Zink in [28] and thus is also known as the Rapoport–Zink spectral sequence.

Let $\mathfrak{X}$ be a relative curve over $\mathcal{O}_K$. Then we can immediately calculate that

$$Gr_{-1}^M R\Psi(\Lambda) = a_{1*}\Lambda[-1],$$
$$Gr_0^M R\Psi(\Lambda) = a_{0*}\Lambda,$$
$$Gr_1^M R\Psi(\Lambda) = a_{1*}\Lambda[-1](-1).$$

The E_1-page of the weight spectral sequence is thus given by
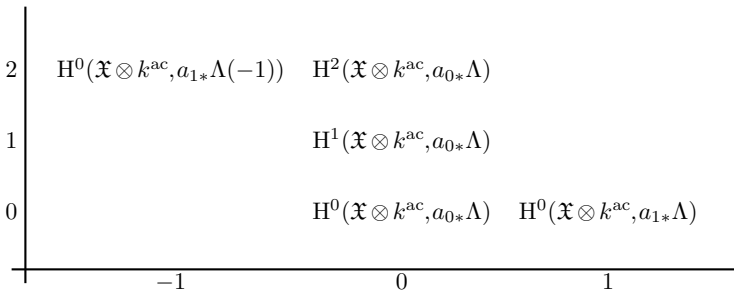
and it clearly degenerates at the E_2-page. We therefore have the monodromy filtration

$$0 \subset^{\mathrm{E}_2^{1,0}} M_1\mathrm{H}^1(\mathfrak{X} \otimes K^{\mathrm{ac}}, \Lambda) \subset^{\mathrm{E}_2^{0,1}} M_0\mathrm{H}^1(\mathfrak{X} \otimes K^{ac}, \Lambda) \subset^{\mathrm{E}_2^{-1,2}} M_{-1}\mathrm{H}^1(\mathfrak{X} \otimes K^{\mathrm{ac}}, \Lambda)$$
$$= \mathrm{H}^1(\mathfrak{X} \otimes K^{\mathrm{ac}}, \Lambda)$$

with the graded pieces given by

$$Gr_{-1}^M\mathrm{H}^1(\mathfrak{X} \otimes K^{\mathrm{ac}}, \Lambda) = \ker[\mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{1*}\Lambda(-1)) \xrightarrow{\tau} \mathrm{H}^2(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{0*}\Lambda)]$$
$$Gr_0^M\mathrm{H}^1(\mathfrak{X} \otimes K^{\mathrm{ac}}, \Lambda) = \mathrm{H}^1(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{0*}\Lambda) \qquad (2.11)$$
$$Gr_1^M\mathrm{H}^1(\mathfrak{X} \otimes K^{\mathrm{ac}}\Lambda) = \mathrm{coker}[\mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{0*}\Lambda) \xrightarrow{\rho} \mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{1*}\Lambda)]$$

where τ is the *Gysin morphism* and ρ is the *restriction morphism*. Note that the monodromy action on $\mathrm{H}^1(X \otimes K^{\mathrm{ac}}, \Lambda(1))$ can be understood using the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{H}^1(\mathfrak{X} \otimes K^{\mathrm{ac}}, \Lambda(1)) & \longrightarrow & \ker[\mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{1*}\Lambda) \xrightarrow{\tau} \mathrm{H}^2(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{0*}\Lambda(1))] \\
\downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} \\
\mathrm{H}^1(\mathfrak{X} \otimes K^{\mathrm{ac}}, \Lambda) & \longleftarrow & \mathrm{coker}[\mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{0*}\Lambda) \xrightarrow{\rho} \mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{1*}\Lambda)].
\end{array}
$$

In this case, we recover the *Picard–Lefschetz formula* if we identify $\mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{1*}\Lambda)$ with the *vanishing cycles* $\bigoplus_x R\Phi(\Lambda)_x$ on $X_{k^{\mathrm{ac}}}$ where x runs through the singular points $X_{k^{\mathrm{ac}}}^{(1)}$ on $X_{k^{\mathrm{ac}}}$. Let M be a G_K-module over Λ; then we have the following exact sequence of Galois cohomology groups:

$$0 \to \mathrm{H}^1_{\mathrm{fin}}(K, \mathrm{M}) \to \mathrm{H}^1(K, \mathrm{M}) \xrightarrow{\partial_p} \mathrm{H}^1_{\mathrm{sing}}(K, \mathrm{M}) \to 0 \tag{2.12}$$

where $\mathrm{H}^1_{\mathrm{fin}}(K, \mathrm{M}) = \mathrm{H}^1(k, \mathrm{M}^{I_K})$ is called the *unramified* or *finite* part of the cohomology group $\mathrm{H}^1(K, \mathrm{M})$ and $\mathrm{H}^1_{\mathrm{sing}}(K, \mathrm{M})$ defined as the quotient of $\mathrm{H}^1(K, \mathrm{M})$ by its finite part is called the *singular quotient* of $\mathrm{H}^1(K, \mathrm{M})$. The natural quotient map $\mathrm{H}^1(K, \mathrm{M}) \xrightarrow{\partial_p} \mathrm{H}^1_{\mathrm{sing}}(K, \mathrm{M})$ will be referred to as the *singular quotient map*. The element $\partial_p(x)$ will be referred to as the singular residue of x for $x \in \mathrm{H}^1(K, \mathrm{M})$. Let $\mathrm{M} = \mathrm{H}^n(X_{K^{\mathrm{ac}}}, \Lambda(r))$ be the rth twist of the middle degree cohomology of $X_{K^{\mathrm{ac}}}$. We need the following elementary lemma.

Lemma 2.7. *Let* $\mathrm{M} = \mathrm{H}^n(X_{K^{\mathrm{ac}}}, \Lambda(r))$*; then we have*

$$\mathrm{H}^1_{\mathrm{fin}}(K, \mathrm{M}) \cong \frac{\mathrm{M}^{I_K}}{(\mathrm{Frob}_p - 1)}, \quad \mathrm{H}^1_{\mathrm{sing}}(K, \mathrm{M}) \cong \left(\frac{\mathrm{M}(-1)}{N\mathrm{M}} \right)^{G_k}.$$

Proof. This is well-known. The details can be found, for example, in [20, Lemma 2.6]. $\quad\square$

For $\mathrm{M} = \mathrm{H}^1(X_{K^{\mathrm{ac}}}, \Lambda(1))$, we can use the Picard–Lefschetz formula to calculate $\mathrm{H}^1_{\mathrm{sing}}(K, \mathrm{M})$; more precisely, we have

$$\mathrm{H}^1_{\mathrm{sing}}(K, \mathrm{M}) \cong \left(\frac{\mathrm{M}(-1)}{N\mathrm{M}} \right)^{G_k} \cong \left(\frac{\mathrm{coker}[\mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{0*}\Lambda) \xrightarrow{\rho} \mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{1*}\Lambda)]}{N\ker[\mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{1*}\Lambda) \xrightarrow{\tau} \mathrm{H}^2(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{0*}\Lambda(1))]} \right)^{G_k}. \tag{2.13}$$

Composing the isomorphism (2.13) with τ, we obtain

$$\mathrm{H}^1_{\mathrm{sing}}(K, \mathrm{M}) \cong \left(\frac{\mathrm{coker}[\mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{0*}\Lambda) \xrightarrow{\rho} \mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{1*}\Lambda)]}{N\ker[\mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{1*}\Lambda) \xrightarrow{\tau} \mathrm{H}^2(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{0*}\Lambda(1))]} \right)^{G_k}$$

$$\cong \mathrm{coker}[\mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{0*}\Lambda) \xrightarrow{\rho} \mathrm{H}^0(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{1*}\Lambda) \xrightarrow{\tau} \mathrm{H}^2(\mathfrak{X} \otimes k^{\mathrm{ac}}, a_{0*}\Lambda(1))]^{G_k}. \tag{2.14}$$

Next we consider the curve $\mathfrak{X}_d(p)$ over $\mathrm{Spec}(\mathbf{Z}_{p^2})$. Let $\mathcal{A}_d(p) \to \mathfrak{X}_d(p)$ be the universal abelian surface over $\mathfrak{X}_d(p)$. Then we define the *Kuga–Sato variety* of weight k by the $\frac{k-2}{2}$-fold fibre product of $\mathcal{A}_d(p)$ over $\mathfrak{X}_d(p)$; that is,

$$\mathcal{W}_{k,d}(p) := \mathcal{A}_d(p) \times_{\mathfrak{X}_d(p)} \cdots \times_{\mathfrak{X}_d(p)} \mathcal{A}_d(p). \tag{2.15}$$

Then there is a semi-stable model $\widetilde{\mathcal{W}}_{k,d}(p)$ constructed in [32, Lemma 7.1] and the action of the projector ϵ_k extends naturally to the semi-stable model $\widetilde{\mathcal{W}}_{k,d}(p)$. Moreover, the first page of the weight spectral sequence converging to

$$\epsilon_k \mathrm{H}^{k-1}\left(\widetilde{\mathcal{W}}_{k,d}(p) \otimes \mathbf{Q}_p^{\mathrm{ac}}, \Lambda\left(\frac{k-2}{2} \right) \right) = \mathrm{H}^1(\mathfrak{X}_d(p) \otimes \mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\Lambda))$$

takes the following form by [32, (8.10)]:

$$
\begin{array}{c|ccc}
2 & \mathrm{H}^0(\overline{X}_d(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{1*}\mathcal{L}_{k-2}(\Lambda)(-1)) & \mathrm{H}^2(\overline{X}_d(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) & \\
1 & & \mathrm{H}^1(\overline{X}_d(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) & \\
0 & & \mathrm{H}^0(\overline{X}_d(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) & \mathrm{H}^0(\overline{X}_d(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{1*}\mathcal{L}_{k-2}(\Lambda)) \\
\hline
& -1 & 0 & 1
\end{array}
$$

.

This means that the weight spectral sequence of the Kuga–Sato variety agrees with the weight spectral sequence of the base curve with certain nontrivial coefficients. We have to remark that the author works with a different Shimura curve in [32], but his method adapts to our situation easily and gives the same result. By further applying the projector ϵ_d, we obtain the following first page of the weight spectral sequence converging to

$$
\mathrm{H}^1(\mathfrak{X}_0(p) \otimes \mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\Lambda))
$$

$$
\begin{array}{c|ccc}
2 & \mathrm{H}^0(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{1*}\mathcal{L}_{k-2}(\Lambda)(-1)) & \mathrm{H}^2(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) & \\
1 & & \mathrm{H}^1(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) & \\
0 & & \mathrm{H}^0(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) & \mathrm{H}^0(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{1*}\mathcal{L}_{k-2}(\Lambda)) \\
\hline
& -1 & 0 & 1
\end{array}
$$

.

Note that we can make it explicit for the terms in the above spectral sequence:

(1) $\mathrm{H}^0(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{1*}\mathcal{L}_{k-2}(\Lambda)) = \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\Lambda))$;

(2) $\mathrm{H}^1(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) = \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\Lambda)) \oplus \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\Lambda))$;

(3) $\mathrm{H}^2(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\Lambda))$ is Eisenstein;

(4) $\mathrm{H}^0(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\Lambda))$ is Eisenstein.

2.5. Unramified level raising on the Kuga–Sato varities

Let $f \in S_k^{\mathrm{new}}(N)$ be a newform of level $\Gamma_0(N)$ with even weight k and with Fourier expansion $f = \sum a_n(f)q^n$. We denote by $E = \mathbf{Q}(f)$ the Hecke field of f. Let λ be a place of E over l and E_λ be the completion of E at λ. Let ϖ be a uniformiser of the ring of integers $\mathcal{O} := \mathcal{O}_{E_\lambda}$ of E_λ and $\mathbf{F}_\lambda$ be its residue field. We will set $\mathcal{O}_n = \mathcal{O}/\varpi^n$. Let K be

an imaginary quadratic field whose discriminant is $-D_K$ with $D_K > 0$. We assume that N admits a factorisation $N = N^+N^-$ where N^+ consists of prime factors that are split in K and N^- consists of prime factors that are inert in K. Let $\rho_{f,\lambda} : G_{\mathbf{Q}} \to \mathrm{GL}_2(E_\lambda)$ be the λ-adic Galois representation attached to the form f characterised by the fact that the trace of Frobenius at $p \nmid N$ agrees with $a_p(f)$ and the determinant of $\rho_{f,\lambda}$ is ϵ_l^{k-1} where ϵ_l is the l-adic cyclotomic character. We shall consider the twist $\rho_{f,\lambda}^* = \rho_{f,\lambda}(\frac{2-k}{2})$. Let $V_{f,\lambda}$ be the representation space for $\rho_{f,\lambda}^*$. We normalise the construction of $\rho_{f,\lambda}$ such that it occurs in the cohomology $\mathrm{H}^1(X_{\mathbf{Q}^{\mathrm{ac}}}, \mathcal{L}_{k-2}(E_\lambda)(\frac{k}{2}))$ and therefore $\rho_{f,\lambda}^*$ occurs in the cohomology $\mathrm{H}^1(X_{\mathbf{Q}^{\mathrm{ac}}}, \mathcal{L}_{k-2}(E_\lambda)(1))$. Let $\mathbb{T} = \mathbb{T}(N^+, N^-)$ be the l-adic completion of the integral Hecke algebra that acts faithfully on the subspace of $S_k(N)$ consisting of forms that are new at N^- and let $\mathbb{T}^{[p]} = \mathbb{T}(N^+, pN^-)$ be the l-adic completion of the integral Hecke algebra that acts faithfully on the subspace of $S_k(pN)$ consisting of modular forms that are new at pN^-. The modular form f gives rise to a homomorphism $\phi_f : \mathbb{T} \to \mathcal{O}$ corresponding to the Hecke eigensystem of f. For $n \geq 1$, we define $\phi_{f,n} : \mathbb{T} \to \mathcal{O}_n := \mathcal{O}/\varpi^n$ to be the natural reduction of ϕ_f by ϖ^n. We will define $I_{f,n}$ to be the kernel of the morphism $\phi_{f,n}$ and by $\mathfrak{m}_f$ the unique maximal ideal of $\mathbb{T}$ containing $I_{f,n}$. We fix a $G_{\mathbf{Q}}$-stable lattice $\mathrm{T}_{f,\lambda}$ in $V_{f,\lambda}$ and denote by $\mathrm{T}_{f,n}$ the reduction $\mathrm{T}_{f,\lambda} \bmod \varpi^n$. We will always assume that the residual Galois representation $\bar{\rho}_{f,\lambda}$ satisfies the following assumption.

Assumption 5 (CR*). The residual Galois representation $\bar{\rho}_{f,\lambda}$ satisfies the following assumptions:

(1) $k < l - 1$ and $|(\mathbf{F}_l^\times)^{k-1}| > 5$;

(2) $\bar{\rho}_{f,\lambda}$ is absolutely irreducible when restricted to $G_{\mathbf{Q}(\sqrt{p^*})}$ where $p^* = (-1)^{\frac{p-1}{2}} p$;

(3) If $q \mid N^-$ and $q \equiv \pm 1 \bmod l$, then $\bar{\rho}_{f,\lambda}$ is ramified at q;

(4) If $q \parallel N^+$ and $q \equiv 1 \bmod l$, then $\bar{\rho}_{f,\lambda}$ is ramified at q;

(5) The Artin conductor $N_{\bar{\rho}}$ of $\bar{\rho}_{f,\lambda}$ is prime to $N/N_{\bar{\rho}}$;

(6) There is a place $q \parallel N$ such that $\bar{\rho}_{f,\lambda}$ is ramified at q.

We will now prove a level-raising result for the modular form f. First, we recall the following notion of n-admissible prime for f.

Definition 2.8. We say a prime p is n-admissible for f if

(1) $p \nmid Nl$;

(2) p is an inert prime in K;

(3) l does not divide $p^2 - 1$;

(4) ϖ^n divides $p^{\frac{k}{2}} + p^{\frac{k-2}{2}} - \epsilon_p a_p(f)$ with $\epsilon_p \in \{\pm 1\}$.

We consider the special fibre $\overline{W}_{k,d}$ of $\mathcal{W}_{k,d,\mathbf{Z}_{p^2}}$ and define its supersingular locus by $W_{k,d}^{\mathrm{ss}} := \pi_{k,d}^{-1}(X_d^{\mathrm{ss}})$. Similarly, we define the supersingular locus of $\mathcal{W}_{k,d}(p)$ by $W_{k,d}^{\mathrm{ss}}(p) := \pi_{k,d}^{-1}(p)(X_d^{\mathrm{ss}}(p))$. We will consider the following *ordinary-supersingular* excision exact sequence

$$\mathrm{H}^{\frac{k}{2}}\left(\overline{W}_{k,d,\mathbf{F}_p^{\mathrm{ac}}},\Lambda\left(\frac{k}{2}\right)\right) \to \mathrm{H}^{\frac{k}{2}}\left(\overline{W}_{k,d,\mathbf{F}_p^{\mathrm{ac}}} - W_{k,d,\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}},\Lambda\left(\frac{k}{2}\right)\right)$$

$$\to \mathrm{H}^{\frac{k}{2}+1}_{W_{k,d,\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}}\left(\overline{W}_{k,d,\mathbf{F}_p^{\mathrm{ac}}},\Lambda\left(\frac{k}{2}\right)\right).$$

Apply the projector ϵ_k and localise at the maximal ideal $\mathfrak{m}_f$; we in fact obtain an exact sequence

$$0 \to \mathrm{H}^1(\overline{X}_{d,\mathbf{F}_p^{\mathrm{ac}}},\mathcal{L}_{k-2}(\Lambda)(1))_{\mathfrak{m}_f} \to \mathrm{H}^1(X_{d,\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ord}},\mathcal{L}_{k-2}(\Lambda)(1))_{\mathfrak{m}_f} \to \mathrm{H}^0(X_{d,\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}},\mathcal{L}_{k-2}(\Lambda))_{\mathfrak{m}_f} \to 0.$$

This follows from Lemma 2.2 and the following equations:

(1) $\mathrm{H}^{\frac{k}{2}+1}_{W_{k,d}^{\mathrm{ss}}}(\mathcal{W}_{k,d}\otimes\mathbf{F}_p^{\mathrm{ac}},\Lambda(\frac{k}{2})) \cong \bigoplus_{x\in X_d^{\mathrm{ss}}} \mathrm{H}^{\frac{k-2}{2}}(\mathcal{A}_{d,x}^{\frac{k-2}{2}},\Lambda(\frac{k-2}{2}))$

(2) $\bigoplus_{x\in X_d^{\mathrm{ss}}} \epsilon_k \mathrm{H}^{\frac{k-2}{2}}(\mathcal{A}_{d,x}^{\frac{k-2}{2}},\Lambda(\frac{k-2}{2})) \cong \bigoplus_{x\in X_d^{\mathrm{ss}}} L_{k-2}(\Lambda)$

by the definition of ϵ_k. Here $\mathcal{A}_{d,x}$ is the fibre of $\pi_d : \mathcal{A}_d \to \mathfrak{X}_d$ above x. The ordinary-supersingular excision exact sequence induces the following connecting homomorphism:

$$\Phi_d : \mathrm{H}^0(X_{d,\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}},\mathcal{L}_{k-2}(\Lambda))_{\mathfrak{m}_f}^{G_{\mathbf{F}_{p^2}}} \to \mathrm{H}^1(\mathbf{F}_{p^2},\mathrm{H}^1(\overline{X}_{d,\mathbf{F}_p^{\mathrm{ac}}},\mathcal{L}_{k-2}(\Lambda)(1))_{\mathfrak{m}_f}).$$

By applying the projector ϵ_d to the above map, we have

$$\Phi : \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}},\mathcal{L}_{k-2}(\Lambda))_{\mathfrak{m}_f}^{G_{\mathbf{F}_{p^2}}} \to \mathrm{H}^1(\mathbf{F}_{p^2},\mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}},\mathcal{L}_{k-2}(\Lambda)(1))_{\mathfrak{m}_f}).$$

By further taking the quotient by $I_{f,n}$, we have

$$\Phi_n : \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}},\mathcal{L}_{k-2}(\Lambda))_{/I_{f,n}}^{G_{\mathbf{F}_{p^2}}} \to \mathrm{H}^1(\mathbf{F}_{p^2},\mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}},\mathcal{L}_{k-2}(\Lambda)(1))_{/I_{f,n}}).$$

The following theorem is known as the arithmetic level raising for the Kuga–Sato variety.

Theorem 2.9. *Let p be an n-admissible prime for f. We assume that the residual Galois representation $\bar{\rho}_{f,\lambda}$ satisfies $(\mathrm{CR}^\star)$. Then we have the following statements:*

(1) *There exists a morphism $\phi_{f,n}^{[p]} : \mathbb{T}^{[p]} \to \mathcal{O}_n$ which agrees with $\phi_{f,n} : \mathbb{T} \to \mathcal{O}_n$ for all of the Hecke operators away from p and sends U_p to $\epsilon_p p^{\frac{k-2}{2}}$.*

(2) *Let $I_{f,n}^{[p]}$ be the kernel of the morphism $\phi_{f,n}^{[p]}$ and $\mathfrak{m}_f^{[p]}$ be the maximal ideal containing $I_{f,n}$. When $n = 1$, there exists a modular form $f^{[p]} \in S_k^{\mathrm{new}}(pN)$ such that the morphism $\phi_{f,1}^{[p]}$ lifts to $f^{[p]}$.*

(3) *The Hecke module $\mathrm{H}^1(\mathfrak{X}\otimes\mathbf{Q}^{\mathrm{ac}},\mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}$ is free of rank 2 over $\mathbb{T}_{\mathfrak{m}_f}$ and the Hecke module $S_k^B(N^+,\mathcal{O})_{\mathfrak{m}_f^{[p]}}$ is free of rank 1 over $\mathbb{T}_{\mathfrak{m}_f^{[p]}}^{[p]}$.*

(4) *We have a canonical isomorphism*

$$\Phi_n : \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}},\mathcal{L}_{k-2}(\mathcal{O}))_{/I_{f,n}}^{G_{\mathbf{F}_{p^2}}} \xrightarrow{\cong} \mathrm{H}^1(\mathbf{F}_{p^2},\mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}},\mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}}) \qquad (2.16)$$

which can be identified with an isomorphism

$$\Phi_n : S_k^B(N^+,\mathcal{O})_{/I_{f,n}^{[p]}} \xrightarrow{\cong} \mathrm{H}^1(\mathbf{F}_{p^2},\mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}},\mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}}). \qquad (2.17)$$

Remark 2.10. We will refer to this theorem as the *unramified arithmetic level raising for the Kuga–Sato varieties*. It addresses a question raised in the introduction of [9] about the surjectivity of the Abel–Jacobi map restricted to the supersingular locus. The proof is inspired by lectures of Liang Xiao [38] at the Morningside center; see also [21].

Proof of Theorem 2.9. We first proceed to show that Φ_n is surjective. We consider the localised weight spectral sequence for $\mathrm{H}^1(\mathfrak{X}_0(p) \otimes \mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}$ and its induced monodromy filtration:

$$0 \overset{\mathrm{E}_{2,\mathfrak{m}_f}^{1,0}}{\subset} M_1\mathrm{H}^1(\mathfrak{X}_0(p)\otimes\mathbf{Q}^{\mathrm{ac}},\mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \overset{\mathrm{E}_{2,\mathfrak{m}_f}^{0,1}}{\subset} M_0\mathrm{H}^1(\mathfrak{X}_0(p)\otimes\mathbf{Q}^{\mathrm{ac}},\mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}$$

$$\overset{\mathrm{E}_{2,\mathfrak{m}_f}^{-1,2}}{\subset} M_{-1}\mathrm{H}^1(\mathfrak{X}_0(p)\otimes\mathbf{Q}^{\mathrm{ac}},\mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}.$$

By (2.11), we have

$$\begin{aligned}
\mathrm{E}_{2,\mathfrak{m}_f}^{1,0} &= \ker[\mathrm{H}^0(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{1*}\mathcal{L}_{k-2}(\mathcal{O})) \overset{\tau}{\to} \mathrm{H}^2(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\mathcal{O})(1))]_{\mathfrak{m}_f} \\
&= \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}; \\
\mathrm{E}_{2,\mathfrak{m}_f}^{0,1} &= \mathrm{H}^1(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \\
&= \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}^{\oplus 2}; \\
\mathrm{E}_{2,\mathfrak{m}_f}^{-1,2} &= \mathrm{coker}[\mathrm{H}^0(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\mathcal{O})(1)) \overset{\rho}{\to} \mathrm{H}^0(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, a_{1*}\mathcal{L}_{k-2}(\mathcal{O})(1))]_{\mathfrak{m}_f} \\
&= \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O}))_{\mathfrak{m}_f}.
\end{aligned}$$

Next we consider the pushforward map

$$\mathrm{H}^1(\mathfrak{X}_0(p)\otimes\mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \xrightarrow{(\pi_{1*},\pi_{2*})} \mathrm{H}^1(\mathfrak{X}\otimes\mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}^{\oplus 2}.$$

This is surjective by 'Ihara's lemma', Theorem 2.5 and Nakayma's lemma. It is well-known that the composite

$$\mathrm{E}_{2,\mathfrak{m}_f}^{1,0} \hookrightarrow \mathrm{H}^1(\mathfrak{X}_0(p)\otimes\mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \xrightarrow{(\pi_{1*},\pi_{2*})} \mathrm{H}^1(\mathfrak{X}\otimes\mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}^{\oplus 2}$$

is zero. Therefore, we obtain the following commutative diagram where we have omitted the coefficient $\mathcal{L}_{k-2}(\mathcal{O})$ in all of the terms:

$$\begin{array}{ccccc}
\mathrm{H}^1(\mathfrak{X}\otimes\mathbf{F}_p^{\mathrm{ac}})(1)_{\mathfrak{m}_f}^{\oplus 2} & \xrightarrow{(i_{1*},i_{2*})} & \mathrm{H}^1(\mathfrak{X}_0(p)\otimes\mathbf{Q}_p^{\mathrm{ac}})(1)_{\mathfrak{m}_f} & \longrightarrow & \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}})_{\mathfrak{m}_f} \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle(\pi_{1*},\pi_{2*})} & & \downarrow{\scriptstyle\Phi'} \\
\mathrm{H}^1(\mathfrak{X}\otimes\mathbf{Q}_p^{\mathrm{ac}})(1)_{\mathfrak{m}_f}^{\oplus 2} & \xrightarrow{\nabla} & \mathrm{H}^1(\mathfrak{X}\otimes\mathbf{Q}_p^{\mathrm{ac}})(1)_{\mathfrak{m}_f}^{\oplus 2} & \longrightarrow & \mathrm{coker}(\nabla)
\end{array}\quad.$$

Here the top row of the diagram is the monodromy filtration of $\mathrm{H}^1(\mathfrak{X}\otimes\mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}$ which is exact on the right. The map Φ' is the one naturally induced by (π_{1*},π_{2*}). The map ∇ is by definition given by the composite of (π_{1*},π_{2*}) and (i_{1*},i_{2*}). By (2.9), the map ∇ is given by the matrix

$$\begin{pmatrix} \mathrm{id} & \mathrm{Frob}_p \\ \mathrm{Frob}_p & \mathrm{id} \end{pmatrix},$$

since the central element S_p has the trivial action. It then follows that we have an isomorphism

$$\mathrm{coker}(\nabla) = \mathrm{H}^1(\mathbf{F}_{p^2}, \mathrm{H}^1(\mathfrak{X} \otimes \mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}).$$

Since (π_{1*}, π_{2*}) is surjective, the map Φ' is surjective as well. Let Φ'_n be the reduction of Φ' modulo $I_{f,n}$. Therefore, we are left to show that Φ'_n agrees with the map Φ_n. To show this, we rely on some results proved in [16]. More precisely, the natural quotient map induced by the monodromy filtration

$$\mathrm{H}^1(\mathfrak{X}_0(p) \otimes \mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \to \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O}))_{\mathfrak{m}_f}$$

factors through $\mathrm{H}^1(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ord}}, \mathcal{L}_{k-2}(\mathcal{O}))$:

$$\mathrm{H}^1(\mathfrak{X}_0(p) \otimes \mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \xrightarrow{i_1^*} \mathrm{H}^1(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ord}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}$$
$$\to \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O}))_{\mathfrak{m}_f} \to 0$$

where the map $\mathrm{H}^1(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ord}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \to \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O}))_{\mathfrak{m}_f}$ comes from the natural excision exact sequence for $\mathrm{H}^1(\mathfrak{X} \otimes \mathbf{F}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}$ and i_1^* is the pullback of the cohomology of nearby cycles

$$\mathrm{H}^1(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, R\Psi(\mathcal{L}_{k-2}(\mathcal{O}))(1))_{\mathfrak{m}_f} \xrightarrow{i_1^*} \mathrm{H}^1(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ord}}, R\Psi(\mathcal{L}_{k-2}(\mathcal{O}))(1))_{\mathfrak{m}_f}.$$

For the proof of these facts, see [16, Proposition 1.3], which extends to nontrivial coefficients. Let $x \in \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O}))_{\mathfrak{m}_f}$ and let $\tilde{x}$ be a preimage of x in $\mathrm{H}^1(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ord}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}$. Since $i_1^* i_{1*}$ is the identity map, we can take $i_{1*}(\tilde{x})$ as a preimage of $\tilde{x}$ in

$$\mathrm{H}^1(\overline{X}_0(p)_{\mathbf{F}_p^{\mathrm{ac}}}, R\Psi(\mathcal{L}_{k-2}(\mathcal{O}))(1))_{\mathfrak{m}_f}.$$

Therefore, for $x \in \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O}))$, we have $\Phi'(x) = (\pi_{1*} i_{1*}(\tilde{x}), \pi_{2*} i_{2*}(\tilde{x})) = (\tilde{x}, \mathrm{Frob}_p(\tilde{x}))$. Since the natural quotient map

$$\mathrm{H}^1(\mathfrak{X} \otimes \mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \oplus \mathrm{H}^1(\mathfrak{X} \otimes \mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \to \mathrm{coker}(\nabla)$$

is given by sending

$$(x,y) \in \mathrm{H}^1(\mathfrak{X} \otimes \mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f} \oplus \mathrm{H}^1(\mathfrak{X} \otimes \mathbf{Q}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{\mathfrak{m}_f}$$

to $(x - \mathrm{Frob}_p(y))$ in light of the definition of ∇, we have $\Phi'(x) = (1 - \mathrm{Frob}_p^2)\tilde{x}$. But this is precisely the definition of $\Phi(x)$. Note that we have an isomorphism

$$\mathrm{T}_{f,n}^r \cong \mathrm{H}^1(\mathfrak{X} \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}}$$

for some positive integer r. By Definition 2.8 (3), we have $\mathrm{T}_{f,n}|_{G_{\mathbf{Q}_p}} \cong \mathcal{O}_n(1) \oplus \mathcal{O}_n$. Then it follows that

$$\mathrm{H}^1(\mathbf{F}_{p^2}, \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}}) \cong \mathrm{H}^1(\mathbf{F}_{p^2}, \mathcal{O}_n(1) \oplus \mathcal{O}_n)^r.$$

Therefore, Frob_p acts by ϵ_p on $\mathrm{H}^1(\mathbf{F}_{p^2}, \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}})$. By [30, Proposition 3.8], which also applies to our more general setting, we know that Frob_p acts by U_p on X^{ss} and thus by $p^{\frac{2-k}{2}}U_p$ on $\mathrm{H}^0(X^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O}))$. From the above discussion and by identifying $S_k^B(N^+, \mathcal{O})$ with $\mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}, \mathcal{L}_{k-2}(\mathcal{O}))$, we conclude that we have a surjective morphism

$$\Phi_n : S_k^B(N^+, \mathcal{O}) \twoheadrightarrow \mathrm{H}^1(\mathbf{F}_{p^2}, \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}})$$
$$\cong \mathrm{H}^1(\mathbf{F}_{p^2}, \mathcal{O}_n(1) \oplus \mathcal{O}_n)^r$$
$$\cong \mathcal{O}_n^r.$$

This gives us the desired morphism $\phi_{f,n}^{[p]} : \mathbb{T}^{[p]} \to \mathcal{O}_n$ by projecting to any copy of $\mathcal{O}_n^r$ in the above equation. This finishes the proof of (1).

The statement in (2) follows from the main results of [12, Theorem 2] with trivial modification to cover the higher weight case. Note that when $n \geq 2$, this lift may not exist.

The statement in (3) follows from [8, Proposition 6.8] and the slight modification in [7, Proposition 5.9], which replaces the ordinary local condition at l by the more general local conditions given by Fontaine–Laffaille theory. It follows then $S_k^B(N^+, \mathcal{O})/I_{f,n}^{[p]}$ is of rank 1 over $\mathcal{O}_n$. Since we have a surjective map $\Phi_n : S_k^B(N^+, \mathcal{O}) \twoheadrightarrow \mathrm{H}^1(\mathbf{F}_{p^2}, \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}})$, the rank of $\mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}}$ has to be 2.

The statement in (4) follows from the previous discussions. More precisely, by (3), the module $S_k^B(N^+, \mathcal{O})_{/I_{f,n}^{[p]}}$ is free of rank 1 over $\mathcal{O}_n$ and we have a surjective map Φ_n. This concludes the proof of this theorem.

2.6. Ramified level raising on Kuga–Sato varieties

Let B'' be the indefinite quaternion algebra with discriminant $pp'N^-$. Let $\mathcal{O}_{B'',N^+}$ be an Eichler order of level N^+ contained in a fixed maximal order $\mathcal{O}_{B''}$. Then we define the Shimura curve $X'' = X_{N^+, pp'N^-}^{B''}$ the same way as we define $X = X_{N^+, N^-}^{B'}$. Then we define an integral model $\mathfrak{X}''$ of X'' over $\mathbf{Z}_{p'}$. For a $\mathbf{Z}_{p'}$-scheme S, $\mathfrak{X}''$ is the set of triples (A, ι, C) where

(1) A is an S-abelian scheme of relative dimension 2;

(2) $\iota : \mathcal{O}_{B''} \hookrightarrow \text{End}_S(A)$ is an embedding which is special in the sense of [4, pg. 131-132];

(3) C is an $\mathcal{O}_{B''}$-stable locally cyclic subgroup of $A[N^+]$ of order N^{+2}.

This moduli problem is coarsely represented by a projective scheme $\mathfrak{X}''$ of relative dimension 1 over $\mathbf{Z}_{p'}$. We can similarly rigidify the moduli problem by adding a full level-d structure for a square-free positive integer d with $(d, pp'N) = 1$ as we did in (2.2). Then we will write the resulting moduli problem by $\mathfrak{X}_d''$. The formal completion of $\mathfrak{X}_d''$ along the special fibre at p' admits the Cerednick–Drinfeld uniformisation after base change to $\mathbf{Z}_{p'^2}$. The Cerednick–Drinfeld uniformisation theorem asserts that the $\mathfrak{X}_d''^\wedge$ can be uniformised by the formal scheme $\mathcal{M}$, which is a disjoint union of the Drinfeld upper half-planes

$$\mathfrak{X}_d''^\wedge \xrightarrow{\sim} G(\mathbf{Q}) \backslash \mathcal{M} \times G(\mathbf{A}^{(\infty, p')})/K_d^{p'}. \tag{2.18}$$

Here K is the open compact subgroup given by the Eichler order $\mathcal{O}_{B,N+}$ and K_d is given by

$$K_d = \{g = (g_v)_v \in K : g_v \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod v \text{ for all } v \mid d\}.$$

Let $\overline{X}''_d$ be the special fibre of $\mathfrak{X}''_d$. Then we have the following proposition.

Proposition 2.11. *We have the following descriptions of the scheme $\overline{X}''_d$:*

(1) *The scheme $\overline{X}''_d$ is a union of $\mathbf{P}^1$-bundles over Shimura sets*

$$\overline{X}''_d = \mathbf{P}^1(X^B_+) \cup \mathbf{P}^1(X^B_-)$$

where both X^B_+ and X^B_- are isomorphic to the Shimura set X^B_d as in (2.8).
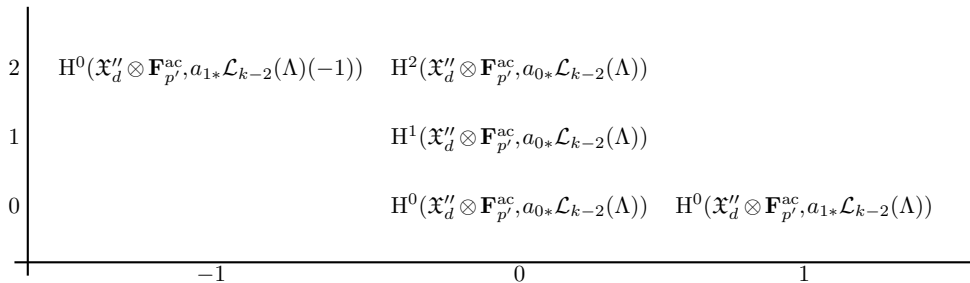
(2) *The intersection points of the two $\mathbf{P}^1$-bundles $\mathbf{P}^1(X^B_+)$ and $\mathbf{P}^1(X^B_-)$ are given by*

$$\mathbf{P}^1(X^B_+) \cap \mathbf{P}^1(X^B_-) = X^B_d(p).$$

This also can be identified with the set of singular points on $\overline{X}''_d$. A similar statement holds for the curve $\overline{X}''$ replacing X^B_d by X^B and $X^B_d(p)$ by $X^B_0(p)$.
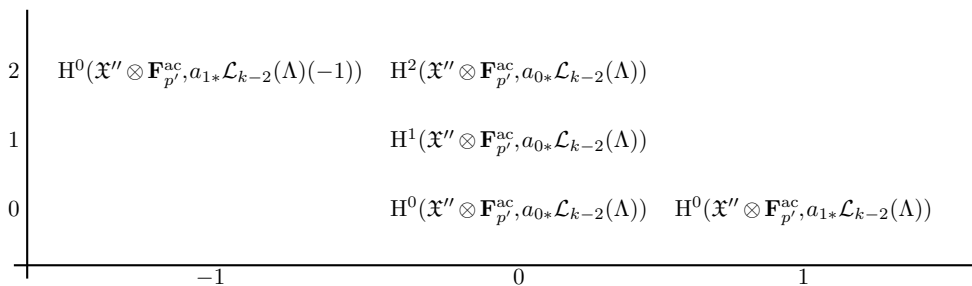
Proof. This is well-known. See [36, Proposition 3.2] for an exposition of this result. $\square$

Let $\pi''_{k,d} : \mathcal{W}''_{k,d} \to \mathfrak{X}''_d$ be the Kuga–Sato variety defined similarly as in (2.15). Then there is a semi-stable model $\widetilde{\mathcal{W}}''_{k,d}$ constructed in [32, Lemma 7.1] and the action of the projector ϵ_k extends naturally to the semi-stable model $\widetilde{\mathcal{W}}''_{k,d}$. Moreover, the first page of the weight spectral sequence converges to $\epsilon_k \mathrm{H}^{k-1}(\widetilde{\mathcal{W}}''_{k,d} \otimes \mathbf{Q}^{\mathrm{ac}}_{p'}, \Lambda) = \mathrm{H}^1(\mathfrak{X}''_d \otimes \mathbf{Q}^{\mathrm{ac}}_{p'}, \mathcal{L}_{k-2}(\Lambda))$ and takes the following form by [32, (8.10)]

$$
\begin{array}{c|ccc}
2 & \mathrm{H}^0(\mathfrak{X}''_d \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{1*}\mathcal{L}_{k-2}(\Lambda)(-1)) & \mathrm{H}^2(\mathfrak{X}''_d \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) & \\
1 & & \mathrm{H}^1(\mathfrak{X}''_d \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) & \\
0 & & \mathrm{H}^0(\mathfrak{X}''_d \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) & \mathrm{H}^0(\mathfrak{X}''_d \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{1*}\mathcal{L}_{k-2}(\Lambda)) \\
\hline
 & -1 & 0 & 1
\end{array}
$$

This means that the weight spectral sequence of the Kuga–Sato variety agrees with the weight spectral sequence of the base curve with certain nontrivial coefficients. By further applying the projector ϵ_d, we obtain the first page of the weight spectral sequence converging to

$$\mathrm{H}^1(\mathfrak{X}'' \otimes \mathbf{Q}^{\mathrm{ac}}_{p'}, \mathcal{L}_{k-2}(\Lambda)) \tag{2.19}$$

The spectral sequence diagram:

2	$\mathrm{H}^0(\mathfrak{X}'' \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{1*}\mathcal{L}_{k-2}(\Lambda)(-1))$	$\mathrm{H}^2(\mathfrak{X}'' \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{0*}\mathcal{L}_{k-2}(\Lambda))$	
1		$\mathrm{H}^1(\mathfrak{X}'' \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{0*}\mathcal{L}_{k-2}(\Lambda))$	
0		$\mathrm{H}^0(\mathfrak{X}'' \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{0*}\mathcal{L}_{k-2}(\Lambda))$	$\mathrm{H}^0(\mathfrak{X}'' \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{1*}\mathcal{L}_{k-2}(\Lambda))$
	-1	0	1

.

Note that we can make it explicit for the terms in the above spectral sequence:

(1) $\mathrm{H}^0(\mathfrak{X}'' \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{1*}\mathcal{L}_{k-2}(\Lambda)) = \mathrm{H}^0(X_0^B(p)_{\mathbf{F}^{\mathrm{ac}}_{p'}}, \mathcal{L}_{k-2}(\Lambda))$;

(2) $\mathrm{H}^1(\mathfrak{X}'' \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) = 0$;

(3) $\mathrm{H}^0(\mathfrak{X}'' \otimes \mathbf{F}^{\mathrm{ac}}_{p'}, a_{0*}\mathcal{L}_{k-2}(\Lambda)) = \mathrm{H}^0(X^B_{\mathbf{F}^{\mathrm{ac}}_{p'}}, \mathcal{L}_{k-2}(\Lambda))^{\oplus 2}$.

Let $\mathbb{T}^{[pp']}$ be the l-adic completion of the integral Hecke algebra that acts faithfully on the subspace of $S_k(pp'N)$ consisting of forms that are new at $pp'N^-$.

Theorem 2.12. *Let (p,p') be a pair of distinct n-admissible primes for f. We assume that the residual Galois representation $\bar{\rho}_{f,\lambda}$ satisfies the assumption $(\mathrm{CR}^\star)$. Then we have the following statements:*

(1) *There exists a surjective homomorphism $\phi^{[pp']}_{f,n} : \mathbb{T}^{[pp']} \to \mathcal{O}_n$ such that $\phi^{[pp']}_{f,n}$ agrees with $\phi_{f,n}$ at all Hecke operators away from pp' and sends $(U_p, U_{p'})$ to $(\epsilon_p p^{\frac{k-2}{2}}, \epsilon_{p'} p'^{\frac{k-2}{2}})$. We will denote by $I^{[pp']}_{f,n}$ the kernel of $\phi^{[pp']}_{f,n}$.*

(2) *We have an isomorphism of $\mathcal{O}_n$-modules of rank 1*
$$\Xi_n : S^B_k(N^+, \mathcal{O})/I^{[p]}_{f,n} \xrightarrow{\cong} \mathrm{H}^1_{\mathrm{sing}}(\mathbf{Q}_{p'^2}, \mathrm{H}^1(\mathfrak{X}'' \otimes \mathbf{Q}^{\mathrm{ac}}_{p'}, \mathcal{L}_{k-2}(\mathcal{O})(1))/I^{[pp']}_{f,n}).$$

Proof. Following the formulas proved in (2.14), we have that
$$\mathrm{H}^1_{\mathrm{sing}}(\mathbf{Q}_{p'^2}, \mathrm{H}^1(\mathfrak{X}'' \otimes \mathbf{Q}^{\mathrm{ac}}_{p'}, \mathcal{L}_{k-2}(\mathcal{O})(1))) \tag{2.20}$$

is isomorphic to

$$\mathrm{coker}[\mathrm{H}^0(X''_{\mathbf{F}^{\mathrm{ac}}_{p'}}, a_{0*}\mathcal{L}_{k-2}(\mathcal{O})) \xrightarrow{\rho} \mathrm{H}^0(X''_{\mathbf{F}^{\mathrm{ac}}_{p'}}, a_{1*}\mathcal{L}_{k-2}(\mathcal{O})) \xrightarrow{\tau} \mathrm{H}^2(X''_{\mathbf{F}^{\mathrm{ac}}_{p'}}, a_{0*}\mathcal{L}_{k-2}(\mathcal{O}(1)))]^{G_{\mathbf{F}_{p'^2}}}. \tag{2.21}$$

Here we have

$$\mathrm{H}^0(X''_{\mathbf{F}^{\mathrm{ac}}_{p'}}, a_{0*}\mathcal{L}_{k-2}(\mathcal{O})) = \mathrm{H}^0(X^B_{\mathbf{F}^{\mathrm{ac}}_{p'}}, \mathcal{L}_{k-2}(\mathcal{O}))^{\oplus 2}$$

and we can identify it with the space $S^B_k(N^+, \mathcal{O})^{\oplus 2}$. Similarly, under the Poincare duality, we can also identify

$$\mathrm{H}^2(X''_{\mathbf{F}^{\mathrm{ac}}_{p'}}, a_{0*}\mathcal{L}_{k-2}(\mathcal{O})(1)) = \mathrm{H}^2(\mathbf{P}^1(X^B_{\mathbf{F}^{\mathrm{ac}}_{p'}}), \mathcal{L}_{k-2}(\mathcal{O})(1))^{\oplus 2}$$

with $S_k^B(N^+,\mathcal{O})^{\oplus 2}$. The space $\mathrm{H}^0(X''_{\mathbf{F}_{p'}^{\mathrm{ac}}}, a_{1*}\mathcal{L}_{k-2}(\mathcal{O}))$ can be identified with $S_k^B(pN^+,\mathcal{O})$. Under these identifications, the composition

$$\mathrm{H}^0(X''_{\mathbf{F}_{p'}^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\mathcal{O})) \xrightarrow{\rho} \mathrm{H}^0(X''_{\mathbf{F}_{p'}^{\mathrm{ac}}}, a_{1*}\mathcal{L}_{k-2}(\mathcal{O})) \xrightarrow{\tau} \mathrm{H}^2(X''_{\mathbf{F}_{p'}^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\mathcal{O}(1)))$$

is given by the *intersection matrix*

$$\begin{pmatrix} -p'^{\frac{k-2}{2}}(p'+1) & T_{p'} \\ T_{p'} & -p'^{\frac{k-2}{2}}(p'+1) \end{pmatrix},$$

which we will also denote by ∇. Since p,p' are n-admissible for f, the module

$$\mathrm{coker}\left[S_k^B(N^+,\mathcal{O})^{\oplus 2}_{/I_{f,n}^{[p]}} \xrightarrow{\nabla} S_k^B(N^+,\mathcal{O})^{\oplus 2}_{/I_{f,n}^{[p]}} \right]$$

is of rank 1 over $\mathcal{O}_n$ and is isomorphic to $S_k^B(N^+,\mathcal{O})_{/I_{f,n}^{[p]}}$. Note here the isomorphism between

$$\mathrm{coker}\left[S_k^B(N^+,\mathcal{O})^{\oplus 2}_{/I_{f,n}^{[p]}} \xrightarrow{\nabla} S_k^B(N^+,\mathcal{O})^{\oplus 2}_{/I_{f,n}^{[p]}} \right]$$

and $S_k^B(N^+,\mathcal{O})^{\oplus 2}_{/I_{f,n}^{[p]}}$ is induced by the map $(x,y) \mapsto \frac{1}{2}(x+\epsilon_{p'}y)$ for $(x,y) \in S_k^B(N^+,\mathcal{O})^{\oplus 2}_{/I_{f,n}^{[p]}}$. By [1, Theorem 5.8] and [9, §3.5] adapted to the higher weight case, the natural U_p-action on

$$\mathrm{H}^2(X''_{\mathbf{F}_{p'}^{\mathrm{ac}}}, a_{0*}\mathcal{L}_{k-2}(\mathcal{O})(1)) \cong S_k^B(N^+,\mathcal{O})^{\oplus 2}$$

is given by $(x,y) \mapsto (-p'^{\frac{k}{2}}y, p'^{\frac{k-2}{2}}x + T_{p'}y)$. We consider the automorphism

$$\delta : S_k^B(N^+,\mathcal{O})^{\oplus 2} \to S_k^B(N^+,\mathcal{O})^{\oplus 2}$$

given by $(x,y) \mapsto (p'^{\frac{k-2}{2}}x + T_{p'}y, p'^{\frac{k-2}{2}}y)$. Then a quick calculation gives us that $\nabla \circ \delta = p'^{-(k-2)}U_{p'}^2 - 1$. This means that the quotient

$$\frac{S_k^B(N^+,\mathcal{O})^{\oplus 2}}{(I_{f,n}^{[p]}, U_{p'}^2 - p'^{k-2})} \cong \mathrm{coker}\left[S_k^B(N^+,\mathcal{O})^{\oplus 2}_{/I_{f,n}^{[p]}} \xrightarrow{\nabla} S_k^B(N^+,\mathcal{O})^{\oplus 2}_{/I_{f,n}^{[p]}} \right]$$

is of rank 1. Since p' is n-admissible for f, we see immediately that $U_{p'} + \epsilon_{p'}p'^{\frac{k-2}{2}}$ is invertible on $S_k^B(N^+,\mathcal{O})^{\oplus 2}_{/I_{f,n}^{[p']}}$. Therefore, we have

$$\frac{S_k^B(N^+,\mathcal{O})^{\oplus 2}}{(I_{f,n}^{[p]}, U_{p'}^2 - p'^{k-2})} \cong \frac{S_k^B(N^+,\mathcal{O})^{\oplus 2}}{(I_{f,n}^{[p]}, U_{p'} - \epsilon_{p'}p'^{\frac{k-2}{2}})}$$

and the latter quotient is of rank 1 over $\mathcal{O}_n$. Then the action of $\mathbb{T}^{[pp']}$ on this rank 1 quotient gives the desired morphism $\phi_{f,n}^{[pp']} : \mathbb{T}^{[pp']} \to \mathcal{O}_n$. This finishes the proof of the part (1). Part (2) follows directly from part (1) using the isomorphism between (2.20) and (2.21). $\qquad\square$

Remark 2.13. We remark that a similar ramified arithmetic level-raising theorem was first proved by Chida in [7, Theorem 5.11].

3. Heegner cycles over Shimura curves

3.1. Heegner cycles over Shimura curves

Let K be an imaginary quadratic field with discriminant $-D_K < 0$ and set $\delta_K = \sqrt{-D_K}$. Let $z \mapsto \bar{z}$ be the complex conjugate action on K. We define $\boldsymbol{\theta}$ by

$$\boldsymbol{\theta} = \frac{D' + \delta_K}{2}, \quad D' = \begin{cases} D_K & \text{if } 2 \nmid D_K; \\ D_K/2 & \text{if } 2 \mid D_K. \end{cases}$$

We always fix a positive integer N such that $N = N^+ N^-$ with N^+ consists of prime factors that are split in K, while N^- consists of prime factors that are inert in K. We will assume the following generalised Heegner hypothesis:

N^- is square-free and consists of even number of prime factors that are inert in K.

$$\text{(Heeg)}$$

Let B' be the indefinite quaternion algebra of discriminant N^-. We can regard K as a subalgebra of B' via an embedding $\iota : K \hookrightarrow B'$. Let m be a positive integer such that $(m, Nl) = 1$. We will chose an element J such that

$$B' = K \oplus K \cdot J \tag{3.1}$$

and satisfies the following properties:

(1) $J^2 = \beta \in \mathbf{Q}^\times$ with $\beta < 0$ and $Jt = \bar{t}J$ for all $t \in K$;
(2) $\beta \in (\mathbf{Z}_q^\times)^2$ for all $q \mid N^+$ and $\beta \in \mathbf{Z}_q^\times$ for $q \mid D_K$.

We define $\varsigma_q \in G'(\mathbf{Q}_q)$ as follows:

$$\varsigma_q = \begin{cases} 1 & \text{if } q \nmid mN^+; \\ \delta^{-1} \begin{pmatrix} \boldsymbol{\theta} & \bar{\boldsymbol{\theta}} \\ 1 & 1 \end{pmatrix} & \text{if } q = \mathfrak{q}\bar{\mathfrak{q}} \text{ is split with } \mathfrak{q} \mid \mathfrak{N}^+; \\ \begin{pmatrix} q^n & 0 \\ 0 & 1 \end{pmatrix} & \text{if } q \mid m \text{ and } q \text{ is inert in } K \text{ with } n = \operatorname{ord}_q(m); \\ \begin{pmatrix} 1 & q^{-n} \\ 0 & 1 \end{pmatrix} & \text{if } q \mid m \text{ and } q \text{ is split in } K \text{ with } n = \operatorname{ord}_q(m), \end{cases} \tag{3.2}$$

and we define the element $\varsigma \in G'(\mathbf{A}^{(\infty)})$ by $\varsigma = \prod_q \varsigma_q$.

Recall that we have defined the *Atkin–Lehner involution* at q to be

$$\tau_q = \begin{cases} \begin{pmatrix} 0 & 1 \\ -N^+ & 0 \end{pmatrix} & \text{for } q \mid N^+; \\ J & \text{for } q \mid \infty N^-; \\ 1 & \text{for } q \nmid N. \end{cases} \tag{3.3}$$

We abuse the notation and put $\tau^{N^+} = \prod_{q \text{ finite}} \tau_q$ as an element in $G'(\mathbf{A}^{(\infty)})$. Next we define the set of CM points on X. We define z' to be the fixed point in $\mathcal{H}^{\pm}$ by $\iota_{\infty}(K) \subset \mathrm{GL}_2(\mathbf{R})$. We define the set of CM points on the Shimura curve X by

$$\mathrm{CM}_K(X) = \{[z',b']_{\mathbf{C}} : b' \in G'(\mathbf{A}^{(\infty)})\}.$$

Let $\mathrm{rec}_K : \widehat{K}^{\times} \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ be the geometrically normalised reciprocity law. The Shimura reciprocity law says that

$$\mathrm{rec}_K(a)[z',b'] = [z',\iota(a)b'].$$

Let m be a positive integer that is prime to N. Let $\mathcal{O}_{K,m} = \mathbf{Z} + m\mathcal{O}_K$ be the order of K with conductor m. Let $\mathcal{G}_m = K^{\times}\backslash\widehat{K}^{\times}/\widehat{\mathcal{O}}_{K,m}^{\times}$ be the Galois group of the ring class field K_m of conductor m over K. Let $a \in \widehat{K}^{\times}$ and p be a prime which is inert in K; then we define the *CM point of level m* on X by

$$P_m(a) = [z',a^{(p)}\varsigma\tau^{N^+}]_{\mathbf{C}} \in X(\mathbf{C}). \tag{3.4}$$

We set $P_m := P_m(1)$ and call this the *Heegner point of level m*. By definition, this gives a point in $X(K_m)$ which has the following moduli interpretation. The point P_m corresponds to a triple (A_m,ι_m,C_m) such that $\mathrm{End}(A_m,\iota_m)$ is isomorphic to $\mathcal{O}_{K,m}$. We let $(\tilde{P}_m(a),\tilde{P}_m)$ be an arbitrary lift of $(P_m(a),P_m)$ in $X_d(K_m)$.

We define Heegner cycles over X_d and X following [25] in the classical modular curve case and [17], [7] in the Shimura curve case. Consider the point $\tilde{P}_m = (A_m,\iota_m,C_m,\nu_m)$ and the Neron–Severi group $\mathrm{NS}(A_m)$ of A_m. There is a natural action of $B'^{\times}$ on $\mathrm{NS}(A_m)_{\mathbf{Q}}$ given by $\mathcal{L} \cdot b = \iota_m(b)^*\mathcal{L}$. Since A_m admits an action of $\mathcal{O}_{B'} \otimes \mathcal{O}_{K,m} \cong \mathrm{M}_2(\mathcal{O}_{K,m})$, it is clear that A_m is isomorphic to a product $E_m \times E_m$ with E_m an elliptic curve with CM by $\mathcal{O}_{K,m}$. Let Γ_m be the graph of $m\sqrt{-D_K}$ in $A_m = E_m \times E_m$. Then we define Z_m to be the image of the divisor given by $[\Gamma_m] - [E_m \times 0] - mD_K[0 \times E_m]$ in $\mathrm{NS}(A_m)$. It lies in the rank 1 submodule of $\mathrm{NS}(A_m)$ generated by $\langle [0 \times E_m],[E_m \times 0],\Delta_m \rangle$ where Δ_m is the diagonal. Let $y_m \in \mathrm{NS}(A_m) \otimes \mathbf{Z}_l$ be the class representing $m^{-1}Z_m$. This is the unique class up to sign satisfying

(1) $\iota_m(b)^*(y_m) = \mathrm{Nrd}(b)y_m$ for any $b \in B'$;
(2) The self-intersection number of y_m is $2D_K$.

Taking the $\frac{k-2}{2}$th exterior product of the element $\epsilon_2 y_m \in \epsilon_2\mathrm{NS}(A_m) \otimes \mathbf{Z}_l \cong \epsilon_2\mathrm{CH}^1(A_m) \otimes \mathbf{Z}_l$, we obtain an element $\epsilon_k y_m^{\frac{k-2}{2}} \in \epsilon_k\mathrm{CH}^{\frac{k-2}{2}}(A_m^{\frac{k-2}{2}}) \otimes \mathbf{Z}_l$. Denote by the embedding $j_{k,d}$:

$A_m^{\frac{k-2}{2}} \hookrightarrow \mathcal{W}_{k,d}$ which is of codimension 1. We have the pushforward map

$$\epsilon_k \mathrm{CH}^{\frac{k-2}{2}}(A_m^{\frac{k-2}{2}}) \otimes \mathbf{Z}_l \xrightarrow{j_{k,d*}} \epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d}) \otimes \mathbf{Z}_l. \tag{3.5}$$

Then we define the *Heegner cycle* $Y_{m,k}$ in $\epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d} \otimes K_m) \otimes \mathbf{Z}_l$ by

$$Y_{m,k} := j_{k,d*}(\epsilon_k y_m^{\frac{k-2}{2}}). \tag{3.6}$$

Next we consider the Abel–Jacobi map for X_d and the local system $\mathcal{L}_{k-2}$:

$$\mathrm{AJ}_{k,d} : \epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d} \otimes K_m) \otimes \mathbf{Z}_l \to \mathrm{H}^1\left(K_m, \epsilon_k \mathrm{H}^{k-1}\left(\mathcal{W}_{k,d} \otimes \mathbf{Q}^{\mathrm{ac}}, \mathbf{Z}_l\left(\frac{k}{2}\right)\right)\right)$$
$$\cong \mathrm{H}^1(K_m, \mathrm{H}^1(\mathfrak{X}_d \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathbf{Z}_l)(1))).$$

We can further apply the projector ϵ_d, and it induces the following Abel–Jacobi map for X and the local system $\mathcal{L}_{k-2}$:

$$\mathrm{AJ}_k : \epsilon_d \epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d} \otimes K_m) \otimes \mathbf{Z}_l \to \mathrm{H}^1(K_m, \mathrm{H}^1(\mathfrak{X} \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathbf{Z}_l)(1))).$$

Finally, we compose this map with the canonical map from $\mathrm{H}^1(\mathfrak{X} \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathbf{Z}_l)(1))$ to

$$\mathrm{H}^1(\mathfrak{X} \otimes \mathbf{Q}^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathbf{Z}_l)(1))_{\mathfrak{m}_f} \otimes_{\mathbb{T}_{\mathfrak{m}_f}} \mathcal{O} \cong \mathrm{T}_{f,\lambda}$$

where the tensor product is induced by $\phi_f : \mathbb{T}_{\mathfrak{m}_f} \to \mathcal{O}$. We therefore have the following Abel–Jacobi map for the representation $\mathrm{T}_{f,\lambda}$:

$$\mathrm{AJ}_{f,k} : \epsilon_d \epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d} \otimes K_m) \otimes \mathbf{Z}_l \to \mathrm{H}^1(K_m, \mathrm{T}_{f,\lambda}). \tag{3.7}$$

We will define the *level m Heegner cycle class* by

$$\kappa(m) := \mathrm{AJ}_{f,k}(\epsilon_d \epsilon_k Y_{m,k}) \in \mathrm{H}^1(K_m, \mathrm{T}_{f,\lambda}).$$

We will refer to the following class simply as the *Heegner cycle class*:

$$\kappa := \mathrm{Cor}_{K_1/K}\kappa(1) \in \mathrm{H}^1(K, \mathrm{T}_{f,\lambda}). \tag{3.8}$$

For $n \geq 1$, we define similarly the Abel–Jacobi map for the representation $\mathrm{T}_{f,n}$:

$$\mathrm{AJ}_{k,n} : \epsilon_d \epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d} \otimes K_m) \otimes \mathbf{Z}_l \to \mathrm{H}^1(K_m, \mathrm{T}_{f,n}). \tag{3.9}$$

Reducing the classes $\kappa(m)$ and κ modulo ϖ^n, we define

$$\kappa_n(m) \in \mathrm{H}^1(K_m, \mathrm{T}_{f,n});$$
$$\kappa_n \in \mathrm{H}^1(K, \mathrm{T}_{f,n}). \tag{3.10}$$

3.2. Theta element and special value formula

Let p be a prime away from N and consider the definite quaternion algebra B over $\mathbf{Q}$ with discriminant pN^-. We denote by G the algebraic group over $\mathbf{Q}$ given by $B^\times$. We will choose the element J' as in (3.1) such that $B = K \oplus K \cdot J'$. For each $a \in \widehat{K}$, we define the *Gross points* of conductor m associated to K by

$$x_m(a) := a \cdot \varsigma \in G(\mathbf{A}). \tag{3.11}$$

Recall that we have the fixed embedding $\iota_l : \mathbf{Q}^{\mathrm{ac}} \hookrightarrow \mathbf{C}_l$ and it induces the place $\mathfrak{l}$ of K and the place λ of $\mathbf{Q}^{\mathrm{ac}}$. We define an embedding

$$i_K : B \to \mathrm{M}_2(K), \ a + bJ' \mapsto \begin{pmatrix} a & b\beta \\ \bar{b} & \bar{a} \end{pmatrix}, \ a,b \in K \tag{3.12}$$

and let $i_{\mathbf{C}} := \iota_\infty \circ i_K$ and $i_{K_{\mathfrak{l}}} = \iota_l \circ i_K$ be the composition. Let $\rho_{k,\infty}$ be the representation

$$\rho_{k,\infty} : G(\mathbb{R}) \xrightarrow{i_{\mathbf{C}}} \mathrm{GL}_2(\mathbf{C}) \to \mathrm{Aut}_{\mathbf{C}} L_{k-2}(\mathbf{C}). \tag{3.13}$$

Then $\mathbf{C} \cdot \mathbf{v}_r$ is the line on which $\rho_{k,\infty}(t)$ acts by $(\bar{t}/t)^r$ for $t \in (K \otimes \mathbf{C})^\times$. For a K-algebra A we define the space $\mathbf{S}_k^B(U,A)$ of modular forms on B of weight k and level U to be

$$\{ h : G(\mathbf{A}^{(\infty)}) \to L_{k-2}(A) : h(\alpha g u) = \rho_{k,\infty}(\alpha) h(g) \text{ for } \alpha \in G(\mathbf{Q}) \text{ and } u \in U \}.$$

Let $\mathbf{S}_k^B(\mathbf{C}) = \varinjlim_U \mathbf{S}_k^B(U,\mathbf{C})$ and $\mathcal{A}(G)$ be the space of automorphic forms on $G(\mathbf{A})$. We define a morphism

$$\Psi : L_{k-2}(\mathbf{C}) \otimes \mathbf{S}_k^B(\mathbf{C}) \to \mathcal{A}(G)$$

by the following recipe:

$$\Psi(\mathbf{v} \otimes f)(g) := \langle \rho_{k,\infty}(g_\infty) \mathbf{v}, f(g^\infty) \rangle_{k-2}$$

for $\mathbf{v} \in L_{k-2}(\mathbf{C})$. Let π be the automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ corresponding to $f^{[p]}$ and π' be the automorphic representation of $G(\mathbf{A})$ that corresponds to π via the Jacquet–Langlands correspondence. Let $f_{\pi'}^{[p]}$ be a generator of $\mathbf{S}^B(N^+,\mathbf{C})[\pi'_f]$. We define an automorphic form in $\mathcal{A}(G)$ by

$$\varphi_{\pi'}^{[p]} := \Psi(\mathbf{v}_0^* \otimes f_{\pi'}^{[p]}) \text{ for } \mathbf{v}_0^* = D_K^{\frac{k-2}{2}} \mathbf{v}_0. \tag{3.14}$$

Let $\rho_{k,l}$ to be the representation defined by

$$\rho_{k,l} : G(\mathbf{Q}) \xrightarrow{i_{K_{\mathfrak{l}}}} \mathrm{GL}_2(\mathbf{C}_l) \to \mathrm{Aut}_{\mathbf{C}_l} L_{k-2}(\mathbf{C}_l). \tag{3.15}$$

It is easy to check that ρ_k and $\rho_{k,l}$ are compatible in the sense that

$$\rho_{k,l}(g) = \rho_k(\gamma_{\mathfrak{l}} i_l(g) \gamma_{\mathfrak{l}}^{-1}), \text{ where } \gamma_{\mathfrak{l}} := \begin{pmatrix} \sqrt{\beta} & -\sqrt{\beta}\boldsymbol{\theta} \\ -1 & \boldsymbol{\theta} \end{pmatrix} \in \mathrm{GL}_2(K_{\mathfrak{l}}). \tag{3.16}$$

If l is invertible in A, then we in fact have an isomorphism

$$\mathbf{S}_k^B(N^+,A) \xrightarrow{\cong} S_k^B(N^+,A), \ h \mapsto \widehat{h}(g) := \rho_k(\gamma_{\mathfrak{l}}^{-1}) \rho_{k,l}(g_l^{-1}) h(g) \tag{3.17}$$

and we say $\widehat{h}$ is an *l-adic avatar* of h. We will say $f_{\pi'}^{[p]}$ is *l*-adically normalised if $\widehat{f}_{\pi'}^{[p]}$ is a generator of the rank 1 module $S_k^B(N,\mathcal{O})[\pi'_f] := S_k^B(N,\mathcal{O}) \cap S_k^B(N,\mathbf{C}_l)[\pi'_f]$. We can now define the theta element associated to f and K. Let $f_{\pi'}^{[p]}$ be λ-adically normalised. We define the theta element $\Theta_m(f_{\pi'}^{[p]}) \in \mathcal{O}[\mathcal{G}_m]$ by

$$\Theta_m(f_{\pi'}^{[p]}) = \sum_{\sigma \in \mathcal{G}_m} \varphi_{\pi'}^{[p]}(\sigma \cdot x_m(1))[\sigma]. \tag{3.18}$$

We will denote the theta element simply by $\Theta(f_{\pi'}^{[p]})$ if $m = 1$. The following theorem relates the central critical value of the L-function of $f^{[p]}$ over K twisted by a ring class character χ of $\mathcal{G}_m$ to the theta element above.

Theorem 3.1 (Chida–Hsieh, Hung). *Let χ be character of $\mathcal{G}_m$ and $N^+ = \mathfrak{N}^+ \cdot \overline{\mathfrak{N}^+}$. Then we have*

$$\chi(\Theta_m(f_{\pi'}^{[p]})^2) = \Gamma(k/2)^2 \cdot \frac{L(f^{[p]}/K,\chi,k/2)}{\Omega_{\pi',N^-}} \cdot (-1)^{k/2} \cdot m \cdot D_K^{k-1} \cdot \frac{|\mathcal{O}_K^\times|^2}{8} \cdot \sqrt{-D_K}^{-1} \cdot \chi(\mathfrak{N}^+)$$

where $\Omega_{f^{[p]},pN^-}$ is the l-adically normalised period for $f^{[p]}$ given by

$$\Omega_{f^{[p]},pN^-} := \frac{4^{k-1}\pi^k \|f^{[p]}\|_{\Gamma_0(pN)}}{\langle f_{\pi'}^{[p]}, f_{\pi'}^{[p]}\rangle_B}.$$

Proof. This follows from the main result of [15] generalising [8] to ramified characters. $\square$

Note here the period $\Omega_{f^{[p]},pN^-}$ is not the canonical period $\Omega_{f^{[p]}}^{\mathrm{can}}$ of Hida defined by

$$\Omega_{f^{[p]}}^{\mathrm{can}} := \frac{4^{k-1}\pi^k \|f^{[p]}\|_{\Gamma_0(pN)}}{\eta_{f^{[p]}}(pN)}$$

where $\eta_{f^{[p]}}(pN)$ is the congruence number of $f^{[p]}$ in $S_k(pN)$. We record the following result comparing these two periods, which we will use at a later occasion. Let q be a prime and recall the local Tamagawa ideal $\mathrm{Tam}_q(\mathrm{T}_{f,\lambda})$ at q is defined by

$$\mathrm{Tam}_q(\mathrm{T}_{f,\lambda}) = \mathrm{Fitt}_{\mathcal{O}}(\mathrm{H}^1(K_q^{\mathrm{ur}},\mathrm{T}_{f,\lambda})_{\mathrm{tor}})$$

and the local Tamagawa exponent at q is defined by the number $t_q(f)$ such that

$$\mathrm{Tam}_q(\mathrm{T}_{f,\lambda}) = (\varpi^{t_q(f)}).$$

Proposition 3.2 (Kim-Ota). *The following equation holds under the assumption* $(\mathrm{CR}^\star)$:

$$v_\varpi\left(\frac{\Omega_{f^{[p]},pN^-}}{\Omega_{f^{[p]}}^{\mathrm{can}}}\right) = \sum_{q|pN^-} t_q(f^{[p]}).$$

Proof. This follows from [26, Corollary 5.8] generalising the work of Pollack–Weston [27] in weight 2. $\square$

3.3. Explicit reciprocity laws for Heegner cycles

Recall that we have the modular form $f \in S_k^{\mathrm{new}}(N)$ with $N = N^+N^-$ such that N^- is square-free with even number of prime divisors. Let $n \geq 1$; we consider the Abel–Jacobi map for $\mathrm{T}_{f,n}$,

$$\mathrm{AJ}_{k,n} : \epsilon_d \epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d} \otimes K_m) \otimes \mathbf{Z}_l \to \mathrm{H}^1(K_m,\mathrm{T}_{f,n}).$$

We have the Heegner cycle class $\epsilon_d Y_{m,k} \in \epsilon_d \epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d} \otimes K_m) \otimes \mathbf{Z}_l$ with $Y_{m,k} = \epsilon_k y_m^{\frac{k-2}{2}}$ for an element $y_m \in \mathrm{NS}(A_m)$ satisfying

(1) $\iota_m(b)^*(y_m) = \mathrm{Nrd}(b) y_m$ for any $b \in B'$;
(2) The self-intersection number of y_m is $2D_K$.

Here A_m is given by the Heegner point $P_m = (A_m, \iota_m, C_m)$ on X. Let p be an n-admissible prime for f. We consider the following composite map:

$$\epsilon_d \epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d} \otimes K_m) \xrightarrow{\mathrm{AJ}_{k,n}} \mathrm{H}^1(K_m, \mathrm{T}_{f,n}) \xrightarrow{\mathrm{loc}_p} \mathrm{H}^1(K_{m,p}, \mathrm{T}_{f,n}).$$

The image of $\epsilon_d Y_{m,k} \in \epsilon_d \epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d} \otimes K_m) \otimes \mathbf{Z}_l$ under the above map is by definition given by $\mathrm{loc}_p(\kappa_n(m))$ and it lands in $\mathrm{H}^1_{\mathrm{fin}}(K_{m,p}, \mathrm{T}_{f,n})$ as $\mathcal{W}_{k,d}$ has good reduction at p. Note that there is an isomorphism

$$\mathrm{H}^1_{\mathrm{fin}}(K_{m,p}, \mathrm{T}_{f,n}) \cong \mathrm{H}^1_{\mathrm{fin}}(K_p, \mathrm{T}_{f,n}) \otimes \mathcal{O}_n[\mathcal{G}_m] \tag{3.19}$$

by [1, Lemma 2.4 and 2.5] and [9, Lemma 1.4]. Therefore, Theorem 2.9 implies the following isomorphism:

$$\Phi_n : S_k^B(N^+, \mathcal{O})_{/I_{f,n}^{[p]}} \otimes \mathcal{O}_n[\mathcal{G}_m] \xrightarrow{\cong} \mathrm{H}^1\left(\mathbf{F}_{p^2}, \mathrm{H}^1(\overline{X} \otimes \mathbf{F}_p^{\mathrm{ac}}, \mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}}\right) \otimes \mathcal{O}_n[\mathcal{G}_m]$$

$$\xrightarrow{\cong} \mathrm{H}^1_{\mathrm{fin}}(K_{m,p}, \mathrm{T}_{f,n}). \tag{3.20}$$

It follows then that $\mathrm{loc}_p(\kappa_n(m))$ can be regarded as an element in $S_k^B(N^+, \mathcal{O})_{/I_{f,n}^{[p]}} \otimes \mathcal{O}_n[\mathcal{G}_m]$. Recall that $\tilde{P}_m = (A_m, \iota_m, C_m, \nu_m) \in X_d(K_m)$ is a lift of the Heegner point $P_m = (A_m, \iota_m, C_m) \in X(K_m)$ and $A_m \cong E_m \times E_m$ for a CM elliptic curve E_m. We have the following commutative diagram:

$$
\begin{array}{ccc}
\epsilon_d \epsilon_k \mathrm{CH}^{\frac{k-2}{2}}(A_m^{\frac{k-2}{2}} \otimes K_m) \otimes \mathbf{Z}_l & \xrightarrow{cl} & \epsilon_d \epsilon_k \mathrm{H}^{k-2}\left(A_m^{\frac{k-2}{2}} \otimes K_m, \mathbf{Z}_l\left(\frac{k-2}{2}\right)\right) \\
\downarrow{\scriptstyle j_{k,d*}} & & \downarrow{\scriptstyle j_{k,d*}} \\
\epsilon_d \epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k-2,d} \otimes K_m) \otimes \mathbf{Z}_l & \xrightarrow{cl} & \epsilon_d \epsilon_k \mathrm{H}^k\left(\mathcal{W}_{k,d} \otimes K_m, \mathbf{Z}_l\left(\frac{k}{2}\right)\right).
\end{array}
$$

On the first line of the diagram, we have the following isomorphism:

$$\epsilon_d \epsilon_k \mathrm{H}^{k-2}\left(A_m^{\frac{k-2}{2}} \otimes K_m, \mathbf{Z}_l\left(\frac{k-2}{2}\right)\right) = \epsilon_d \epsilon_k \mathrm{H}^{k-2}\left(E_m^{k-2} \otimes K_m, \mathbf{Z}_l\left(\frac{k-2}{2}\right)\right)$$

$$= \mathrm{Sym}^{k-2} \mathrm{H}^1\left(E_m \otimes K_m, \mathbf{Z}_l\left(\frac{k-2}{2}\right)\right)$$

$$\cong L_{k-2}(\mathbf{Z}_l),$$

while on the second line, we have

$$\epsilon_d\epsilon_k\mathrm{H}^k\left(\mathcal{W}_{k,d}\otimes K_m,\mathbf{Z}_l\left(\frac{k}{2}\right)\right)_{\mathfrak{m}_f} \cong \mathrm{H}^1\left(K_m,\epsilon_d\epsilon_k\mathrm{H}^{k-1}\left(\mathcal{W}_{k,d}\otimes\mathbf{Q}^{\mathrm{ac}},\mathbf{Z}_l\left(\frac{k}{2}\right)\right)_{\mathfrak{m}_f}\right)$$
$$\cong \mathrm{H}^1(K_m,\mathrm{H}^1(X\otimes\mathbf{Q}^{\mathrm{ac}},\mathcal{L}_{k-2}(\mathbf{Z}_l)(1))_{\mathfrak{m}_f}).$$

Here the first isomorphism follows from the Hochschild–Serre spectral sequence and the fact that outside of the middle degree $k-1$, $\epsilon_d\epsilon_k\mathrm{H}^*(\mathcal{W}_{k,d}\otimes\mathbf{Q}^{\mathrm{ac}},\mathbf{Z}_l(\frac{k}{2}))$ is Eisenstein and hence vanishes after localising at $\mathfrak{m}_f$.

Lemma 3.3. *The image of the element* $\epsilon_d\epsilon_k y_m^{\frac{k-2}{2}} \in \epsilon_d\epsilon_k\mathrm{CH}^{\frac{k-2}{2}}(A_m^{\frac{k-2}{2}}\otimes K_m)\otimes\mathbf{Z}_l$ *under the cycle class map to* $L_{k-2}(\mathbf{Z}_l)$ *can be identified with the vector* $\mathbf{v}_0^*$ *up to sign.*

Proof. This follows from the fact that

(1) $\mathbf{v}_0^*$ and $\epsilon_d\epsilon_k cl(y_m^{\frac{k-2}{2}})$ are the eigenvector of the action by K with eigenvalue 1;

(2) $\langle\epsilon_d\epsilon_k cl(y_m^{\frac{k-2}{2}}),\epsilon_d\epsilon_k cl(y_m^{\frac{k-2}{2}})\rangle = \langle\mathbf{v}_0^*,\mathbf{v}_0^*\rangle = D_K^{k-2}$.

These properties characterise an element in $L_{k-2}(\mathbf{Z}_l)$ up to sign. See [7, Lemma 7.2]. $\qquad\square$

We recall the class $\kappa(m)$ is given by the extension class obtained by pulling back the exact sequence

$$0\to\epsilon_d\epsilon_k\mathrm{H}^{k-1}\left(\mathcal{W}_{k,d}\otimes\mathbf{Q}^{\mathrm{ac}},\mathbf{Z}_l\left(\frac{k}{2}\right)\right)_{\mathfrak{m}_f}\to\epsilon_d\epsilon_k\mathrm{H}^{k-1}\left((\mathcal{W}_{k,d}-Y_{m,k})\otimes\mathbf{Q}^{\mathrm{ac}},\mathbf{Z}_l\left(\frac{k}{2}\right)\right)_{\mathfrak{m}_f}$$
$$\to\epsilon_d\epsilon_k\mathrm{H}^k_{Y_{m,k}\otimes\mathbf{Q}^{\mathrm{ac}}}\left(\mathcal{W}_{k,d}\otimes\mathbf{Q}^{\mathrm{ac}},\mathbf{Z}_l\left(\frac{k}{2}\right)\right)_{\mathfrak{m}_f}\to 0 \tag{3.21}$$

along the map $\mathbf{Z}_l\to\epsilon_d\epsilon_k\mathrm{H}^k_{Y_{m,k}\otimes\mathbf{Q}^{\mathrm{ac}}}(\mathcal{W}_{k,d}\otimes\mathbf{Q}^{\mathrm{ac}},\mathbf{Z}_l(\frac{k}{2}))_{\mathfrak{m}_f}$ sending $1\in\mathbf{Z}_l$ to the fundamental class of $Y_{m,k}\otimes\mathbf{Q}^{\mathrm{ac}}$. The above exact sequence is equivalent to

$$0\to\mathrm{H}^1(X\otimes\mathbf{Q}^{\mathrm{ac}},\mathcal{L}_{k-2}(\mathbf{Z}_l)(1))_{\mathfrak{m}_f}\to\mathrm{H}^1(X\otimes\mathbf{Q}^{\mathrm{ac}}-P_m\otimes\mathbf{Q}^{\mathrm{ac}},\mathcal{L}_{k-2}(\mathbf{Z}_l)(1))_{\mathfrak{m}_f}$$
$$\to\mathrm{H}^2_{P_m\otimes\mathbf{Q}^{\mathrm{ac}}}(X\otimes\mathbf{Q}^{\mathrm{ac}},\mathcal{L}_{k-2}(\mathbf{Z}_l)(1))_{\mathfrak{m}_f}\to 0 \tag{3.22}$$

and the last term is the same as $\epsilon_d\epsilon_k\mathrm{H}^{k-2}(A_m^{\frac{k-2}{2}}\otimes\mathbf{Q}^{\mathrm{ac}},\mathbf{Z}_l(\frac{k-2}{2}))_{\mathfrak{m}_f}\cong L_{k-2}(\mathbf{Z}_l)$, as explained in the paragraph after [7, Lemma 8.1]. Therefore, Lemma 3.3 implies that the class $\kappa(m)$ is the extension class obtained by pulling back (3.22) along the map sending $1\in\mathbf{Z}_l$ to the vector $\mathbf{v}_0^*\in L_{k-2}(\mathbf{Z}_l)$ up to sign. The class $\mathrm{loc}_p(\kappa(m))$ is obtained similarly by considering the special fibre of X; that is, we pull back the following exact sequence:

$$0\to\mathrm{H}^1(\overline{X}\otimes\mathbf{F}_p^{\mathrm{ac}},\mathcal{L}_{k-2}(\mathbf{Z}_l)(1))_{\mathfrak{m}_f}\to\mathrm{H}^1(\overline{X}\otimes\mathbf{F}_p^{\mathrm{ac}}-P_m\otimes\mathbf{F}_p^{\mathrm{ac}},\mathcal{L}_{k-2}(\mathbf{Z}_l)(1))_{\mathfrak{m}_f}$$
$$\to\mathrm{H}^2_{P_m\otimes\mathbf{F}_p^{\mathrm{ac}}}(\overline{X}\otimes\mathbf{F}_p^{\mathrm{ac}},\mathcal{L}_{k-2}(\mathbf{Z}_l)(1))_{\mathfrak{m}_f}\to 0 \tag{3.23}$$

along the map sending $1\in\mathbf{Z}_l$ to $\mathbf{v}_0^*\in L_{k-2}(\mathbf{Z}_l)$ up to sign.

Recall the pairing

$$\langle\ ,\ \rangle_B : S_k^B(N^+,\mathcal{O})\times S_k^B(N^+,\mathcal{O})\to\mathcal{O}$$

defined as in (2.6). It induces a pairing

$$\langle\ ,\ \rangle_B : S_k^B(N^+,\mathcal{O})/I_{f,n}^{[p]} \times S_k^B(N^+,\mathcal{O})[I_{f,n}^{[p]}] \to \mathcal{O}_n.$$

Theorem 3.4 (Second reciprocity law). *Let p be an n-admissible prime for f and assume that $\bar{\rho}_{f,\lambda}$ satisfies the assumption* (CR*). *Let $f_{\pi'}^{[p]}$ be l-adically normalised and $\widehat{f}_{\pi',n}^{[p]}$ be a generator of $S_k^B(N^+,\mathcal{O})[I_{f,n}^{[p]}]$. Then we have the following relation between the Heegner cycle class of level $m\kappa_n(m)$ and the theta element $\Theta_m(f_{\pi'}^{[p]})$:*

$$\sum_{\sigma\in\mathcal{G}_m}\langle\mathrm{loc}_p(\sigma\cdot\kappa_n(m)),\widehat{f}_{\pi',n}^{[p]}\rangle_B = u\cdot\Theta_m(f_{\pi'}^{[p]})\mathrm{mod}\,\varpi^n$$

where $u\in\mathcal{O}_n$ is a unit.

Proof. Let $\sigma\in\mathcal{G}_m$. We define $\mathbf{1}_{\sigma(x_m(1))\cdot\tau^{N^+}}^{[\mathbf{v}_0^*]}$ to be the function on X^B supported on the point $\sigma(x_m(1))\cdot\tau^{N^+}$ with its value given by the vector $\mathbf{v}_0^*$. Note that $\mathbf{1}_{\sigma(x_m(1))\cdot\tau^{N^+}}^{[\mathbf{v}_0^*]}$ can be considered as an element of $S_k^B(N^+,\mathcal{O})$. By the definition of Φ_n and Theorem 2.9, all classes of

$$\mathrm{H}^1(\mathbf{F}_{p^2},\mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}},\mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}})$$

are obtained by pulling back the exact sequence below:

$$0 \to \mathrm{H}^1(\overline{X}_{\mathbf{F}_p^{\mathrm{ac}}},\mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}} \to \mathrm{H}^1(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ord}},\mathcal{L}_{k-2}(\mathcal{O})(1))_{/I_{f,n}}$$
$$\to \mathrm{H}^0(X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}},\mathcal{L}_{k-2}(\mathcal{O}))_{/I_{f,n}} \to 0. \tag{3.24}$$

Since p is n-admissible and in particular inert in K, the point P_m which is represented by a product of CM elliptic curves has supersingular reduction. Moreover, by our parametrisation (3.11) and (3.4), the reduction of P_m is given by $x_m(1)\tau^{N^+}$ when we identify $X_{\mathbf{F}_p^{\mathrm{ac}}}^{\mathrm{ss}}$ with the Shimura set X^B. It follows that the element $\mathrm{loc}_p(\sigma\cdot\kappa_n(m))$ as the pullback of (3.24) factors through the exact sequence (3.23) modulo $I_{f,n}$ and is therefore given by the element $\mathbf{1}_{\sigma(x_m(1))\cdot\tau^{N^+}}^{[\mathbf{v}_0^*]}[\sigma]$ up to a sign. Therefore, we have the following equation:

$$\sum_{\sigma\in\mathcal{G}_m}\left\langle\mathrm{loc}_p(\sigma\cdot\kappa_n(m)),\widehat{f}_{\pi',n}^{[p]}\right\rangle_B = \pm\sum_{\sigma\in\mathcal{G}_m}\left\langle\mathbf{1}_{\sigma(x_m(1))\cdot\tau^{N^+}}^{[\mathbf{v}_0^*]},\widehat{f}_{\pi',n}^{[p]}\right\rangle_B[\sigma]$$

$$= \pm\sum_{\sigma\in\mathcal{G}_m}\left\langle\mathbf{v}_0^*,\widehat{f}_{\pi',n}^{[p]}(\sigma\cdot x_m(1))\right\rangle_k[\sigma]$$

$$= u\cdot\Theta_m(f_{\pi'}^{[p]})\mathrm{mod}\,\varpi^n.$$

$\square$

Next, let (p,p') be a pair of distinct n-admissible primes for f. Then we can consider the Shimura curves X'' and X_d'' and the corresponding Kuga–Sato varieties $\mathcal{W}_{k,d}''$ defined in Subsection 2.6. Note that they correspond to the indefinite quaternion algebra B'' with

discriminant $pp'N^-$. We can define in the same manner as in (3.4) the Heegner point

$$P_m^{[pp']}(a) = [z', a^{(p')}\varsigma\tau^{N^+}]_{\mathbf{C}} \in X''(K_m) \tag{3.25}$$

for $a \in \widehat{K}^\times$. We again write $P_m^{[pp']}$ for $P_m^{[pp']}(1)$. Using these points, we can define Heegner cycles

$$\epsilon_d Y_{m,k}^{[pp']} \in \epsilon_d\epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d}'' \otimes K_m) \otimes \mathbf{Z}_l \tag{3.26}$$

similarly as in (3.6).

Since (p, p') are n-admissible primes for f, there is a homomorphism $\phi_{f,n}^{[pp']} : \mathbb{T}^{[pp']} \to \mathcal{O}_n$ such that $\phi_{f,n}^{[pp']}$ agrees with $\phi_{f,n}$ at all Hecke operators away from pp' and sends $(U_p, U_{p'})$ to $(\epsilon_p p^{\frac{k-2}{2}}, \epsilon_{p'} p'^{\frac{k-2}{2}})$. Recall that $I_{f,n}^{[pp']}$ is the kernel of $\phi_{f,n}^{[pp']}$.

Lemma 3.5. *We have the following statements:*

(1) *The morphism $\phi_{f,n}^{[pp']}$ can be lifted to a genuine modular form $f^{[pp']} \in S_k^{\mathrm{new}}(pp'N)$ when $n = 1$.*

(2) *There is an isomorphism $\mathrm{H}^1(X_{\mathbf{Q}^{\mathrm{ac}}}'', \mathcal{L}_{k-2}(\mathcal{O})(1))/I_{f,n}^{[pp']} \cong \mathrm{T}_{f,n}$.*

Proof. It again follows from the main results of [11] and [12] that the morphism $\phi_{f,1}^{[pp']}$ can be lifted to a genuine modular form, which we will denote as $f^{[pp']}$.

To prove the second statement, it follows from the main result of [3] that

$$\mathrm{H}^1(X_{\mathbf{Q}^{\mathrm{ac}}}'', \mathcal{L}_{k-2}(\mathcal{O})(1))/I_{f,n}^{[pp']} \cong \mathrm{T}_{f,n}^r$$

for some integer r. Then one can consider the weight spectral sequence converges to (2.19) and use the fact that $S_k^B(N^+, \mathcal{O})_{/I_{f,n}^{[p]}}$ is of rank 1 to conclude that $r = 1$. $\square$

Using the above lemma, we can define the Abel–Jacobi map

$$\mathrm{AJ}_{k,n}^{[pp']} : \epsilon_d\epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d}'' \otimes K_m) \otimes \mathbf{Z}_l \to \mathrm{H}^1(K_m, \mathrm{T}_{f,n}) \tag{3.27}$$

following the same recipe for defining (3.7). We can define the corresponding Heegner cycle class of level m,

$$\kappa_n^{[pp']}(m) = \mathrm{AJ}_{k,n}^{[pp']}(\epsilon_d Y_{k,d}^{[pp']}) \in \mathrm{H}^1(K_m, \mathrm{T}_{f,n}). \tag{3.28}$$

Similarly, we define the class

$$\kappa_n^{[pp']} = \mathrm{Cor}_{K_1/K}\kappa_n^{[pp']}(1) \in \mathrm{H}^1(K, \mathrm{T}_{f,n}). \tag{3.29}$$

By [1, Lemma 2.4 and 2.5] and [9, Lemma 1.4], we have an isomorphism

$$\mathrm{H}_{\mathrm{sing}}^1(K_{m,p}, \mathrm{T}_{f,n}) \cong \mathrm{H}_{\mathrm{sing}}^1(K_p, \mathrm{T}_{f,n}) \otimes \mathcal{O}_n[\mathcal{G}_m].$$

The element $\partial_{p'}\mathrm{loc}_{p'}(\kappa^{[pp']}(m)) \in \mathrm{H}_{\mathrm{sing}}^1(K_{p'}, \mathrm{T}_{f,n}) \otimes \mathcal{O}_n[\mathcal{G}_m]$ under the composite below

$$\mathrm{H}^1(K_m, \mathrm{T}_{f,n}) \xrightarrow{\mathrm{loc}_{p'}} \mathrm{H}^1(K_{m,p'}, \mathrm{T}_{f,n}) \xrightarrow{\partial_{p'}} \mathrm{H}_{\mathrm{sing}}^1(K_{m,p'}, \mathrm{T}_{f,n})$$

can be considered as an element in $S^B(N^+,\mathcal{O})/I^{[p]}_{f,n}\otimes\mathcal{O}_n[\mathcal{G}_m]$ using the map Ξ_n given by Theorem 2.12.

Theorem 3.6 (First reciprocity law). *Let (p,p') be a pair of n-admissible prime for f and assume that $\bar{\rho}_{f,\lambda}$ satisfies assumption $(\mathrm{CR}^\star)$. Let $f^{[p]}_{\pi'}$ be l-adically normalised and $\widehat{f}^{[p]}_{\pi',n}$ be a generator of $S^B_k(N^+,\mathcal{O})[I^{[p]}_{f,n}]$. Then we have the following relation between the Heegner cycle class $\kappa^{[pp']}_n(m)$ and the theta element $\Theta_m(f^{[p]}_{\pi'})$:*

$$\sum_{\sigma\in\mathcal{G}_m}\left\langle\partial_{p'}\mathrm{loc}_{p'}(\sigma\cdot\kappa^{[pp']}_n(m)),\widehat{f}^{[p]}_{\pi',n}\right\rangle_B = u\cdot\Theta_m(f^{[p]}_{\pi'})\mathrm{mod}\,\varpi^n \tag{3.30}$$

where $u\in\mathcal{O}^\times_n$ is a unit.

Remark 3.7. This theorem is proved by [7] in a slightly different setup with a different method, but for completeness we give a proof.

Proof. We will use the formula given by [20, Theorem 2.18] and [20, Proposition 2.19], which are proved for trivial coefficients. In our case, we will apply these results to the cohomology of the Kuga–Sato variety $\mathcal{W}''_{k,d}$ with trivial coefficient. As noted in the discussion above Theorem 2.12, the weight spectral sequence of the Kuga–Sato variety agrees with the weight spectral sequence of the base curve with nontrivial coefficients. Moreover, the Heegner cycle is fibered over the Heegner point over the base Shimura curve; therefore, we can adapt these results in our setting with obvious modifications.

First we would like to apply [20, Proposition 2.19]; therefore, we are led to consider the cycle class

$$Y^{[pp'],\#}_{m,k}\times_{\widetilde{\mathcal{W}}''_{k,d}}\widetilde{\mathcal{W}}''^{(0)}_{k,d,\mathbf{F}^{\mathrm{ac}}_{p'}}$$

in the cohomology group $\epsilon_k\mathrm{H}^k(\widetilde{\mathcal{W}}''^{(0)}_{k,d,\mathbf{F}^{\mathrm{ac}}_{p'}},\mathcal{O}(\frac{k}{2}))$ where $Y^{[pp'],\#}_{m,k}$ is defined as in the paragraph below [20, (2.4.2)]. By applying the projector ϵ_d to it, we can identify this cycle class with an element in

$$\mathrm{H}^2(X''_{\mathbf{F}^{\mathrm{ac}}_{p'}},a_{0*}\mathcal{L}_{k-2}(\mathcal{O})(1))=\mathrm{H}^2(\mathbf{P}^1(X^B_{\mathbf{F}^{\mathrm{ac}}_{p'}}),\mathcal{L}_{k-2}(\mathcal{O})(1))^{\oplus 2}$$

supported on $X''^{(0)}_{\mathbf{F}^{\mathrm{ac}}_{p'}}\times_{\mathfrak{x}''}P^{[pp'],\#}_m$. This follows from Saito's computation of the weight spectral sequences of the Kuga–Sato varieties as we discussed in the paragraph of (2.19). Note the latter space can be identified with $S^B_k(N^+,\mathcal{O})^{\oplus 2}$. Again, by our parametrisation in (3.11) and (3.25) along with Lemma 3.3, this cycle class is given up to sign by a pair of functions on X^B supported on $x_m(1)\tau^{N^+}$ sending $x_m(1)\tau^{N^+}$ to $\mathbf{v}^*_0$ in the first copy and sending $x_m(1)\tau^{N^+}$ to $\epsilon_{p'}\mathbf{v}^*_0$ on the second copy. By the proof of Theorem 2.12 and [20, Theorem 2.18], $\partial_{p'}\mathrm{loc}_{p'}(\sigma\cdot\kappa^{[pp']}_n(m))$ is thus given by $\mathbf{1}^{[\mathbf{v}^*_0]}_{\sigma(x_m(1))\cdot\tau^{N^+}}[\sigma]$ as an element

in $S^B(N^+, \mathcal{O})/I_{f,n}^{[p]} \otimes \mathcal{O}_n[\mathcal{G}_m]$. Therefore, we have the following equation:

$$
\sum_{\sigma \in \mathcal{G}_m} \langle \partial_{p'} \mathrm{loc}_{p'}(\sigma \cdot \kappa_n^{[pp']}(m)), \widehat{f}_{\pi',n}^{[p]} \rangle_B = \pm \sum_{\sigma \in \mathcal{G}_m} \langle \mathbf{1}_{\sigma(x_m(1)) \cdot \tau^{N^+}}^{[\mathbf{v}_0^*]}, \widehat{f}_{\pi',n}^{[p]} \rangle_B [\sigma]
$$

$$
= \pm \sum_{\sigma \in \mathcal{G}_m} \langle \mathbf{v}_0^*, \widehat{f}_{\pi',n}^{[p]}(\sigma \cdot x_m(1)) \rangle_k [\sigma]
$$

$$
= u \cdot \Theta_m(f_{\pi'}^{[p]}) \bmod \varpi^n.
$$

$\square$

4. Converse to Gross–Zagier–Kolyvagin type theorem

4.1. Selmer groups of modular forms

Recall that f is a modular form of weight k level $\Gamma_0(N)$ such that $N = N^+ N^-$. We assume the following generalised Heegner hypothesis:

N^- is square-free and consists of even number of prime factors that are inert in K.

(Heeg)

Let K be an imaginary quadratic field with discriminant $-D_K$ such that $(D_K, N) = 1$. Let $l > 2$ be a prime such that $l \nmid N D_K$. Recall $\rho_{f,\lambda} : G_{\mathbf{Q}} \to \mathrm{GL}_2(E_\lambda)$ is the λ-adic Galois representation attached to the form f, which is characterised by the fact that the trace of Frobenius at $p \nmid N$ agrees with $a_p(f)$ and the determinant of $\rho_{f,\lambda}$ is ϵ_l^{k-1} where ϵ_l is the l-adic cyclotomic character. Recall that we are interested in the twist $\rho_{f,\lambda}^* = \rho_{f,\lambda}(\frac{2-k}{2})$. Let $V_{f,\lambda}$ be the representation space for $\rho_{f,\lambda}^*$. We have normalised the construction of $\rho_{f,\lambda}$ such that it occurs in the cohomology $\mathrm{H}^1(X_{\mathbf{Q}^{\mathrm{ac}}}, \mathcal{L}_{k-2}(E_\lambda)(\frac{k}{2}))$ and therefore $\rho_{f,\lambda}^*$ occurs in the cohomology $\mathrm{H}^1(X_{\mathbf{Q}^{\mathrm{ac}}}, \mathcal{L}_{k-2}(E_\lambda)(1))$. The modular form f gives rise to a homomorphism $\phi_f : \mathbb{T} \to \mathcal{O}$ corresponding to the Hecke eigen-system of f. Let $n \geq 1$; we take $\phi_{f,n} : \mathbb{T} \to \mathcal{O}_n$ to be the natural reduction of ϕ_f by ϖ^n. We define $I_{f,n}$ to be the kernel of the morphism $\phi_{f,n}$ and $\mathfrak{m}_f$ the unique maximal ideal of $\mathbb{T}$ containing $I_{f,n}$. We choose a $G_{\mathbf{Q}}$-stable lattice $\mathrm{T}_{f,\lambda}$ in $V_{f,\lambda}$ and denote by $\mathrm{T}_{f,n}$ the reduction $\mathrm{T}_{f,\lambda}/\varpi^n$. We also recall that the residual Galois representation $\bar{\rho}_{f,\lambda}$ satisfies the assumption $(\mathrm{CR}^\star)$. In light of this assumption and Theorem 2.9 (3), we can choose the lattice $\mathrm{T}_{f,\lambda}$ to be $\mathrm{H}^1(X_{\mathbf{Q}^{\mathrm{ac}}}, \mathcal{L}_{k-2}(\mathcal{O}))_{\mathfrak{m}_f}$. We denote by $\mathrm{A}_{f,\lambda}$ the divisible module given by $V_{f,\lambda}/\mathrm{T}_{f,\lambda}$. We will set

$$
\mathrm{A}_{f,n} = \ker[\mathrm{A}_{f,\lambda} \xrightarrow{\varpi^n} \mathrm{A}_{f,\lambda}]. \tag{4.1}
$$

Note that $\mathrm{A}_{f,n}$ is the Kummer dual of $\mathrm{T}_{f,n}$.

Let $\mathrm{M} = \mathrm{T}_{f,n}$ or $\mathrm{A}_{f,n}$. For $v \mid N^-$, we set $F_v^+ \mathrm{M}$ to be the unique line of M such that $G_{\mathbf{Q}_v}$ acts by $\epsilon_v \tau_v$ with τ_v the nontrivial unramified character of $G_{\mathbf{Q}_v}$. Then we define the ordinary part of $\mathrm{H}^1(K_v, \mathrm{M})$ to be

$$
\mathrm{H}^1_{\mathrm{ord}}(K_v, \mathrm{M}) = \ker[\mathrm{H}^1(K_v, M) \to \mathrm{H}^1(K_v, \mathrm{M}/F_v^+ \mathrm{M})].
$$

Let $p \nmid N$ be an n-admissible prime for f; then we set $F_p^+ \mathrm{M}$ to be the unique line such that Frob_p acts by $\epsilon_p p$ and $F_p^- \mathrm{M}$ is the line such that Frob_p acts by ϵ_p. Then

$$
\begin{aligned}
\mathrm{H}^1(K_p, \mathrm{M}) &= \mathrm{H}^1(K_p, F_p^- \mathrm{M}) \oplus \mathrm{H}^1(K_p, F_p^+ \mathrm{M}) \\
&\cong \mathrm{H}^1_{\mathrm{fin}}(K_p, \mathrm{M}) \oplus \mathrm{H}^1_{\mathrm{ord}}(K_p, \mathrm{M}).
\end{aligned} \tag{4.2}
$$

In order to apply the results from Iwasawa theory, we assume that f is *l-ordinary*. For a place $v \mid l$ in K, let $F_v^+ \mathrm{M}$ be the unique line of M such that $G_{\mathbf{Q}_l}$ acts by $\epsilon_l^{\frac{k}{2}}$. We define

$$
\mathrm{H}^1_{\mathrm{ord}}(K_v, \mathrm{M}) = \ker[\mathrm{H}^1(K_v, M) \to \mathrm{H}^1(K_v, \mathrm{M}/F_v^+ \mathrm{M})]. \tag{4.3}
$$

Following the notation in [14], we define the local conditions $\mathcal{F}_b^c(a)$ for a triple of integers (a,b,c) and l by

$$
\mathrm{H}^1_{\mathcal{F}_b^a(c)}(K_v, \mathrm{M}) = \begin{cases} \mathrm{H}^1_{\mathrm{fin}}(K_v, \mathrm{M}) & \text{if } v \nmid abcl; \\ \mathrm{H}^1(K_v, \mathrm{M}) & \text{if } v \mid a; \\ 0 & \text{if } v \mid b; \\ \mathrm{H}^1_{\mathrm{ord}}(K_v, \mathrm{M}) & \text{if } v \mid c; \\ \mathrm{H}^1_{\mathrm{ord}}(K_v, \mathrm{M}) & \text{if } v \mid l. \end{cases} \tag{4.4}
$$

In other words, at places dividing a, we use the relaxed local condition; at places dividing b, we use the strict local condition; at places dividing cl, we use the ordinary local condition. If any of (a,b,c) is 1, then we omit it from the notation. We define the Selmer group for M by

$$
\mathrm{Sel}_{\mathcal{F}_b^a(c)}(K, \mathrm{M}) = \{s \in \mathrm{H}^1(K, \mathrm{M}) : \mathrm{loc}_v(s) \in \mathrm{H}^1_{\mathcal{F}_b^a(c)}(K_v, \mathrm{M}) \text{ for all } v\}.
$$

In this article, we will be mainly concerned with the Selmer group $\mathrm{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{M})$. Notice that the Abel–Jacobi map

$$
\mathrm{AJ}_{k,n} : \epsilon_d \epsilon_k \mathrm{CH}^{\frac{k}{2}}(\mathcal{W}_{k,d} \otimes K) \otimes \mathbf{Z}_l \to \mathrm{H}^1(K, \mathrm{T}_{f,n})
$$

factors through $\mathrm{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,n})$. This is well-known except for a justification for primes dividing N^-. Suppose that $v \mid N^-$ and that $\bar{\rho}_{f,\lambda}$ is ramified at v. Then it follows from our assumption $(\mathrm{CR}^\star)$ that $v \not\equiv 1 \bmod l$ and a simple calculation using [10, Theorem 2.17] shows that $|\mathrm{H}^1(K_v, \mathrm{T}_{f,n})| = |\mathrm{T}_{f,n}^{G_{K_v}}|^2 = 0$. Next, suppose that $v \mid N^-$ and that $\bar{\rho}_{f,\lambda}$ is unramified at v; then we have a decomposition

$$
\mathrm{H}^1(K_v, \mathrm{T}_{f,n}) = \mathrm{H}^1_{\mathrm{ord}}(K_v, \mathrm{T}_{f,n}) \oplus \mathrm{H}^1_{\mathrm{fin}}(K_v, \mathrm{T}_{f,n}).
$$

Then our claim follows from the proof of the ramified level raising in Theorem 2.12.

Recall the definition of a bipartite Euler system of odd type in [14, Definition 2.3.2]. The following result follows from the first reciprocity law in Theorem 3.6 and the second reciprocity law in Theorem 3.4 proved before.

Corollary 4.1. *The theta elements of Chida–Hsieh defined in (3.18) and the Heegner cycle classes defined in (3.8) form a bipartite Euler system of odd type for the Selmer structures given by $\mathcal{F}(N^-)$ over K.*

4.2. The proof of the main result

Now we can state and prove the main result of this article.

Theorem 4.2. *Suppose (f, K) is a pair that satisfies the generalised Heegner hypothesis* (Heeg) *and f is ordinary at l. Assume that $\bar{\rho}_{f,\lambda}$ satisfies the hypothesis* (CR*). *If* $\operatorname{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1})$ *is of dimension 1 over $\mathbf{F}_\lambda$, then the class κ_1 is nonzero in* $\operatorname{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1})$.

Remark 4.3. The above theorem can be considered as a generalisation of the converse to Gross–Zagier–Kolyvagin type theorem proved by Wei Zhang [40] and Skinner [34] to the higher weight case.

Let p be a 1-admissible prime for f and let $f^{[p]}$ be the level raising of the modular form f constructed in Theorem 2.9. Since the residual representations of $f^{[p]}$ and f are isomorphic, we can regard $\operatorname{Sel}_{\mathcal{F}(pN^-)}(K, \mathrm{T}_{f,1})$ as the residual Selmer group for $f^{[p]}$. Then we have the following result comparing the rank of the Selmer group of f and that of $f^{[p]}$.

Proposition 4.4. *Suppose that* $\operatorname{loc}_p : \operatorname{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1}) \to \mathrm{H}^1_{\mathrm{fin}}(K_p, \mathrm{T}_{f,1})$ *is surjective (equivalently nontrivial). Then we have*

$$\dim_k \operatorname{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1}) = \dim_k \operatorname{Sel}_{\mathcal{F}(pN^-)}(K, \mathrm{T}_{f,1}) + 1.$$

Moreover, we have in this case

$$\operatorname{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1}) = \operatorname{Sel}_{\mathcal{F}^p(N^-)}(K, \mathrm{T}_{f,1}), \;\; \operatorname{Sel}_{\mathcal{F}(pN^-)}(K, \mathrm{T}_{f,1}) = \operatorname{Sel}_{\mathcal{F}_p(N^-)}(K, \mathrm{T}_{f,1}).$$

Proof. This follows from [14, Proposition 2.2.9, Corollary 2.2.10]. More precisely, we have the following Cartesian diagram of Selmer structures:

$$
\begin{array}{ccc}
\operatorname{Sel}_{\mathcal{F}^p(N^-)} & \xleftarrow{\quad x \quad} & \operatorname{Sel}_{\mathcal{F}(N^-)} \\
{\scriptstyle y} \Big\uparrow & & {\scriptstyle y} \Big\uparrow \\
\operatorname{Sel}_{\mathcal{F}(pN^-)} & \xleftarrow{\quad x \quad} & \operatorname{Sel}_{\mathcal{F}_p(N^-)}.
\end{array}
\tag{4.5}
$$

Here, the labels x and y on the arrows stand for the length of the respective quotients. We have $x + y = 1$ by [14, Proposition 2.2.9]. Since p is 1-admissible, the local conditions $\mathrm{H}^1_{\mathrm{ord}}(K_p, \mathrm{T}_{f,1})$ and $\mathrm{H}^1_{\mathrm{fin}}(K_p, \mathrm{T}_{f,1})$ are dual to each other under the local Tate duality. Therefore, if

$$\operatorname{loc}_p : \operatorname{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1}) \to \mathrm{H}^1_{\mathrm{fin}}(K_p, \mathrm{T}_{f,1})$$

is surjective, then $y = 1$ and $x = 0$. $\qquad\square$

Next we combine results from [9] and [35] to deduce a special value formula for the modular form $f^{[p]}$. For this, let $\operatorname{Sel}(K, A_{f^{[p]}}) = \varinjlim_n \operatorname{Sel}(K, A_{f^{[p]}, n})$ be the minimal Selmer group of $f^{[p]}$ defined as in [9, Introduction]. Here $A_{f^{[p]}}$ and $A_{f^{[p]}, n}$ are defined the exact same way as in (4.1). We will also use the Selmer group $\operatorname{Sel}_{pN^-}(K, A_{f^{[p]}, n})$ defined in [9, Definition 1.2], which consists of classes in $\mathrm{H}^1(K, A_{f^{[p]}, n})$ that are ordinary at primes

dividing pN^-l and unramified at other places. Also, we recall the following technical assumption imposed on the main result of [9]:

$$a_l(f) \not\equiv 1 \bmod l \text{ if } k = 2. \qquad (\mathrm{PO})$$

Theorem 4.5. *Suppose* (f,K) *is a pair that satisfies the generalised Heegner hypothesis* (Heeg) *and* (PO). *Assume that* $\bar{\rho}_{f,\lambda}$ *satisfies the hypothesis* (CR*) *and, in addition, assume that* f *is l-ordinary. Then* $L(f^{[p]}/K,1) \neq 0$ *if and only if* $\mathrm{Sel}(K, A_{f^{[p]}})$ *is finite and we have*

$$v_\varpi\left(\frac{L(f^{[p]}/K,1)}{\Omega^{\mathrm{can}}_{f^{[p]}}} \right) = \mathrm{leng}_{\mathcal{O}} \mathrm{Sel}(K, A_{f^{[p]}}) + \sum_{q|pN} t_q(f^{[p]}).$$

Proof. Since $\bar{\rho}_{f,\lambda}$ satisfies the hypothesis (CR*), the form $f^{[p]}$ satisfies the hypothesis (CR$^+$) of [9]. Since f satisfies (PO), $f^{[p]}$ also satisfies (PO). Therefore, we can combine [9, Corollary 2] and the main result of [35] to obtain the following equation:

$$v_\varpi\left(\frac{L(f^{[p]}/K,1)}{\Omega_{f^{[p]},pN^-}} \right) = \mathrm{leng}_{\mathcal{O}} \mathrm{Sel}(K, A_{f^{[p]}}) + \sum_{q|N^+} t_q(f^{[p]}).$$

It follows from [26, Corollary 5.8] that

$$v_\varpi\left(\frac{\Omega_{f^{[p]},pN^-}}{\Omega^{\mathrm{can}}_{f^{[p]}}} \right) = \sum_{q|pN^-} t_q(f^{[p]}).$$

The result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.6. Instead of using the one-sided divisibility of Chida–Hsieh [9], one can apply the main result of [18] to f and its quadratic twist f^K to get the same result. This is the approach used in [40], and one can then avoid the assumption (PO).

Proof of Theorem 4.2. Suppose c is a generator of $\mathrm{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1})$. Then we can find a 1-admissible prime p for f such that $\mathrm{loc}_p(c) \in \mathrm{H}^1_{\mathrm{fin}}(K_p, \mathrm{T}_{f,1})$ is nonzero by using the same proof of [9, Theorem 6.3]. Then Proposition 4.4 implies that

$$\dim_k \mathrm{Sel}_{\mathcal{F}(pN^-)}(K, \mathrm{T}_{f,1}) = \dim_k \mathrm{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1}) - 1 = 0.$$

Since $\mathrm{Sel}_{pN^-}(K, \mathrm{T}_{f^{[p]},1})$ can be regarded as a subspace of $\mathrm{Sel}_{\mathcal{F}(N^-)}(K, \mathrm{T}_{f,1})$, we know that

$$\mathrm{Sel}_{pN^-}(K, \mathrm{A}_{f^{[p]},1}) = 0.$$

Then by the control theorem of [9, Proposition 1.9 (2)], we have $\mathrm{Sel}_{pN^-}(K, \mathrm{A}_{f^{[p]}}) = 0$. Therefore, $\mathrm{Sel}(K, \mathrm{A}_{f^{[p]}}) = 0$ and $\sum_{q|N^+} t_q(f^{[p]}) = \sum_{q|N^+} t_q(f) = 0$, by the proof of [9, Corollary 6.15]. Then we can apply Theorem 4.5 and conclude that

$$v_\varpi\left(\frac{L(f^{[p]}/K,1)}{\Omega_{f^{[p]},pN^-}} \right) = 0.$$

The second reciprocity law in Theorem 3.4 and the specialisation formula for the theta element in Theorem 3.1 allow us to conclude that $\mathrm{loc}_p(\kappa_1)$ is nonzero in $\mathrm{H}^1_{\mathrm{fin}}(K_p, \mathbf{T}_{f,1})$. Therefore, κ_1 is nonzero in $\mathrm{Sel}_{\mathcal{F}(N^-)}(K, \mathbf{T}_{f,1})$ and we are done. $\qquad\square$

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