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Relative hyperbolicity of free-by-cyclic extensions

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Abstract

Given a finitely generated free group $\mathbb{F}$ of $\mathsf{rank}(\mathbb{F}) \geq 3$, we show that the mapping torus of ϕ is (strongly) relatively hyperbolic if ϕ is exponentially growing. As a corollary of our work, we give a new proof of Brinkmann's theorem which proves that the mapping torus of an atoroidal outer automorphism is hyperbolic. We also give a new proof of the Bridson–Groves theorem that the mapping torus of a free group automorphism satisfies the quadratic isoperimetric inequality. Our work also solves a problem posed by Minasyan and Osin: the mapping torus of an outer automorphism is not virtually acylindrically hyperbolic if and only if ϕ has finite order.

1. Introduction

Fix a finitely generated free group $\mathbb{F}$ with $\mathsf{rank}(\mathbb{F}) \geq 3$. Any element $\phi \in \mathrm{Out}(\mathbb{F})$ (the outer automorphism group of $\mathbb{F}$) gives us a short exact sequence

$$1 \to \mathbb{F} \to \Gamma \to \langle \phi \rangle \to 1,$$

where the group Γ is referred to as the *extension group* for the short exact sequence. When ϕ has infinite order, we say that Γ is an extension of $\mathbb{F}$ by $\mathbb{Z}$ or a *free-by-cyclic extension.* By choosing a lift $\Phi \in \mathrm{Aut}(\mathbb{F})$ of ϕ, we get the group $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ which we call *a mapping torus* of ϕ. Choosing a different lift will give us an isomorphic group (hence, quasi-isometric). When ϕ has infinite order, the groups Γ and $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ are isomorphic (hence, quasi-isometric). Significant work has been done in understanding geometry of $\mathbb{F} \rtimes_\Phi \mathbb{Z}$. Bestvina, Feighn and Handel first proved that the mapping torus is hyperbolic if ϕ is fully irreducible and atoroidal [BHF97, Theorem 5.1]. It follows from work of Bestvina and Feighn [BF92] and Brinkmann [Bri00] that $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ is hyperbolic if and only if ϕ does not have any periodic conjugacy classes (i.e. ϕ is *atoroidal*). We contribute to this study by proving the following result.

THEOREM 3.15. *Let $\phi \in \mathrm{Out}(\mathbb{F})$ and $\Phi \in \mathrm{Aut}(\mathbb{F})$ be any lift of ϕ. Then the group $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ is strongly relatively hyperbolic if and only if ϕ is exponentially growing.*

One interesting observation that comes out of this result is connected to the mapping class group theory of surfaces with boundaries. It is well known that pseudo-Anosov maps of such surfaces give us (strongly) relatively hyperbolic extension groups. Our work here shows that pseudo-Anosovs are not the only type of maps which give relatively hyperbolic extensions.

The 'only if' direction follows from the work of Makura [Mac02] and Hagen [Hag19]. The 'if' direction is our main contribution to the above theorem. We prove this in Proposition 3.13, where we show that $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ will be (strongly) hyperbolic relative to a collection of subgroups which correspond to the mapping torus of (representatives of) components of the nonattracting subgroup system of certain attracting lamination of ϕ. This connection between nonattracting subgroup system and relative hyperbolicity was first established in [Gho18]. This connection is very natural in the sense that a conjugacy class is attracted to some lamination under iteration of ϕ if and only if it is not carried by the nonattracting subgroup system; and being attracted to any lamination implies exponential growth for the conjugacy class.

As a corollary of Proposition 3.13, we give a new proof of Brinkmann's result by a simple inductive argument (inducting on the rank of the subgroups of $\mathbb{F}$ chosen as representatives of components of the nonattracting subgroup system) on the peripheral subgroups obtained in Proposition 3.13.

COROLLARY 3.14. *An outer automorphism $\phi \in \mathrm{Out}(\mathbb{F})$ is atoroidal if and only if the group $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ is a hyperbolic group for some (hence, every) lift $\Phi \in \mathrm{Aut}(\mathbb{F})$ of ϕ.*

We then combine our work with a result of Genevois and Horbez [GH21] to answer a question asked by Minasyan and Osin in [MO15, Problem 8.2].

COROLLARY 4.2. *If $\phi \in \mathrm{Out}(\mathbb{F})$ is an infinite order element and $\Phi \in \mathrm{Aut}(\mathbb{F})$ is some lift of ϕ, then the group $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ is acylindrically hyperbolic but not relatively hyperbolic if and only if ϕ is polynomially growing.*

The proof of Theorem 3.15 uses the completely split train-track theory from Feighn and Handel's recognition theorem work [FH11] and the weak attraction theory from Handel and Mosher's subgroup decomposition for $\mathrm{Out}(\mathbb{F})$ body of work [HM20].

Idea of proof. Given an exponentially growing ϕ, one always has an attracting lamination Λ^+ associated to it. The nonattracting subgroup system $\mathcal{A}_{\mathrm{na}}(\Lambda^+)$ is a malnormal subgroup system that carries all conjugacy classes which are not attracted to Λ^+ under iterations of ϕ (see §2.1). As Γ and $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ are isomorphic when ϕ is exponentially growing, we set $\Gamma = \mathbb{F} \rtimes_\Phi \mathbb{Z}$ for the rest of the paper for convenience.

We perform a partial electrocution of Γ with respect to a collection of certain lifts of the components of $\mathcal{A}_{\mathrm{na}}(\Lambda^+)$ and form a new metric space $(\widehat{\Gamma}, |\cdot|_{\mathrm{el}})$. After this we use the weak attraction theorem to show the under iteration of either ϕ or ϕ^{-1}, we gain enough legality (Lemma 3.5 to proceed (primarily following the technique in Bestvina, Feighn and Handel's work in [BHF97]) to prove that we have flaring (Proposition 3.10. This combined with the Mj–Reeves strong combination theorem [MR08, Theorem 4.6] proves that Γ will be strongly relatively hyperbolic in Proposition 3.13.

When ϕ is atoroidal, a simple inductive argument by repeatedly applying Proposition 3.13 on the peripheral subgroups which are being electrocuted to form the coned-off graph, shows that Γ will be hyperbolic, thus giving a new proof of Brinkmann's theorem.

In the last section we show an application of our work which generalizes a theorem of [BHF97] and appears in the form of the following theorem.

THEOREM 5.6. *Let ϕ, ψ be outer automorphisms which satisfy the standing assumptions 5.1. Then there exists some $M > 0$ such that for every $m, n \geq M$, the group $Q := \langle \phi^m, \psi^n \rangle$ is a free group of rank 2 and the extension group $\mathbb{F} \rtimes \widetilde{Q}$ is hyperbolic for any lift $\widetilde{Q}$ of Q.*

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The proof of this theorem proceeds by proving a version of Mosher's 3-of-4 stretch lemma [Mos97], which in our setting is Proposition 5.5, and using it together with the Bestvina–Feighn combination theorem. An alternative proof of this theorem can be obtained by a recent work of Uyanik [Uya19].

Hyperbolic extensions of free groups have also been produced by Dowdall and Taylor [DT18] in their work on convex cocompact subgroups of $\mathrm{Out}(\mathbb{F})$ and by Uyanik in a recent work [Uya19]. Uyanik does not have any assumptions of fully irreducibility on the elements of the quotient group. Theorem 5.6 gives a new class of examples of free-by-free hyperbolic extensions and we hope that this will be useful in the future when further research is done to give even weaker conditions on Q so that the extension group is hyperbolic. This result is a significant generalization of a theorem of Bestvina, Feighn and Handel [BHF97, Theorem 5.2] where they prove a similar result by assuming ϕ, ψ to be fully irreducible and atoroidal.

As an application, we use our work together with the polynomial growth case of Bridson and Groves' theorem [BG10], to give a new proof of the general case of the Bridson–Groves theorem [BG10] which shows that the mapping torus of any free group automorphism satisfies the quadratic isoperimetric inequality, implying that the conjugacy problem is solvable for such groups.

THEOREM 5.8. *The mapping tori of a free group automorphism satisfies the quadratic isoperimetric inequality.*

2. Preliminaries

We recall some of the basic notions and tools used in the study of $\mathrm{Out}(\mathbb{F})$. The definitions and results presented in this section have been developed over a significant period of time in [BH92, BFH00, FH11] by Bestvina, Feighn and Handel.

2.1 Marked graphs, circuits and path

A *marked graph* is a finite graph G which has no valence 1 vertices and is equipped with a homotopy equivalence, called a *marking*, to the rose R_n given by $\rho : G \to R_n$ (where $n = \mathrm{rank}(\mathbb{F})$). The homotopy inverse of the marking is denoted by the map $\overline{\rho} : R_n \to G$. A *circuit* in a marked graph is an immersion (i.e. a locally injective continuous map) of S^1 into G. Here I denotes an interval in $\mathbb{R}$ that is closed as a subset. A *path* is a locally injective, continuous map $\alpha : I \to G$, such that any lift $\widetilde{\alpha} : I \to \widetilde{G}$ is proper. When I is compact, any continuous map from I can be homotoped, relative to its endpoints, by a process called *tightening* to a unique path (up to reparametrization) with domain I. If I is noncompact, then each lift $\widetilde{\alpha}$ induces an injection from the ends of I to the ends of $\widetilde{G}$. In this case, there is a unique path $[\alpha]$ which is homotopic to α such that both $[\alpha]$ and α have lifts to $\widetilde{G}$ with the same finite endpoints and the same infinite ends. If I has two infinite ends, then α is called a *line* in G; otherwise, if I has only one infinite end, then α is called a *ray*. Given a homotopy equivalence of marked graphs $f : G \to G'$, $f_{\#}(\alpha)$ denotes the tightened image $[f(\alpha)]$ in G'. Similarly, we define $\tilde{f}_{\#}(\widetilde{\alpha})$ by lifting to the universal cover.

Free factor systems. Given a collection of finitely generated subgroups of $\mathbb{F}$, say $F^1, \ldots, F^k$, the collection of their conjugacy classes denoted by $\{[F^1], \ldots, [F^k]\}$ is called a *subgroup system*. We say that a circuit $[c]$ is carried by this subgroup system if and only if there exists a representative H^i of some $[F^i]$ such that $c \in H^i$. The subgroup system is called a *malnormal subgroup system* if $H^i \cap H^j = \emptyset$ for every $i \neq j$, where H^i is any representative of $[F^i]$.

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A subgroup system $\{[F^1], [F^2], \ldots, [F^k]\}$ is called a *free-factor system* if $\mathbb{F} = F^1 * F^2 * \cdots * F^k * B$, with B possibly trivial. Note that every free-factor system is always a malnormal subgroup system. Note that given any subgraph $H \subset G$, the fundamental groups of the noncontractible components of H gives rise to a free-factor system, which is denoted by $[H]$.

2.2 Weak topology

Given any finite graph G, let $\widehat{\mathcal{B}}(G)$ denote the compact space of equivalence classes of lines, circuits, paths and rays in G. We give this space the *weak topology*. Namely, for each finite path γ in G, we have one basis element $\widehat{N}(G, \gamma)$ which contains all paths and circuits in $\widehat{\mathcal{B}}(G)$ which have γ as its subpath. Let $\mathcal{B}(G) \subset \widehat{\mathcal{B}}(G)$ be the compact subspace of all lines in G with the induced topology. One can give an equivalent description of $\mathcal{B}(G)$ following [BFH00]. A line is completely determined, up to reversal of direction, by two distinct points in $\partial\mathbb{F}$, because there is only one line that joins these two points. We can then induce the weak topology on the set of lines coming from the Cantor set $\partial\mathbb{F}$. More explicitly, let $\widetilde{\mathcal{B}} = \{\partial\mathbb{F} \times \partial\mathbb{F} - \triangle\}/(\mathbb{Z}_2)$, where $\triangle$ is the diagonal and $\mathbb{Z}_2$ acts by interchanging factors. We can put the weak topology on $\widetilde{\mathcal{B}}$, induced by Cantor topology on $\partial\mathbb{F}$. The group $\mathbb{F}$ acts on $\widetilde{\mathcal{B}}$ with a compact but non-Hausdorff quotient space $\mathcal{B} = \widetilde{\mathcal{B}}/\mathbb{F}$. The quotient topology is also called the *weak topology*. Elements of $\mathcal{B}$ are called *lines*. A lift of a line $\gamma \in \mathcal{B}$ is an element $\widetilde{\gamma} \in \widetilde{\mathcal{B}}$ that projects to γ under the quotient map and the two elements of $\widetilde{\gamma}$ are called its endpoints.

One can naturally identify the two spaces $\mathcal{B}(G)$ and $\mathcal{B}$ by considering a homeomorphism between the two Cantor sets $\partial\mathbb{F}$ and set of ends of universal cover of G, where G is a marked graph. Here $\mathrm{Out}(\mathbb{F})$ has a natural action on $\mathcal{B}$. The action comes from the action of $\mathrm{Aut}(\mathbb{F})$ on $\partial\mathbb{F}$. Given any two marked graphs G, G' and a homotopy equivalence $f : G \to G'$ between them, the induced map $f_{\#} : \widehat{\mathcal{B}}(G) \to \widehat{\mathcal{B}}(G')$ is continuous and the restriction $f_{\#} : \mathcal{B}(G) \to \mathcal{B}(G')$ is a homeomorphism. With respect to the identification $\mathcal{B}(G) \approx \mathcal{B} \approx \mathcal{B}(G')$, if f preserves the marking, then $f_{\#} : \mathcal{B}(G) \to \mathcal{B}(G')$ is equal to the identity map on $\mathcal{B}$. When $G = G'$, $f_{\#}$ agree with their homeomorphism $\mathcal{B} \to \mathcal{B}$ induced by the outer automorphism associated to f.

A line (path) γ is said to be *weakly attracted* to a line (path) β under the action of $\phi \in \mathrm{Out}(\mathbb{F})$, if the $\phi^k(\gamma)$ converges to β in the weak topology. This is same as saying, for any given finite subpath of β, $\phi^k(\gamma)$ contains that subpath for all sufficiently large values of k; similarly if we have a homotopy equivalence $f : G \to G$, a line (path) γ is said to be *weakly attracted* to a line (path) β under the action of $f_{\#}$ if the $f_{\#}^k(\gamma)$ converges to β in the weak topology.

Topological representative. Given $\phi \in \mathrm{Out}(\mathbb{F})$ a *topological representative* is a homotopy equivalence $f : G \to G$ (where $\rho : G \to R_n$ is a marked graph), f takes vertices to vertices and edges to edge paths and $\rho \circ f \circ \overline{\rho} : R_n \to R_n$ represents ϕ. A nontrivial path γ in G is a *periodic Nielsen path* if there exists a k such that $f_{\#}^k(\gamma) = \gamma$; the minimal such k is called the period and if $k = 1$, we call such a path *Nielsen path*. A periodic Nielsen path is *indivisible* if it cannot be written as a concatenation of two or more nontrivial periodic Nielsen paths.

2.3 EG strata and NEG strata

A *filtration* of a marked graph G is a strictly increasing sequence of subgraphs $G_0 \subset G_1 \subset \cdots \subset G_k = G$, each with no isolated vertices. The individual terms G_r are called *filtration elements*, and if G_r is a core graph (i.e. a graph without valence 1 vertices), then it is called a *core filtration element*. The subgraph $H_r = G_r \setminus G_{r-1}$ together with the vertices which occur as endpoints of edges in H_r is called the *stratum of height r*. The *height* of subset of G is the minimum r such that the subset is contained in G_r. The height of a map to G is the height of the image of

the map. A *connecting path* of a stratum H_r is a nontrivial finite path γ of height $< r$ whose endpoints are contained in H_r.

Given a topological representative $f : G \to G$ of $\phi \in \mathrm{Out}(\mathbb{F})$, we say that f *respects* the filtration or that the filtration is f-*invariant* if $f(G_r) \subset G_r$ for all r. Given an f-invariant filtration, for each stratum H_r with edges $\{E_1, \ldots, E_m\}$, define the *transition matrix* of H_r to be the square matrix whose jth column records the number of times $f(E_j)$ crosses the other edges. If M_r is the zero matrix, then we say that H_r is a *zero stratum*. If M_r is irreducible, meaning that for each i, j there exists p such that the i, j entry of the pth power of the matrix is nonzero, then we say that H_r is *irreducible*; and if one can furthermore choose p independently of i, j, then H_r is *aperiodic*. Assuming that H_r is irreducible, by the Perron–Frobenius theorem, the matrix M_r a unique eigenvalue $\lambda \geq 1$, called the *Perron–Frobenius eigenvalue*, for which some associated eigenvector has positive entries: if $\lambda > 1$, then we say that H_r is an *exponentially growing* (EG) stratum; whereas if $\lambda = 1$, then H_r is a *nonexponentially growing* (NEG) stratum. An edge in an NEG stratum will be sometimes referred to as *NEG edge*. Similarly for edges in EG stratum we shall sometimes use the term *EG edge*.

2.4 Relative train-track maps

Let $f : G \to G$ be a topological representative for $\phi \in \mathrm{Out}(\mathbb{F})$ and consider a filtration $G_0 \subset G_1 \subset \cdots \subset G_k$ which is preserved by f. One can define a map T_f by setting $T_f(E)$ to be the first edge of the edge path $f(E)$. We say $T_f(E)$ is the *direction* of $f(E)$. If E_1, E_2 are two edges in G with the same initial vertex, then the unordered pair (E_1, E_2) is called a *turn* in G. Define $T_f(E_1, E_2) = (T_f(E_1), T_f(E_2))$. Thus, T_f is a map that takes turns to turns. A turn is said to have height r if both the edges defining the turn are of height r.

We say that a nondegenerate turn (i.e. $E_1 \neq E_2$) is *illegal* if for some $k > 0$ the turn $T_f^k(E_1, E_2)$ becomes degenerate (i.e. $T_f^k(E_1) = T_f^k(E_2)$); otherwise, the turn is *legal*. A path is said to be a *legal path* if it contains only legal turns. A path is $r - legal$ if it is of height r and all its height r turns are legal.

Relative train-track map [FH11, §2.6]. Given $\phi \in \mathrm{Out}(\mathbb{F})$ and a topological representative $f : G \to G$ with a filtration $G_0 \subset G_1 \subset \cdots \subset G_k$ which is preserved by f. The topological representative f is a relative train-track map if every stratum is either a zero stratum or irreducible stratum and, in addition, the following conditions are satisfied for every EG stratum H_r:

(i) if E is edge in H_r, then $T_f(E)$ is also an edge of H_r;
(ii) f maps r-legal paths to legal r-paths;
(iii) if γ is a nontrivial path in G of height less than r with its endpoints in H_r, then $f_\#(\gamma)$ has its end points in H_r.

2.5 Completely split train-track maps

Given relative train-track map $f : G \to G$, *splitting* of a line, path or a circuit γ is a decomposition of γ into subpaths $\ldots \gamma_0 \gamma_1 \cdots \gamma_k \ldots$ such that for all $i \geq 1$ the path $f_\#^i(\gamma) = \cdots f_\#^i(\gamma_0) f_\#^i(\gamma_1) \cdots f_\#^i(\gamma_k) \ldots$. The terms γ_i are called the *terms* of the splitting of γ. Henceforth, the notation $\alpha \cdot \beta$ will be used to denote a splitting and $\alpha\beta$ will denote a concatenation of nontrivial paths α, β.

Given two NEG edges E_1, E_2 and a root-free closed Nielsen path ρ such that $f_\#(E_i) = E_i \cdot \rho^{p_i}$, then E_1, E_2 are said to be in the *same linear family* and any path of the form $E_1 \rho^m \overline{E_2}$ for some integer m is called an *exceptional path*.

Complete splittings. A splitting of a path or circuit $\gamma = \gamma_1 \cdot \gamma_2 \cdot \cdots \cdot \gamma_k$ is called *complete splitting* if each term γ_i falls into one of the following categories:

(i) γ_i is an edge in some irreducible stratum;
(ii) γ_i is an indivisible Nielsen path;
(iii) γ_i is an exceptional path;
(iv) γ_i is a maximal subpath of γ in a zero stratum H_i and γ_i is taken (see [FH11, Definition 4.4]).

Completely split improved relative train-track maps. A *CT* or a completely split improved relative train-track map is a topological representative with particularly nice properties. However, CTs do not exist for all outer automorphisms. Only the *rotationless* (see [FH11, Definition 3.13]) outer automorphisms are guaranteed to have a CT representative as has been shown in the following theorem from [FH11, Theorem 4.28].

LEMMA 2.1. *For each rotationless $\phi \in \mathrm{Out}(\mathbb{F})$ and each increasing sequence $\mathcal{F}$ of ϕ-invariant free-factor systems, there exists a CT $f : G \to G$ that is a topological representative for ϕ and f realizes $\mathcal{F}$.*

The following results are some properties of CTs defined in the recognition theorem work of Feighn and Handel in [FH11]. We only state those we need here.

(i) (*Rotationless*) Each principal vertex is fixed by f and each periodic direction at a principal vertex is fixed by Tf.
(ii) (*Completely split*) For each edge E in each irreducible stratum, the path $f(E)$ is completely split.
(iii) (*Vertices*) The endpoints of all indivisible Nielsen paths are vertices. The terminal endpoint of each nonfixed NEG edge is principal.
(iv) (*Periodic edges*) Each periodic edge is fixed.
(v) (*Zero strata*) Each zero strata H_i is contractible and enveloped by an EG strata H_s, $s > i$, such that every edge of H_i is a taken in H_s. Each vertex of H_i is contained in H_s and link of each vertex in H_i is contained in $H_i \cup H_s$.
(vi) (*Linear edges*) For each linear edge E_i there exists a root-free Nielsen path w_i such that $f_\#(E_i) = E_i w_i^{d_i}$ for some $d_i \neq 0$.
(vii) (*Nonlinear NEG edges*) [FH11, Lemma 4.21] Each non-fixed NEG stratum H_i is a single edge with its NEG orientation and has a splitting $f_\#(E_i) = E_i \cdot u_i$, where u_i is a closed nontrivial completely split circuit and is an Nielsen path if and only if H_i is linear.

We shall call any nonfixed, nonlinear NEG edge a *superlinear NEG edge*. The advantage of using CT maps rather than using the regular relative train-track maps is the greater control that we get when we iterate the train-track maps, which is very much a necessity here.

2.6 Attracting laminations

A closed subset of $\mathcal{B} = \mathcal{B}(\mathbb{F})$ is called a *lamination*. An element of a lamination is called a *leaf*. The action of $\mathrm{Out}(\mathbb{F})$ on $\mathcal{B}$ induces an action on the set of laminations.

For each marked graph G the homeomorphism $\mathcal{B} \approx \mathcal{B}(G)$ induces a bijection between $\mathbb{F}$-laminations and closed subsets of $\mathcal{B}(G)$. The closed subset of $\mathcal{B}(G)$ corresponding to a lamination $\Lambda \subset \mathcal{B}$ is called the *realization* of Λ in G; we generally use the same notation Λ for its realizations in marked graphs. In addition, we occasionally use the term lamination to refer to $\mathbb{F}$-invariant, closed subsets of $\widetilde{\mathcal{B}}$; the quotient map $\widetilde{\mathcal{B}} \mapsto \mathcal{B}$ puts these in natural bijection with laminations in $\mathcal{B}$.

Given $\phi \in \mathrm{Out}(\mathbb{F})$ and a lamination $\Lambda \subset \mathcal{B}$, we say that Λ is an *attracting lamination* for ϕ if there exists a leaf $\ell \in \Lambda$ satisfying the following:

(i) Λ is the weak closure of ℓ;
(ii) ℓ is a birecurrent, i.e. every finite subpath of ℓ occurs infinitely many times in either direction of ℓ;
(iii) ℓ is not the axis of the conjugacy class of a generator of a rank-one free factor of F_n;
(iv) there exists $p \geq 1$ and a weak open set $U \subset \mathcal{B}$ such that $\phi^p(U) \subset U$ and such that $\{\phi^{kp}(U) \mid k \geq 1\}$ is a weak neighborhood basis of ℓ.

Any such leaf ℓ is called a *generic leaf* of Λ, and any such neighborhood U is called an *attracting neighborhood of Λ for the action of ϕ^p*. Let $\mathcal{L}(\phi)$ denote the set of attracting laminations for ϕ.

Bestvina, Feighn and Handel [BFH00] showed that there is bijection between the elements of $\mathcal{L}(\phi)$ and the exponentially growing strata of G. In fact, they showed that given a relative train-track map, there is a unique way of associating each attracting lamination to an exponentially growing stratum of G. Moreover, they established a bijection between the sets $\mathcal{L}(\phi)$ and $\mathcal{L}(\phi^{-1})$ by using the notion of *free-factor support*.

Free factor support. The free-factor support of a line ℓ is the conjugacy class of the smallest (with respect to inclusion) free factor F^k such that $\partial \ell \in \partial F^k$. For an attracting lamination associated to some EG stratum H_r of G, the free-factor support of Λ_ϕ^+ is the free-factor support of any generic leaf of Λ_ϕ^+ and equals to $[G_r]$ (the free-factor system defined by G_r as a subgraph of G) (see [BFH00, Definition 3.2.3, Lemma 3.2.4]).

In this particular case, one can think of a free-factor support of an attracting lamination, $\Lambda^+ \in \mathcal{L}(\phi)$, to be the conjugacy class of the smallest (in terms of subgroup inclusion) free factor that carries Λ^+. Two laminations $\Lambda^+ \in \mathcal{L}(\phi)$ and $\Lambda^- \in \mathcal{L}(\phi^{-1})$ are said to be *dual* if and only if they have the same free-factor support. In [BFH00], it is shown that duality induces a bijection between the sets $\mathcal{L}(\phi)$ and $\mathcal{L}(\phi^{-1})$.

Given a rotationless outer automorphism $\phi \in \mathrm{Out}(\mathbb{F})$ we list a few properties which will be important for us.

LEMMA 2.2. *[FH11, Lemma 3.30, Corollary 3.31] For a rotationless $\phi \in \mathrm{Out}(\mathbb{F})$ the following are true:*

(i) *If F is a ϕ invariant free factor, then $\phi|_F$ is rotationless.*
(ii) *Each $\Lambda^+ \in \mathcal{L}(\phi)$ is invariant under ϕ.*
(iii) *Every free factor, conjugacy class which is periodic under ϕ is fixed by ϕ.*

2.7 Relatively hyperbolic groups

Given a group Γ and a collection of subgroups $H_\alpha < \Gamma$, the *coned-off Cayley graph* of Γ or the *electric space* of Γ relative to the collection $\{H_\alpha\}$ is a metric space which consists of the Cayley graph of Γ and a collection of vertices v_α (one for each left coset of H_α) such that each point of H_α is joined to (or coned-off at) v_α by an edge of length $1/2$. The resulting metric space is denoted by $(\widehat{\Gamma}, |\cdot|_{\mathrm{el}})$.

A group Γ is said to be *(weakly) hyperbolic* relative to a *finite* collection of *finitely generated* subgroups $\{H_\alpha\}$ if $\widehat{\Gamma}$ is a δ-hyperbolic metric space, in the sense of Gromov. Here Γ is said to be strongly hyperbolic relative to the collection $\{H_\alpha\}$ if the coned-off space $\widehat{\Gamma}$ is hyperbolic and it satisfies the *bounded coset penetration* property (see [Far98]), but this bounded coset penetration property is a very hard condition to check for random groups Γ. However, it is well known that if the group Γ is weakly relatively hyperbolic with respect to the collection of subgroups $\{H_\alpha\}$ and

this collection is *mutually malnormal* and *quasiconvex*, then Γ is strongly relatively hyperbolic. We shall be using this result for our constructions here.

In our main result, Proposition 3.13, the bounded coset penetration property is verified by the cone-bounded hallways strictly flare condition due to [MR08]. It was shown in [MR08] that this flaring property establishes a condition (namely, *mutual coboundedness*) due to Bowditch [Bow12] which implies the strong relative hyperbolicity.

3. Exponential growth case

We begin this section by recalling the construction of the nonattracting subgroup system and list a few of the properties we will be using. We give the definitions and some results about the two key concepts, nonattracting subgroup system and weak attraction theorem, from the subgroup decomposition work of Handel and Mosher [HM20, §1 and Theorem F] which are central to the proofs in this paper.

Recall from [FH11] that there exists some $K > 0$ such that given any $\phi \in \text{Out}(\mathbb{F})$, ϕ^K is *rotationless*. Hence, given any ϕ we may pass on to a rotationless power to make use of the rich CT structure. We show that the mapping torus of ϕ^K is relatively hyperbolic. Then using Drutu's work [Dru09], we conclude that the mapping torus of ϕ is also relatively hyperbolic, because mapping torus of ϕ^K is quasi-isometric to mapping torus of ϕ.

Topmost lamination. For an outer automorphism ϕ, we call an attracting lamination Λ to be *topmost* if there are no attracting lamination of ϕ that contain Λ as a proper subset.

It is easy to see that for any exponentially growing outer automorphism, if we choose a relative train-track map, then the attracting lamination associated to the highest EG stratum is always topmost (by observing that there are no other EG strata above it and using [HM09, Proposition 2.31]). Thus, every exponentially growing outer automorphism has at least one topmost attracting lamination. From [BFH00, Corollary 6.0.11] we know that if $\Lambda^{\pm}$ is a dual lamination pair for ϕ, then Λ^+ is topmost lamination for ϕ if and only if Λ^- is topmost for ϕ^{-1}.

3.1 Nonattracting subgroup system

The *nonattracting subgroup system* of an attracting lamination contains information about lines and circuits which are not attracted to the lamination. This is a crucial construction from the train-track theory that lies in the heart of our proof here. First introduced by Bestvina, Feighn and Handel in their Tit's alternative work [BFH00], it was later studied in more details by Handel and Mosher in [HM20]. We urge the reader unfamiliar with this construction to look into [HM20] where it has been explored in great detail.

The construction of the *nonattracting subgraph* is as follows.

Suppose $\phi \in \text{Out}(\mathbb{F})$ is rotationless and $f : G \to G$ is a CT representing ϕ such that Λ_ϕ^+ is an invariant attracting lamination which corresponds to the EG stratum $H_s \subset G$. The *nonattracting subgraph* Z of G is defined as a union of irreducible strata H_i of G such that no edge in H_i is weakly attracted to Λ_ϕ^+. This is equivalent to saying that a strata H_i is contained in $G \setminus Z$ if and only if there exists some $k \geq 0$ such that for some edge E_i of H_i, a term in the complete splitting of $f_\#^k(E_i)$ is an edge of H_s.

Define the path $\widehat{\rho}_s$ to be trivial path at an arbitrarily chosen vertex if there does not exist any indivisible Nielsen path of height s, otherwise $\widehat{\rho}_s$ is the unique indivisible path of height s (see [FH11, Corollary 4.19]).

The groupoid $\langle \mathcal{Z}, \widehat{\rho}_s \rangle$. Let $\langle \mathcal{Z}, \widehat{\rho}_s \rangle$ be the set of lines, rays, circuits and finite paths in G which can be written as a concatenation of subpaths, each of which is an edge in Z, the path $\widehat{\rho}_s$ or

its inverse. Under the operation of tightened concatenation of paths in G, this set forms a groupoid [HM20, Lemma 5.6]. We say that a path, circuit, ray or line is *carried by* $\langle \mathcal{Z}, \widehat{\rho}_s \rangle$ if it can be written as a concatenation of paths in $\langle \mathcal{Z}, \widehat{\rho}_s \rangle$.

Define the graph K by setting $K = Z$ if $\widehat{\rho}_s$ is trivial and let $h : K \to G$ be the inclusion map. Otherwise, define an edge E_ρ representing the domain of the Nielsen path $\rho_s : E_\rho \to G_s$, and let K be the disjoint union of Z and E_ρ with the following identification. Given an endpoint $x \in E_\rho$, if $\rho_s(x) \in Z$, then identify $x \sim \rho_s(x)$. Given distinct endpoints $x, y \in E_\rho$, if $\rho_s(x) = \rho_s(y) \notin Z$, then identify $x \sim y$. In this case, define $h : K \to G$ to be the inclusion map on K and the map ρ_s on E_ρ. It is not difficult to see that the map h is an immersion. Hence, restricting h to each component of K, we get an injection at the level of fundamental groups. The *nonattracting subgroup system* $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$ is defined to be the subgroup system defined by this immersion.

Handel and Mosher proved that the nonattracting subgroup system has a maximality property with respect to subgroup inclusion (see [HM20, Corollary 1.8]).

We leave it to the reader to look it up in [HM20] where it is explored in detail. We do, however, list some key properties which we will use and these results exhibit the importance of this subgroup system. For ease of notation, we simply write $\hat{\sigma} = \widehat{\rho}_s$ when it is clear what height or Nielsen path we are working with.

LEMMA 3.1 [Lemmas 1.5 and 1.6 and Corollary 1.8, HM20].

(i) *A path is carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$ if and only if it lifts to an edge path in K (under the immersion defined earlier). A line or conjugacy class is carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$ if and only if it is carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$.*

(ii) *The set of lines carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$ is closed in the weak topology.*

(iii) *A conjugacy class $[c]$ is not attracted to Λ_ϕ^+ if and only if it is carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$.*

(iv) *For a finite-rank subgroup $B < \mathbb{F}$, if each conjugacy class is not weakly attracted to Λ_ϕ^+, then there exists some $A < \mathbb{F}$ such that $B < A$ and $[A] \in \mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$.*

(v) *The nonattracting subgroup system $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$ does not depend on the choice of the CT representing ϕ.*

(vi) *Given $\phi, \phi^{-1} \in \mathrm{Out}(\mathbb{F})$ both rotationless elements and a dual lamination pair $\Lambda_\phi^\pm$ we have*
$$\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+) = \mathcal{A}_{\mathrm{na}}(\Lambda_\phi^-)$$

(vii) *The nonattracting subgroup system $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$ is a free-factor system if and only if the stratum H_r is not geometric.*

(viii) *The nonattracting subgroup system $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$ is a malnormal subgroup system. More specifically, it is a vertex group system.*

In light of this lemma, we sometimes use the phrase 'α is an element of $\langle \mathcal{Z}, \hat{\sigma} \rangle$' and '$\alpha$ is carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$' interchangeably when α is a path, line or circuit in G. Similarly, when α is a line or a circuit, the phrase 'α is carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$' and '$\alpha$ is carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$' mean the same thing.

3.2 Weak attraction theorem

LEMMA 3.2 [HM20, Corollary 2.17]. *Let $\phi \in \mathrm{Out}(\mathbb{F})$ be a rotationless and exponentially growing. Let $\Lambda_\phi^\pm$ be a dual lamination pair for ϕ. Then for any line $\gamma \in \mathcal{B}$ not carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$, at least one of the following hold:*

(i) γ *is attracted to Λ_ϕ^+ under iterations of ϕ;*

(ii) γ *is attracted to Λ_ϕ^- under iterations of ϕ^{-1}.*

Moreover, if V_ϕ^+ and V_ϕ^- are attracting neighborhoods for the laminations Λ_ϕ^+ and Λ_ϕ^- respectively, there exists an integer $M \geq 0$ (depending on the choice of attracting neighborhoods) such that for every $\gamma \in \mathcal{B}$, at least one of the following holds:

(a) $\gamma \in V_\phi^-$;
(b) $\phi^M(\gamma) \in V_\phi^+$;
(c) γ is carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$.

3.3 Free-by-cyclic extensions for exponentially growing ϕ

The method of proof followed in this work was developed by the author in [Gho18], where examples of free-by-free (strongly) relatively hyperbolic extensions are constructed, which, in turn, was inspired by the work of Bestvina, Feighn and Handel in [BHF97], where they constructed free-by-free hyperbolic extensions.

For the rest of this section, we assume that $\phi \in \mathrm{Out}(\mathbb{F})$ is an exponentially growing and rotationless outer automorphism. Let $\Lambda_\phi^\pm$ be a dual lamination pair associated to this automorphism. In addition, let $f : G \to G$ be a CT map representing ϕ and H_r be the *unique* exponentially growing strata associated to Λ_ϕ^+ and $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$ be the nonattracting subgroup system of Λ_ϕ^+. Recall that, by construction, any conjugacy class is not weakly attracted to Λ_ϕ^+ under iterates of ϕ if and only if it is carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$. Similarly, let $f' : G' \to G'$ be a CT representing ϕ^{-1} and H_s' be the unique exponentially growing strata in G' that is associated to Λ_ϕ^-.

We use the term *generic leaf segment of* α repeatedly, by which we mean we are considering a subpath of α of height r (or height s) which is also a subpath of some generic leaf of Λ_ϕ^+ (or Λ_ϕ^-) and the first and last edges of this subpath are in H_r (or H_s', depending on the context). In general, when we talk about subpaths of generic leaves we are always considering leaf segments with their initial and terminal edges in H_r (or H_s').

Convention. We use the notation $|\alpha|_{H_r}$ to denote the r-length of a path α in G, i.e. we only count the edges of α contained in H_r for computing this length.

Bounded cancelation constant. Let $f : G \to G$ be a CT representing some $\phi \in \mathrm{Out}(\mathbb{F})$. The *bounded cancelation constant* for f, denoted by $BCC(f)$ is a constant such that given any concatenation of paths $\alpha\beta$ (with endpoints at vertices) in G, $f_\#(\alpha\beta)$ is obtained from the concatenation $f_\#(\alpha)f_\#(\beta)$ by canceling at most $BCC(f)$ edges from terminal end of $f_\#(\alpha)$ with at most $BCC(f)$ edges from initial end of $f_\#(\beta)$ (see [BFH00, Lemma 2.3.1]).

We recall the notion of critical constant from [BHF97, p. 219].

Critical constant. Let $f : G \to G$ be a CT for some exponentially growing $\phi \in \mathrm{Out}(\mathbb{F})$ with H_r being an exponentially growing strata with associated Perron–Frobenius eigenvalue λ and Λ_ϕ^+ be the attracting lamination associated to H_r. The number $2BCC(f)/(\lambda - 1)$ is called the *critical constant* for f corresponding to H_r.

Let C be some number greater than critical constant for f corresponding to H_r. Suppose $\alpha\beta\gamma$ is a concatenation of height r paths where β is some height r legal segment with r-length $\geq C$ and α, γ are height r legal paths. Note that $f_\#(\beta)$ is a legal path of height r. Then bounded cancelation implies that $f_\#^k(\alpha\beta\gamma)$ contains a legal segment of height r. It can be easily seen that because C exceeds the critical constant, there is some $1 \geq \mu > 0$ (depending on C) such that after tightening operation, the subpath of $f_\#^k(\beta)$ which survives as a height r legal segment in $f_\#^k(\alpha\beta\gamma)$ has r-length $\geq \mu\lambda^k|\beta|_{H_r} \geq \mu\lambda^k C$.

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It is perhaps worth noting here that the r-length of β grows exponentially fast under iteration by f, whereas the cancelations due to tightening at both ends of iterates of β happen at most by a fixed amount determined by the bounded cancelation constant of f. Thus, we may remove the requirement that α, γ are r-legal paths and instead assume that they are height r paths such that for some M we have $|f_\#^M(\alpha)|, |f_\#^M(\gamma)| > BCC(f^M)$. Then for that same M, the subpath of $f_\#^M(\beta)$ which survives as height r legal segment in $f_\#^M(\alpha\beta\gamma)$ has r-length at least $\mu\lambda^M C$.

To summarize: if we have a path in G which has some r-legal 'central' subsegment (such as β above) of length greater than the critical constant, with sufficient 'protection' on both sides (such as α, γ absorbing the cancelations above), then the bounded cancelation lemma protects this 'central' segment from completely canceling out under the tightening operation while iterating under f.

Following the work of Bestvina, Feighn and Handel [BHF97] we define the following notion of *legality* for any number $C > 0$ which exceeds the critical constant for f.

Notation. For any path α in G, we decompose $\alpha = \epsilon_1\delta_1 \cdots \epsilon_k\delta_k$. where each ϵ_i is a path which does not contain any subsegment that is an element of $\langle \mathcal{Z}, \hat{\sigma} \rangle$ and each δ_i is a path that is entirely carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$.

Let $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} = \sum |\epsilon_i|_G$, where $|\alpha_i|_G$ denotes the length of α_i in G. Recall that no edge of H_r is individually an element by $\langle \mathcal{Z}, \hat{\sigma} \rangle$ because it grows exponentially under iteration and limits to a generic leaf of the attracting lamination associated to H_r. However, a concatenation of edges of H_r can give us an indivisible Nielsen path (in that case, it would be one of the δ_i) which is represented by an element of $\langle \mathcal{Z}, \hat{\sigma} \rangle$.

Therefore, $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ denotes the edge length of α in G relative to $\langle \mathcal{Z}, \hat{\sigma} \rangle$, i.e. length of α in G, not counting the copies of subpaths of α which are carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$, where σ is the unique indivisible Nielsen path of height r (if it exists). In what follows, ρ will denote the unique indivisible Nielsen path of height s in the CT map $f' : G' \to G'$ for ϕ^{-1}. From train-track theory we know that if ρ is a closed Nielsen path, then so is σ and the conjugacy classes of ρ and σ are same up to reversal, because the laminations associated to the strata H_r and H_s' are dual to each other. In addition, recall from Bestvina–Feighn–Handel train-track theory that σ has exactly one illegal turn in H_r and, hence, does not occur as a subpath of any generic leaf of Λ_ϕ^+.

LEMMA 3.3. *Suppose $\phi \in \mathrm{Out}(\mathbb{F})$ is rotationless and exponentially growing with a lamination pair $\Lambda_\phi^\pm$ which are topmost for ϕ, ϕ^{-1}. Let $f : G \to G$ be a CT map representing ϕ and let δ be any path in G not carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$. If λ is Perron–Frobenius eigenvalue corresponding to the exponentially growing stratum H_r associated with Λ_ϕ^+, then $|f_\#^k(\delta)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \leq \lambda^k |\delta|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ for all $k \geq 0$.*

Proof. Recall that if H is an exponentially growing stratum such that no edge in H is weakly attracted to Λ_ϕ^+ under iterates of $f_\#$, then H is carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$. Next, consider an exponentially growing stratum H_t such that some edge, say E, of H_t is weakly attracted to Λ_ϕ^+ under iteration by $f_\#$. By using [HM20, Fact 1.59, item 1(c)] we see that some edge in the complete splitting of $f_\#^k(E)$, for all sufficiently large $k \geq 0$, will be an edge in H_r. Now apply [HM09, Proposition 2.31] to conclude that $t > r$ and the attracting lamination associated with H_t properly contains Λ_ϕ^+.

Therefore, our hypothesis that Λ_ϕ^+ being topmost implies that every exponentially growing strata having height not equal to r is carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$. Suppose edge E is any edge not carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$. This means either E is an edge in H_r or some NEG stratum of height greater than r not carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$. The maximum possible growth rate, measured by $|\cdot|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$, under iterates

of f is λ, i.e. $|f_\#^k(E)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \leq \lambda^k$. If δ is any path in G, we get $|f_\#^k(\delta)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \leq \lambda^k |\delta|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ for every $k \geq 0$, because every edge of δ grows at most by a factor of λ^k. $\qquad \square$

Standing assumptions for this section. For the rest of this section, fix some C much larger than the critical constants of f, f' corresponding to the EG strata H_r, H'_s, respectively. By increasing C if necessary, also assume that $C > \max\{2BCC(f), 2BCC(f')\}$. In addition, fix the μ corresponding to this choice of C arising from the notion of critical constant.

DEFINITION 3.4. For any circuit α in G, decompose α into a concatenation of paths each of which is either contained in the complement of G_r or a path in G_{r-1} or a path of height r. By α_i we denote a component (if it exists) in this decomposition of α which is a generic leaf segment of height r and $|\alpha_i|_{H_r} \geq C$. We look at the following ratio for this decomposition:

$$\frac{\sum |\alpha_i|_{H_r}}{|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}}.$$

The H_r-legality of α is defined as the maximum of the above ratio over all such decompositions of α and denoted by $LEG_{H_r^\sigma}(\alpha)$. Define $LEG_{H_r^\sigma}(\alpha) = 0$ if $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} = 0$.

The following lemma shows that there exists some uniform exponent M, such that iterating α by $\phi^{\pm M}$ gives us enough legality to eventually be proportional to the relative length of $\phi^{\pm M}$ in their respective marked graphs.

LEMMA 3.5 (Legality growth). *Suppose $\phi \in \mathrm{Out}(\mathbb{F})$ is rotationless and exponentially growing with a lamination pair $\Lambda_\phi^\pm$ which are topmost for ϕ, ϕ^{-1}. Then there exists $\epsilon > 0$ and some $M > 0$ such that for every conjugacy class α not carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$ and for every $m \geq M$, either $LEG_{H_r^\sigma}(\phi_\#^m(\alpha)) \geq \epsilon$ or $LEG_{H_s'^\rho}(\phi_\#^{-m}(\alpha)) \geq \epsilon$.*

Proof. We choose a long generic leaf segment γ^+ of some generic leaf of Λ_ϕ^+ and let $|\gamma^+|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \gg 3C$. Use γ^+ to define a weak attracting neighborhood V_ϕ^+ for Λ_ϕ^+. Similarly, choose a long generic leaf segment γ^- such that $|\gamma^-|_{\langle \mathcal{Z}', \hat{\rho} \rangle} \gg 3C$ and define an attracting neighborhood V_ϕ^-. The weak attraction theorem tells us that there exists some $M^+ > 0$ such that for any conjugacy class α that is not carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$ either $\alpha \in V_\phi^-$ or $\phi_\#^m(\alpha) \in V_\phi^+$ for every $m \geq M^+$. By a symmetric argument applied on ϕ^{-1}, we get $M^- > 0$. Let $M = \max\{M^+, M^-\}$. By increasing M if necessary, we also assume that $\mu \lambda^M > 2$, where λ is the Perron–Frobenius eigenvalue for H_r, μ is the constant arising out from the discussion on critical constant. We now show that there exists some $\epsilon_1 > 0$ such that for every conjugacy class α not carried by the nonattracting subgroup system either $LEG_{H_r^\sigma}(\phi_\#^M(\alpha)) \geq \epsilon_1$ or $LEG_{H_s'^\rho}(\phi_\#^{-M}(\alpha)) \geq \epsilon_1$. We use the change of markings map τ to push paths in G to paths in G' as follows: if α is a path in G, then we use the notation α' to denote the path $\tau_\#(\alpha)$ in G'.

Suppose there does not exist any such $\epsilon_1 > 0$ which satisfies the property stated previously, i.e. for each positive integer i, there exists a conjugacy class which is represented by a circuit δ_i in G such that $|\delta_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}, |\delta_i'|_{\langle \mathcal{Z}', \hat{\rho} \rangle} > 0$ (because they are not carried by the nonattracting subgroup system) and legality ratios of $f_\#^M(\delta_i)$ in G and $f'^M_\#(\delta_i'))$ in G' are both less than $1/i$. We argue to a contradiction by constructing a line which violates the weak attraction theorem.

First we handle the case when $|\delta_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ (respectively, $|\delta_i'|_{\langle \mathcal{Z}', \hat{\rho} \rangle}$) is unbounded. Pass to a subsequence, if necessary, and assume that both $|\delta_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ and $|\delta_i'|_{\langle \mathcal{Z}', \hat{\rho} \rangle}$ diverge to ∞ as $i \to \infty$. As δ_i are not carried by the nonattracting subgroup system, we have either $f_\#^M(\delta_i) \in V_\phi^+$ or

$\delta'_i \in V_\phi^-$ (which implies $f'^M_\#(\delta'_i) \in V_\phi^-$). However, given our assumption that $|\delta_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \to \infty$ and that the legality ratio of $f^M_\#(\delta_i)$ is small for all sufficiently large i, the height r leaf segments of $f^M_\#(\delta_i)$ contributing to its legality ratio is small compared with $|f^M_\#(\delta_i)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ (which diverges to ∞). We claim that we can find some sequence of subpaths $\alpha_i \subset \delta_i$ which has the following properties for all sufficiently large i:

(i) α_i not carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$;

(ii) $|\alpha_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ diverges to ∞;

(iii) for all sufficiently large i, $f^M_\#(\alpha_i)$ does not contain any height r generic leaf segment of Λ_ϕ^+ of r-length $\geq C$;

(iv) for all sufficiently large i, α_i cannot be written as a concatenation $\alpha_i = \alpha_{x_i} \alpha_{y_i} \alpha_{z_i}$, where α_{y_i} is a generic leaf segment of Λ_ϕ^+ of height r with $|\alpha_{y_i}|_{H_r} \geq C$ and $\alpha_{x_i}, \alpha_{z_i}$ are paths of height r with $|f^M_\#(\alpha_{x_i})|, |f^M_\#(\alpha_{z_i})| > BCC(f^M)$.

It is clear that items (i) and (ii) are easily satisfied. To see why we can choose the α_i so that item (iii) is also true, we argue by contradiction. If property (iii) fails, then there exists some $L > 0$ such that for each sufficiently large i, and for any subpath β_i of δ_i with $|\beta_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \geq L$, $|f^M_\#(\beta_i)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ contains a height r leaf segment of r-length $\geq C$. For every sufficiently large i, write δ_i as a concatenation of subpaths β_i^j such that $|\beta_i^j|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} = L$ for each j, except perhaps for at most one value of j. We then have that for each j, $f^M_\#(\beta_i^j)$ contains a r-legal generic leaf segment, say γ_i^j, having r-length $\geq C$. Fixing i and summing over the j, we obtain

$$LEG_{H_r^\sigma}(f^M_\#(\delta_i)) \geq \frac{\sum_j |\gamma_i^j|_{H_r}}{|f^M_\#(\delta_i)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}} \geq \frac{C}{|f^M_\#(\delta_i)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}} \frac{|\delta_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}}{L+1} \geq \frac{C}{\lambda^M(L+1)} > 0,$$

where the first inequality is due to the maximality clause in the definition of legality ratio. The third inequality is obtained by using Lemma 3.3. However, our assumption that the legality ratio of $f^M_\#(\delta_i)$ is less than $1/i$ for all sufficiently large i now gives us the desired contradiction.

If α_i satisfy items (i), (ii), (iii), then item (iv) must also be satisfied. Suppose property (iv) fails and let α_{y_i} be a subsegment of α_i such that α_{y_i} is a height r generic leaf segment with r-length $\geq C$. We can then write $\alpha_i = \alpha_{x_i} \alpha_{y_i} \alpha_{z_i}$ where $\alpha_{x_i}, \alpha_{z_i}$ are paths of height r with $|f^M_\#(\alpha_{x_i})|, |f^M_\#(\alpha_{z_i})| > BCC(f^M)$. This implies that $f^M_\#(\alpha_i)$ contains a generic leaf segment of r-length at least $\mu\lambda^M |\alpha_{y_i}|_{H_r}$ (follows from definition of critical constant). Since $|\alpha_{y_i}|_{H_r} \geq C$ and $\mu\lambda^M > 2$, we see that $f^M_\#(\alpha_i)$ contains a generic leaf segment of r-length greater than C, contradicting property (iii).

Suppose that for every such sequence of α_i that satisfies items (i)–(iv), α'_i fails to satisfy item (ii) for the map $f' : G' \to G'$ for all sufficiently large i. This would imply that for any sufficiently large subpath of δ_i whose $f^M_\#$ image does not contain a central generic leaf segment of r-length at least C, the $f'^M_\# \tau_\#$ image contains a central long generic leaf segment of s-length $\geq C$. However, this would violate that the legality ratios of both $f^M_\#(\delta_i)$ and $f'^M_\#(\delta'_i)$ converge to 0. Therefore, we assume that we have a sequence $\alpha_i \subset \delta_i$ which satisfies items (i)–(iv) for $f : G \to G$ and α'_i satisfies similar properties in $f' : G' \to G'$.

As the lengths of α_i in G diverge to ∞, we may assume (after perhaps adding a few edges and changing the length of α_i by at most $|G|$, where $|G|$ denotes the total number of edges in G) that for all sufficiently large i, α_i is a circuit not carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$, because $|\alpha_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \to \infty$.

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In addition, because the circuits α_i are not carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$, [HM20, Lemma 1.11] tells us that there exists a line ℓ which is a weak limit of some subsequence of α_i and ℓ is not carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$. This, by definition, implies that ℓ is not carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$. Therefore, neither is ℓ carried by the nonattracting subgroup system nor does the realization of ℓ in G or G' contain γ^+ or γ^- as a subpath (i.e. ℓ does not belong to the chosen attracting neighborhoods) by using properties (ii) and (iv) of α_i. In addition, we have that $f_\#^M(\ell) \notin V_\phi^+$ and $f'^M_\#(\ell) \notin V_\phi^-$ (by property (iii) of α_i), but this violates the weak attraction theorem.

Next we deal with the case when $|\delta_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ (respectively, $|\delta'_i|_{\langle \mathcal{Z}', \hat{\rho} \rangle}$) are bounded above. It directly follows that $|f_\#^M(\delta_i)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ and $|f'^M_\#(\delta'_i)|_{\langle \mathcal{Z}', \hat{\rho} \rangle}$ are both bounded above. Applying the weak attraction theorem, we also have that $f_\#^M(\delta_i)$ has a central leaf segment of length at least C, hence the numerator in the legality ratio of $f_\#^M(\delta_i)$ is at least 1. Similarly for legality ratio of $f'^M_\#(\delta'_i)$. Hence, this contradicts our assumption that both of these legality ratios converge to zero as $i \to \infty$. Thus, the case when the $|\delta_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ is bounded is ruled out.

Hence, we conclude that an $\epsilon_1 > 0$ does indeed exist so that either $LEG_{H_r^\sigma}(\phi_\#^M(\alpha)) \geq \epsilon_1$ or $LEG_{H_s'^\rho}(\phi_\#^{-M}(\alpha)) \geq \epsilon_1$.

For the final step, take any conjugacy class α not carried by the nonattracting subgroup system and assume without loss that $LEG_{H_r^\sigma}(\phi_\#^M(\alpha)) \geq \epsilon_1$. As we are working with a topmost lamination it follows that $|f_\#^k(\phi_\#^M(\alpha))|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \leq \lambda^k |\phi_\#^M(\alpha)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ for every $k > 0$ by using Lemma 3.3. If $m > M$, it now follows from the definitions that $LEG_{H_r^\sigma}(\phi_\#^m(\alpha)) \geq \epsilon$, where $\epsilon = \epsilon_1 \mu$. Recall that $\mu \leq 1$ and, therefore, we set $\epsilon = \epsilon_1 \mu$ to complete the proof. $\qquad \square$

Given a conjugacy class α not carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$, the following lemma compares the growth of legal segments of α relative to the size of α, when both are being measured in terms of paths not carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$, i.e. $|\cdot|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$.

Before we state the lemma, we would like to point out a small subtlety regarding the *topmost lamination* hypothesis. As Λ_ϕ^+ is a topmost lamination, any exponentially growing strata of height greater than r is carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$ (see proof of Lemma 3.3). Thus, the only strata of height $s > r$ which can get attracted to Λ_ϕ^+ are superlinear NEG edges of the form $E_s \mapsto E_s u_s$, where u_s is a circuit contained in G_{s-1} and gets attracted to Λ_ϕ^+. Hence, neither E_s nor u_s are carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$. In this situation, for any circuit α not carried by the nonattracting subgroup system, the part of α that lies above H_r only grows polynomially and the part that intersects with H_r grows exponentially. This enables us to control the exponent for flaring in the following lemma. However, if we work with a circuit that does not get attracted to any lamination which properly contains Λ_ϕ^+, the lemma will still be true for all such circuits without the topmost assumption.

If we remove the *topmost* assumption on Λ_ϕ^+, then we could have a circuit α that has a long overlap with both Λ^- (which properly contains Λ_ϕ^-) and Λ_ϕ^+. It would be very tricky to control the exponents of the following lemma in that case. However, if we consider a conjugacy classes which are attracted to Λ_ϕ^+, but not weakly attracted to any lamination properly containing Λ_ϕ^+, both Lemmas 3.5 and 3.6 will still be true for such circuits. This observation will be useful in the proof of Proposition 5.5.

LEMMA 3.6. *Suppose $\phi \in \mathrm{Out}(\mathbb{F})$ is rotationless and exponentially growing with a lamination pair $\Lambda_\phi^\pm$ which are topmost for ϕ, ϕ^{-1}. Then for every $\epsilon > 0$ and $A > 0$, there is M_1 depending*

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only on ϵ, A such that if $LEG_{H_r^\sigma}(\alpha) \geq \epsilon$ for some circuit α, then

$$|f_\#^m(\alpha)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \geq A|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$$

for every $m \geq M_1$.

Proof. Using the description of critical constant we can see that

$$|f_\#^m(\alpha)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \geq \mu\lambda^m \left\{ \sum |\text{generic leaf segments of } \alpha \text{ with } r\text{-length} \geq C|_{H_r} \right\}$$

$$\geq \mu\lambda^m \epsilon |\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}.$$

Choose m to be large enough so that $\mu\lambda^m \epsilon \geq A$. $\qquad\square$

Recall the nonattracting subgroup system $\mathcal{A}_{\text{na}}(\Lambda_\phi^\pm) = \{[F^1], [F^2], \ldots, [F^k]\}$ is a mutually malnormal subgroup system of finitely generated subgroups of $\mathbb{F}$ (hence, quasiconvex). Hence, one can start with the Cayley graph of $\mathbb{F}$ and construct the coned-off space $\widehat{\mathbb{F}}$ with respect to the collection $\{F^i\}$ and form a electrocuted metric space denoted by $(\widehat{\mathbb{F}}, |\cdot|_{\text{el}})$. The group $\mathbb{F}$ is then strongly hyperbolic relative to the collection $\{F^i\}$. Note, that this property of $\widehat{\mathbb{F}}$ is still true if we replace any of the F^i with a conjugate of itself. We stress the fact here that $\mathcal{A}_{\text{na}}(\Lambda_\phi^\pm)$ is not necessarily a free-factor system. For a conjugacy class not carried by $\mathcal{A}_{\text{na}}(\Lambda_\phi^\pm)$, by $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ we denote the length (relative to $\langle \mathcal{Z}, \hat{\sigma} \rangle$) of the circuit in G that realizes this conjugacy class. By $\|\alpha\|_{\text{el}}$ we denote the length of the shortest representative of the conjugacy class α in the electrocuted metric space $(\widehat{\mathbb{F}}, |\cdot|_{\text{el}})$. Note that $\|\alpha\|_{\text{el}}$ is independent of the graph G on which we have the relative train-track map. The following lemma establishes that these two measurements are comparable and, hence, our results are independent of the choice of train-track map. Before we begin, we would like to state an example which will perhaps be useful for the reader to understand the way we are computing the bounds.

The first half of the proof gives an upper bound to the electrocuted length of lifts (to the electrocuted universal cover of marked graph G) of circuits α in the graph G in terms of $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ because lifting a circuit can cause its length to blow up in the electrocuted metric of the universal cover, we need to show that we can control this. Consider the automorphism which is realized by a train-track map on the rose with two edges $f(a) = ab$, $f(b) = bab$. Here the indivisible Nielsen path $\sigma = abAB$, where A, B are inverses of a, b, respectively. Consider the conjugacy class of the circuit $\alpha = ab\sigma Ab\sigma$. Then $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} = 4$ and $\|\alpha\|_{\text{el}} = 6$. All the embedded paths in the rose here can be listed as ab, aB, Ab, AB and their inverses. We can get a bound on $\|\alpha\|_{\text{el}}$ in terms of a multiple of $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ by counting the number of such embedded paths and copies of the path σ each of which can appear at most $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ times.

The second half of the proof deals with bounding a similar problem which can happen when we project paths from the electrocuted universal cover to the marked graph G. An example to keep in mind would be the geodesic $a\overline{\sigma}a$ which gives $\|a\overline{\sigma}a\|_{\text{el}} = 3$, but its projection has length 4 in the rose. By rewriting $a\overline{\sigma}a$ as $abaBAa$ we see that when we tighten this path in the projection, there are no copies of σ in it. Thus, each segment of such type will cause a discrepancy of length at most 4 in this example. Then we can bound $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ by a multiple of $\|\alpha\|_{\text{el}}$ by counting the maximum number of times such discrepancies can occur.

Suppose Λ^+ is an attracting lamination for $\phi \in \text{Out}(\mathbb{F})$ and H_p is the EG strata associated to Λ^+. If there is a closed indivisible Nielsen path of height p (equivalently, if H_p is a geometric strata [HM20, Proposition 2.18]), then Λ^+ is minimal (i.e. does not contain any other attracting lamination (see [HM09, Corollary 2.32] and [HM20, Proposition 2.15])). It is due to this property that we do not need the 'topmost lamination' assumption for the following lemma.

LEMMA 3.7 (Length comparison). *Suppose $\phi \in \mathrm{Out}(\mathbb{F})$ is rotationless and exponentially growing with a dual lamination pair $\Lambda_\phi^\pm$ and α is any circuit in G which is not carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$. Then there exists some $K > 0$, independent of α, such that $K \geq |\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} / \|\alpha\|_{\mathrm{el}} \geq 1/K$.*

Proof. As we are working with a conjugacy class which is not carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$, the circuit α representing such a conjugacy class is not carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$, hence $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \geq 1$. Let H_r denote the exponentially growing stratum corresponding to the lamination Λ_ϕ^+.

Let $L_2' = \max\{|\tilde{\epsilon}_i|_{\mathrm{el}}^{\tilde{G}}\}$ where ϵ_i varies over finitely many embedded paths (i.e. no repetition of an edge or its inverse) in G which consist entirely of edges of G which are not carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$. Each such $\tilde{\epsilon}_i$ is a geodesic (in the electrocuted universal cover) that does not pass through any conepoint. The electrocuted geodesic representing $\tilde{\alpha}$ can be written as a concatenation of paths of type $\tilde{\epsilon}_i$ and geodesics of length one connecting two points inside the copy of a coset, via the conepoint. The number of such $\tilde{\epsilon}_i$ appearing in a decomposition of $\tilde{\alpha}$ is at most $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} + 1$ and the number of components of the other type (via the conepoints) is at most $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$. Hence, if α is any circuit in G which is not carried by the nonattracting subgroup system, then

$$|\tilde{\alpha}|_{\mathrm{el}}^{\tilde{G}} \leq L_2'(|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} + 1) + |\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \implies |\tilde{\alpha}|_{\mathrm{el}}^{\tilde{G}} / |\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \leq L_2,$$

for some $L_2 (= 3L_2')$ independent of α.

Suppose next that $w \in \mathbb{F}$ is some cyclically reduced word not in the union of F^i. Let $\tilde{\alpha}$ be a path in electrocuted universal cover $\tilde{G}$ that represents the geodesic connecting the identity element to w, under the lift of the marking on G. Suppose $\tilde{\alpha} = \tilde{u}_1 X_1 \tilde{u}_2 X_2 \cdots \tilde{u}_n X_n$, where X_i are geodesic paths in $\tilde{G}$ connecting two points in a copy of some coset via the attached conepoint and u_i are geodesic paths in $\tilde{G}$ which connect copies of two electrocuted cosets and does not pass through any conepoint. $\tilde{u}_1, X_n$ could possibly be trivial.

Under this setup we have

$$|\tilde{\alpha}|_{\mathrm{el}}^{\tilde{G}} = |\tilde{u}_1|_{\mathrm{el}}^{\tilde{G}} + |\tilde{u}_2|_{\mathrm{el}}^{\tilde{G}} + \cdots + |\tilde{u}_n|_{\mathrm{el}}^{\tilde{G}} + n.$$

Modify $\tilde{\alpha}$ by replacing each X_i with a path $\tilde{v}_i$ inside the corresponding coset, such that $\tilde{v}_i$ is a geodesic in the standard metric on $\tilde{G}$ (i.e. before electrocution). Consider projection of this modified path obtained from $\tilde{\alpha}$ to G, and tighten the projection of $\tilde{u}_i$ to get the path u_i. This gives the inequality $|u_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \leq |\tilde{u}_i|_{\mathrm{el}}^{\tilde{G}}$. The projection of each $\tilde{v}_i$ is a path carried by $\langle \mathcal{Z}, \hat{\sigma} \rangle$, so it does not contribute to the relative length of the tightened projection of $\tilde{\alpha}$ unless the projection of $\tilde{v}_i$, call it v_i^*, is a closed indivisible Nielsen path of height r and there is cancelation between v_i^* and u_i and/or u_{i+1} so that after tightening, the path $u_i v_i^* u_{i+1}$ has no copies of the closed indivisible Nielsen path of height r (this is where we use the property that an EG strata of height $p > r$, which is not contained in $\mathcal{Z}$, has no corresponding closed indivisible Nielsen path of height p). This could bump up the length of the projected path. However, each such v_i^* can contribute a maximum of length $2|G| + 1$ to the tightened (modified) projection of $\tilde{\alpha}$ (call it α).

Thus, α is obtained by tightening of $u_1 v_1^* u_2 v_2^* \cdots u_n v_n^*$, to give us a circuit which represents the conjugacy class of the word determined by $\tilde{\alpha}$. Let j vary over all such indices of v_i^* where v_i^* is the indivisible Nielsen path of height r. We then obtain

$$|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \leq \sum |u_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} + \sum |v_j^*|_G \leq |\tilde{\alpha}|_{\mathrm{el}}^{\tilde{G}} + |\tilde{\alpha}|_{\mathrm{el}}^{\tilde{G}} \cdot (2|G| + 1) \leq 2|\tilde{\alpha}|_{\mathrm{el}}^{\tilde{G}}(1 + |G|).$$

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Therefore, $|\alpha|_{\langle\mathcal{Z},\hat{\sigma}\rangle}/|\tilde{\alpha}|_{\text{el}}^{\tilde{G}} \leq L_1(= 2(1+|G|))$. Combining the two inequalities, we obtain

$$\frac{1}{L_1} \leq |\tilde{\alpha}|_{\text{el}}^{\tilde{G}}/|\alpha|_{\langle\mathcal{Z},\hat{\sigma}\rangle} \leq L_2. \tag{1}$$

For the last step use the lift of marking map on G to $\tilde{G}$ and observe that there is an $\mathbb{F}$-equivariant quasi-isometry between $\tilde{G}$, with electrocuted metric, and the universal cover of the standard rose for $\mathbb{F}$ with the electrocuted metric obtained by electrocuting cosets of the collection of subgroups $\{F^i\}$. Hence, there exists some $K' > 0$ such that for every cyclically reduced word $w \in \mathbb{F} \setminus \cup F^i$ we have

$$\frac{1}{K'} \leq |w|_{\text{el}}/|\tilde{\alpha}_w|_{\text{el}}^{\tilde{G}} \leq K', \tag{2}$$

where $\tilde{\alpha}_w$ is the electrocuted geodesic in $\tilde{G}$ connecting the image of identity element to image of w under the marking map on $\tilde{G}$ and $|w|_{\text{el}}$ is the length of the electrocuted geodesic in $\hat{\mathbb{F}}$ connecting identity element and w. Hence, combining the inequalities (1) and (2) above we can conclude that there exists some $K > 0$ such that

$$K \geq |\alpha|_{\langle\mathcal{Z},\hat{\sigma}\rangle}/\|\alpha\|_{\text{el}} \geq \frac{1}{K},$$

where $\|\alpha\|_{\text{el}}$ is the electrocuted length, in $\hat{\mathbb{F}}$, of the cyclically reduced word whose conjugacy class is represented by α in G. $\square$

COROLLARY 3.8 (Conjugator growth). *If we have a lift Φ such that $\Phi(F^i) = F^i$ and $\Phi(F^j) = x_j^{-1}F^j x_j$, for some $x_j \in \mathbb{F}$, then $\left|\Phi_{\#}^n(x_j)\right|_{\text{el}}$ must grow exponentially fast as $n \to \infty$.*

Proof. First, we note that $x_j \notin F^i$, otherwise we can construct a lift that leaves both F^i and F^j invariant. This would contradict the maximality of the subgroups (see Lemma 3.1(iv)) whose conjugacy classes form the nonattracting subgroup system. Let f_i, f_j be nontrivial elements of F^i, F^j, respectively, and consider a word $w = f_i x_j^{-1} f_j x_j$ and iterate this under Φ. As conjugacy class of w is not carried by the nonattracting subgroup system, $|\phi_{\#}^n[w]|_{\langle\mathcal{Z},\hat{\sigma}\rangle}$ grows exponentially as $n \to \infty$. If x_j has nonexponential growth, in $\hat{\mathbb{F}}$, under iterates of Φ it implies that $|\Phi_{\#}^n(w)|_{\text{el}}$ has nonexponential growth in the electrocuted metric. This would contradict Lemma 3.7. $\square$

Observe that the proof actually tells us that $|x_j\Phi(x_j)\Phi^2(x_j)\cdots\Phi^n(x_j)|_{\text{el}}$ grows exponentially fast as $n \to \infty$. This observation will be useful when we try to prove the cone-bounded hallway flaring condition in Lemma 3.12.

In addition, note that this corollary is trivially true if there are no peripheral subgroups, i.e. if the nonattracting subgroup system is trivial. This is because the triviality of the nonattracting subgroup system implies that every conjugacy class is attracted to the corresponding attracting lamination and, thus, will be exponentially growing.

PROPOSITION 3.9 (Conjugacy flaring). *Suppose $\phi \in \text{Out}(\mathbb{F})$ is rotationless and exponentially growing with a lamination pair $\Lambda_\phi^\pm$ which are topmost. Then there exists some $M_0 > 0$ such that for every conjugacy class α not carried by $\mathcal{A}_{\text{na}}(\Lambda_\phi^\pm)$, we have*

$$3\|\alpha\|_{\text{el}} \leq \max\{\|\phi_{\#}^m(\alpha)\|_{\text{el}}, \|\phi_{\#}^{-m}(\alpha)\|_{\text{el}}\}$$

for every $m \geq M_0$.

Proof. Let M, ϵ be as in Lemma 3.5 and let λ_r, λ_s be stretch factors associated to the EG strata for Λ_ϕ^+ and Λ_ϕ^-, respectively. Every conjugacy class not carried by $\mathcal{A}_{\text{na}}(\Lambda_\phi^\pm)$ is stretched at most

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by a factor of λ_r under ϕ and by factor of λ_s under ϕ^{-1}. Thus, for every conjugacy class α not carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$, we get $|\phi_\#^M(\alpha)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \leq \lambda_r^M |\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$, which implies $|\alpha|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \leq \lambda_r^M |\phi_\#^{-M}(\alpha)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ by replacing α with $\phi^{-M}(\alpha)$. By a symmetric argument we obtain an inequality involving λ_s. Use Lemma 3.7 to choose some number $D > 0$ such that we have $\|\phi_\#^M(\alpha)\|_{\mathrm{el}} \geq \|\alpha\|_{\mathrm{el}}/D$ and $\|\phi_\#^{-M}(\alpha)\|_{\mathrm{el}} \geq \|\alpha\|_{\mathrm{el}}/D$ for every conjugacy class α not carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$. Note that D is chosen arbitrarily here, only to track the ratio of the numbers above and it depends only on M, λ_r, λ_s. We also have $LEG_{H_r^\sigma}(\phi_\#^M(\alpha)) \geq \epsilon$ or $LEG_{H_s'^\rho}(\phi_\#^{-M}(\alpha)) \geq \epsilon$.

By applying Lemma 3.7, we may choose some constant K such that for every conjugacy class α as above, either $K \geq |\phi_\#^M(\alpha)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}/\|\phi_\#^M(\alpha)\|_{\mathrm{el}} \geq 1/K$ or $K \geq |\phi_\#^{-M}(\alpha)|_{\langle \mathcal{Z}', \hat{\rho} \rangle}/\|\phi_\#^{-M}(\alpha)\|_{\mathrm{el}} \geq 1/K$.

For concreteness assume that $LEG_{H_r^\sigma}(\phi_\#^m(\alpha)) \geq \epsilon$. Then by applying Lemma 3.6 with ϵ and $A = 3DK^2$, we get that there exists some M_1 such that for all $m \geq M_0 > M + M_1$, such that

$$
\begin{aligned}
\|\phi_\#^m(\alpha)\|_{\mathrm{el}} &\geq \frac{1}{K} |\phi_\#^m(\alpha)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \\
&\geq \frac{1}{K} 3DK^2 |\phi_\#^M(\alpha)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \\
&\geq 3DK \frac{1}{K} \|\phi_\#^M(\alpha)\|_{\mathrm{el}} = 3D \|\phi_\#^M(\alpha)\|_{\mathrm{el}} \\
&\geq 3D \frac{1}{D} \|\alpha\|_{\mathrm{el}} = 3\|\alpha\|_{\mathrm{el}}.
\end{aligned}
\tag{3}
$$

$\square$

The following lemma proves that conjugacy flaring implies the Mj–Reeves *cone-bounded hallways strictly flaring* condition. The technique used in the proof is similar to that used by Bestvina, Feighn and Handel in [BHF97, Theorem 5.1]. Recall that we are working with an exponentially growing outer automorphism with a dual lamination pair $\Lambda_\phi^\pm$ such that $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm) = \{[F^1], [F^2], \ldots, [F^k]\}$.

PROPOSITION 3.10 (Hallway flaring). *Suppose $\phi \in \mathrm{Out}(\mathbb{F})$ is rotationless and exponentially growing with a lamination pair $\Lambda_\phi^\pm$ which are topmost. Choose a lift $\Phi \in \mathrm{Aut}(\mathbb{F})$ of ϕ. There exists numbers $N_\Phi > 0$ and $L_\Phi > 0$ such that for every word $w \in \mathbb{F} \setminus \bigcup_i F^i$ with $|w|_{\mathrm{el}} \geq L_\Phi$ we have*

$$
2|w|_{\mathrm{el}} \leq \max\{|\Phi_\#^n(w)|_{\mathrm{el}}, |\Phi_\#^{-n}(w)|_{\mathrm{el}}\}
$$

for every $n \geq N_\Phi$. Moreover, if w is cyclically reduced, then the result holds for all w such that $|w|_{\mathrm{el}} > 1$.

Proof. Recall that any subgraph $H \subset G$ determines a free-factor system $\mathcal{F}_r$, the components of $\mathcal{F}_r$ being determined by the connected components of H. Consider the filtration element G_r corresponding to H_r and let $\mathcal{F}_r$ be the free-factor system determined by G_r. The free-factor support of $\Lambda_\phi^\pm$, denoted by $[B_r]$ say, has the property $[B_r] \sqsubset \mathcal{F}_r$ (see [BFH00, § 2.6]). As Φ leaves some conjugate of B_r invariant, let B denote such an invariant free factor. We note here that B is not necessarily a proper free factor. However, we can choose a basis of B so that none of the conjugacy classes of the basis elements are carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$. We work with such a basis.

Let $L' = \max\{|\Phi_\#^i(b_j)|_{\mathrm{el}} \mid i = 0, \pm 1, \pm 2, \ldots, \pm M_0\}$ where $b_j \in B$ varies over all the basis elements of B and M_0 is the constant from Proposition 3.9. By the description of L', we have that $|\Phi_\#^i(b_j)|_{\mathrm{el}} \geq 1/L'$ for all i, j as described previously.

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Assume $w \in \mathbb{F} \setminus \bigcup_i F^i$ and $|w|_{\mathrm{el}} \geq L' - 3$.

The proof is by induction. For the base case let $n = M_0$.

If w is a cyclically reduced word, then conjugacy class of w is not carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$ and so by using Proposition 3.9 we have

$$\max\{|\Phi_\#^n(w)|_{\mathrm{el}}, |\Phi^{-n}(w)|_{\mathrm{el}}\} \geq 3|w|_{\mathrm{el}} \geq 2|w|_{\mathrm{el}}.$$

If w is not cyclically reduced, then we can choose a basis element $k \in B$, such that $kw \in \mathbb{F} \setminus \cup F^i$ is a cyclically reduced word. Hence, we get the same inequality as previously, but with w being substituted by kw.

For concreteness, suppose that $|\Phi_\#^n(kw)|_{\mathrm{el}} \geq 3|kw|_{\mathrm{el}}$. Then we have $3|kw|_{\mathrm{el}} \leq |\Phi_\#^n(kw)|_{\mathrm{el}} \leq |\Phi_\#^n(w)|_{\mathrm{el}} + |\Phi_\#^n(k)|_{\mathrm{el}} \leq |\Phi_\#^n(w)|_{\mathrm{el}} + L'$. This implies that $3 + 3|w|_{\mathrm{el}} - L' \leq 3|kw|_{\mathrm{el}} - L' \leq |\Phi_\#^n(w)|_{\mathrm{el}}$ since $|k|_{\mathrm{el}} = 1$ (because k is a basis element) and there is no cancelation between k and w. As we have $|w|_{\mathrm{el}} \geq L' - 3$, the above inequality then implies $2|w|_{\mathrm{el}} \leq |\Phi_\#^n(w)|_{\mathrm{el}}$ and we are done with the base case for our inductive argument. In addition, note that $L' - 3 < |\Phi_\#^n(w)|_{\mathrm{el}}$ and so $\Phi_\#^n(w) \notin \cup F^i$. By repeatedly applying the above argument, one concludes that $\Phi_\#^{sn}(w) \notin \cup F^i$ for all $s > 0$.

Now assume that $M_0 < n$ for the inductive step. First observe that from what we have proven so far, given any integer $s > 0$ we have either $|\Phi_\#^{sM_0}(w)|_{\mathrm{el}} \geq 2^s|w|_{\mathrm{el}}$ or $|\Phi_\#^{-sM_0}(w)|_{\mathrm{el}} \geq 2^s|w|_{\mathrm{el}}$. Fix some positive integer s_0 such that $2^{s_0} > 2L'$. Any integer $n > s_0 M_0$ can be written as $n = sM_0 + t$ where $0 \leq t < M_0$ and $s_0 \leq s$. If $|\Phi_\#^{sM_0}(w)|_{\mathrm{el}} \geq 2^s|w|_{\mathrm{el}}$, then we can deduce

$$|\Phi_\#^n(w)|_{\mathrm{el}} = |\Phi_\#^{sM_0+t}(w)|_{\mathrm{el}} \geq 2^s|w|_{\mathrm{el}}/L' \geq 2|w|_{\mathrm{el}}.$$

Similarly, when $|\Phi_\#^{-sM_0}(w)|_{\mathrm{el}} \geq 2^s|w|_{\mathrm{el}}$, we can prove by a symmetric argument that $|\Phi_\#^{-n}(w)|_{\mathrm{el}} \geq 2|w|_{\mathrm{el}}$. Set $L_\Phi \geq L' - 3$ and $N_\Phi > s_0 M_0$ to conclude the proof. $\square$

Remark 3.11. We would like to point out that one can be done with the proof of Proposition 3.13 by using the combination theorem due to Gautero [Gau16] because Proposition 3.10 essentially shows that the automorphism Φ is 'relatively hyperbolic' in the sense of Gautero and Lustig [GL07]. However, we take a slightly different approach to leave open some room for further generalizations in the future.

For the rest of the paper, fix some lift $\Phi \in \mathrm{Aut}(\mathbb{F})$ of ϕ. Observe that the construction of the nonattracting subgroup system (in particular, the fact that it is a *vertex group system*) implies that Φ can leave at most one of F^i invariant (see Lemma 3.1(iv)). This implies that if $\Phi(F^i) = w_i F^i w_i^{-1}$ and $\Phi(F^j) = w_j F^j w_j^{-1}$, then $w_i \neq w_j$ if $i \neq j$. For any $w \in \mathbb{F}$, use the notation $w^{-1}\Phi$ to denote the composition of the inner automorphism $x \mapsto w^{-1}xw$ with Φ. Let $\Phi_i = w_i^{-1}\Phi$ and $\Phi_j = w_j^{-1}\Phi$ in Γ. Applying Corollary 3.8, we see that $w_j^{-1}w_i \notin F^j$ and $w_i^{-1}w_j \notin F^i$. Using the action of Φ on $\mathbb{F}$ we get $\Phi_i^n = w_i^{-1}\Phi(w_i^{-1})\cdots\Phi^{(n-1)}(w_i^{-1})\Phi^n$.

To be in sync with the framework for the cone-bounded hallway flaring condition, we work with a lift Φ_i (constructed from Φ) such that $\Phi_i(F^i) = F^i$ and $\Phi_i(F^j) = x_j F^j x_j^{-1}$. Then $\Phi_j = x_j^{-1}\Phi_i$ is a lift which leaves F^j invariant and $\Phi_j(F^i) = x_j^{-1}F^i x_j$. A simple computation using the action of the automorphism Φ_i on Γ, we get that $\Phi_i^{-n}\Phi_j^n = \Phi_i^{-n}(x_j^{-1})\Phi_i^{-(n-1)}(x_j^{-1})\cdots\Phi_i^{-1}(x_j^{-1})$. This means that $\Phi_i^{-n}\Phi_j^n$ is an inner automorphism which is defined by the conjugating word $\Phi_i^{-n}(x_j^{-1})\Phi_i^{-(n-1)}(x_j^{-1})\cdots\Phi_i^{-1}(x_j^{-1}) \in \mathbb{F}$. Recalling Corollary 3.8 (and the remark just after it) we have that this conjugator word grows exponentially as $n \to \infty$.

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We have to bound the exponent to get flaring, which is done in the following lemma, and it gives the *cone-bounded hallways strictly flaring condition* as we soon show.

LEMMA 3.12 (Cone-bounded hallway flaring). *Suppose $\phi \in \mathrm{Out}(\mathbb{F})$ is rotationless and exponentially growing with a lamination pair $\Lambda_\phi^\pm$ which are topmost. Let Φ_i be a lift such that $\Phi_i(F^i) = F^i$ and $\Phi_i(F^j) = x_j F^j x_j^{-1}$ for some $x_j \in \mathbb{F}$ when $j \neq i$. Then there exists some $N_c > 0$ such that*

$$2|x_j|_{\mathrm{el}} \leq \max\{|\Phi_{i\#}^n(x_j)|_{\mathrm{el}}, |\Phi_{i\#}^{-n}(x_j)|_{\mathrm{el}}\}$$

for every $n \geq N_c$ (independent of i, j) and for every such $j \neq i$.

Proof. From Corollary 3.8 we have that $x_j \notin F^i \cup F^j$ and x_j has exponential growth under iterates of Φ_i in the electric metric in $\widehat{\mathbb{F}}$. In addition, note that if y_j is another word in $\mathbb{F}$ such that $\Phi_i(F^j) = y_j F^j y_j^{-1}$, then we have $x_j F^j x_j^{-1} = y_j F^j y_j^{-1}$. The malnormality of the subgroup system immediately implies that $y_j = x_j f_j$ for some $f_j \in F^j$. This shows that once Φ_i is chosen, the corresponding conjugators x_j are uniquely determined up to choosing the tail f_j. For concreteness, we assume that x_j has no tail in F^j. If y_j is another conjugator such that $y_j = x_j f_j$, then $|y_j|_{\mathrm{el}} = |x_j|_{\mathrm{el}} + 1$ and $\Phi_i^k(y_j) = \Phi_i^k(x_j)\Phi_i^{k-1}(x_j) \cdots \Phi_i(x_j)x_j f_{j,k} x_j^{-1} \cdots \Phi_i^{k-1}(x_j^{-1})$ for some $f_{j,k} \in F^j$. Note that for no k can we have $\Phi_i(f_{j,k}) = x_j f_{j,k} x_j^{-1}$ as this violates malnormality of F^j. Corollary 3.8 and the remark after the lemma tells us that electrocuted length of y_j grows exponentially under Φ_j. Our calculations here show that $|\Phi_i^k(y_j)|_{\mathrm{el}}$ grows exponentially if and only if $|\Phi_i^k(x_j)\Phi_i^{k-1}(x_j) \cdots \Phi_i(x_j)x_j|_{\mathrm{el}}$ grows exponentially.

Let N_i', L_i' be the constants we obtain as output from Proposition 3.10 by replacing Φ with Φ_i. Set $L = \max\{L_i'\}$. If $|x_j|_{\mathrm{el}} > L$, then we are done by using Proposition 3.10. Otherwise, let $N_2^{i,j}$ be a constant such that $|\Phi_i^{N_2^{i,j}}(x_j)|_{\mathrm{el}} > 2L$ and $|\Phi_j^{N_2^{i,j}}(x_j)\Phi_j^{N_2^{i,j}-1}(x_j) \cdots \Phi_j(x_j)x_j|_{\mathrm{el}} > 2L$. Such a constant always exists follows from Corollary 3.8. We take N_i to be the maximum of N_i' and the $\{N_2^{i,j}\}$ as j varies over a finite set of indices.

Finally, let N_c be the maximum of all the N_i, as i varies over the indices of Φ_i, where any Φ_j is obtained from Φ_i by precomposing with inner automorphism defined by the word x_j. $\square$

Remark. We would like to point out to the reader how to deduce [BFH00, Theorem 5.1], which deals with case when ϕ is fully irreducible and nongeometric. Proposition 3.10 gives us the flaring condition for all nontrivial cyclically reduced words and words which are not cyclically reduced but have length greater than or equal to L. If w is a nontrivial word which is not cyclically reduced and has length less than or equal to L, then using the weak attraction theorem we can conclude that the word grows exponentially under iterates of any chosen lift of ϕ. As one only has finitely many words of absolute length less than or equal to L in $\mathbb{F}$, we get a uniform exponent for all nontrivial words just as in [BFH00, Theorem 5.1]. Note that for the fully irreducible and nongeometric case, the nonattracting subgroup system of the unique attracting lamination for ϕ is trivial. The difficulty of producing such a strong statement in the case when the F^i are nontrivial is due to presence of words of the form $f_i f_j f_i^{-1}$ where $i \neq j$ and $f_i \in F^i$.

For convenience of the reader we reproduce some of the terminologies used in the Mj–Reeves strong combination theorem for relative hyperbolicity and put it in the context that we have here. Recall that we have fixed some $\Phi \in \mathrm{Aut}(\mathbb{F})$ which is a lift of $\phi \in \mathrm{Out}(\mathbb{F})$.

We start with the short exact sequence

$$1 \to \mathbb{F} \to \Gamma \to \langle \Phi \rangle \to 1.$$

The Cayley graph of the quotient group gives us a tree T, in fact a line in this case, which we shall use to construct a tree of spaces structure for Γ. The map from Γ to T is the obvious one given by the short exact sequence. The preimage of each vertex is a coset of $\mathbb{F}$ and same for the edges.

(i) We perform an electrocution of $\mathbb{F}$ with respect to the collection $\{F^i\}$, denoted by $(\widehat{\mathbb{F}}, |\cdot|_{\mathrm{el}})$, and because $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$ is a malnormal subgroup system, $\mathbb{F}$ is (strongly) relatively hyperbolic with respect to the collection $\{F^i\}$. This gives a (strongly) relatively hyperbolic structure on the edge spaces and vertex spaces associated to T. Hence, conditions (1) and (2) of [MR08, Definition 3.1] are satisfied.

(ii) For each edge of T, its preimage is $\mathbb{F} \times [0,1]$, therefore condition (3) of [MR08, Definition 3.1] is satisfied.

(iii) For each edge of T, the maps from the edge space to the incident vertex spaces are given by identity and Φ, which are both quasi-isometries. Hence, condition (4) (*quasi-isometries embedded condition*) of [MR08, Definition 3.1] is satisfied.

(iv) Condition (5) (strictly type-preserving condition) of [MR08, Definition 3.1] follows from the fact that Φ preserves the F^i up to conjugacy.

(v) As all the vertex and edge spaces are isometric to the same space, which is obtained by electrocuting $\{F^i\}$ inside $\mathbb{F}$, condition (6) of [MR08, Definition 3.1] is also satisfied.

This gives us a tree of strongly relatively hyperbolic spaces structure on Γ. Now we perform partial electrocution of Γ by electrocuting the cosets of $\mathbb{F}$ in Γ by using all the cosets of F^i for each i, and denote this electrocuted metric space by $\widehat{\Gamma}$. The strictly type-preserving condition ensures that $\widehat{\Gamma}$ is a tree of strongly relatively hyperbolic spaces with respect to the same base tree T. The conepoints in edge spaces are mapped to conepoints in vertex spaces and the edge maps used for Γ induce edge maps for $\widehat{\Gamma}$ in the same way as discussed previously. The vertex spaces and edge spaces are again copies of cosets of $\mathbb{F}$ where the map from each edge space to the incident vertex spaces are given by identity and Φ, which are therefore (uniform) quasi-isometries. Hence, the *quasi-isometries-preserving electrocution condition* [MR08, p. 1786] is satisfied.

Induced tree of coned-off spaces. $\hat{\Gamma}$ as described previously is the induced tree of coned-off spaces. This is the first stage of electrocution (called *partial electrocution* in the paper of Mj and Reeves; see [MR08, Definition 3.1] and the discussion that follows after it).

Cone locus (from [MR08, p. 1787]). The cone locus of $\hat{\Gamma}$, induced tree of coned-off spaces, is the forest whose vertex set consists of the conepoints of the vertex spaces of $\hat{\Gamma}$ and whose edge set consists of the conepoints in the edge spaces of $\hat{\Gamma}$. The incidence relations of the cone locus is dictated by the incidence relations in T. In our case the vertex set corresponds to conepoints attached to copies of electrocuted cosets of each F^i in $g\mathbb{F}$, where g varies over all elements of Γ. The incidence relations are dictated by $\Phi(gF^i) = hF^i$ when there is an edge in T such that $g\widehat{\mathbb{F}}$ maps to into the initial vertex space and $h\widehat{\mathbb{F}}$ maps into the terminal vertex space of the edge, under the natural map induced from short exact sequence.

Each connected component of the cone locus is called a *maximal cone subtree*. In our case, the connected components of the cone locus are lines, whose vertices correspond to conepoints attached to $\Phi^m gF^i$ as m varies over all of $\mathbb{Z}$ and $g \in \mathbb{F}$, i are fixed. If Φ_i is some lift of ϕ that leaves F^i invariant, then we see that $f_i\Phi_i^m F^i = \Phi_i^m F^i$ for every word $f_i \in F^i$ and every $m \in \mathbb{Z}$. This shows that the stabilizer of the maximal cone subtree corresponding to the identity coset of F^i in Γ can be identified with the mapping torus $F^i \rtimes \langle \Phi_i \rangle$. One similarly shows that for any

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other coset gF^i, where $g \in \mathbb{F}$, the stabilizer of the corresponding maximal cone subtree will be isomorphic to $F^i \rtimes \langle \Phi_i \rangle$.

The collection of maximal cone subtrees, T_j, is denoted by $\mathcal{T}$. The metric space that T_j gives rise to is denoted by C_j, and the collection of C_j is denoted by $\mathcal{C}$. In our context, the collection $\mathcal{C}$ corresponds to the collection of cosets of $\Gamma_i = F^i \rtimes \langle \Phi_i \rangle$ as a subgroup of Γ, for each i.

Hallway (from [BF92]). A disk $f : [-m, m] \times I \to \hat{\Gamma}$ is a hallway of length $2m$ if it satisfies the following conditions:

(i) $f^{-1}(\cup \hat{\Gamma}_v : v \in T) = \{-m, \dots, m\} \times I$;
(ii) f maps $i \times I$ to a geodesic in $\hat{\Gamma}_v$ for some vertex space;
(iii) f is transverse, relative to condition (i) to $\cup \hat{\Gamma}_e$.

Recall that in our case, the vertex spaces being considered above are just copies of $\hat{\mathbb{F}}$ with the electrocuted metric (obtained from $\mathbb{F}$ by coning-off the collection of subgroups F^i).

Thin hallway. A hallway is δ-thin if $d(f(i, t), f(i + 1, t)) \leq \delta$ for all i, t.

A hallway is λ-hyperbolic if

$$\lambda l(f(\{0\} \times I)) \leq \max\{l(f(\{-m\} \times I)), l(f(\{m\} \times I))\}$$

Essential hallway. A hallway is essential if the edge path in T resulting from projecting $\hat{\Gamma}$ onto T does not backtrack (and, hence, is a geodesic segment in the tree T).

Cone-bounded hallway (from [MR08, Definition 3.4]). An essential hallway of length $2m$ is cone bounded if $f(i \times \partial I)$ lies in the cone locus for $i = \{-m, \dots, m\}$.

Recall that in our case, the connected components of the cone locus are T_j which are the cosets of Γ_i (post electrocuting the cosets of F^i in Γ_i) inside Γ. This condition therefore requires that the points $f(i \times \partial I)$ are conepoints of electrocuted copies of some coset of the peripheral subgroups given by the nonattracting subgroup system.

Hallways flare condition (from [BF92, MR08]). The induced tree of coned-off spaces, $\hat{\Gamma}$, is said to satisfy the hallways flare condition if there exists $\lambda > 1$, $m \geq 1$ such that for all δ there is some constant $C(\delta)$ such that any δ-thin essential hallway of length $2m$ and girth at least $C(\delta)$ is λ-hyperbolic.

In our context, Proposition 3.10 establishes that hallways flare condition is satisfied for $\hat{\Gamma}$ (with $\lambda = 2$ and $C(\delta) = L$), because $\hat{\Gamma}$ is obtained from Γ by electrocuting cosets of F^i, for each i. To see this, consider any two distinct quasi-isometry lifts of T in $\hat{\Gamma}$ which are at least distance L apart. Look at the identity coset of $\widehat{\widehat{\mathbb{F}}}$ and a geodesic path in $\widehat{\widehat{\mathbb{F}}}$ whose end points are in the vertices of the two chosen lifts. If w is the word in $\mathbb{F}$ representing this path, then $|w|_{\text{el}} \geq L$ and, hence, Proposition 3.10 implies the hallway flaring condition.

Thus, $\hat{\Gamma}$ is a hyperbolic metric space by using the Bestvina–Feighn combination theorem and Γ is weakly hyperbolic relative to the collection $\mathcal{T}$ ([MR08, Lemma 3.8]).

Cone-bounded hallways strictly flare condition (from [MR08, Definition 3.6]). The induced tree of coned-off spaces $\hat{\Gamma}$, is said to satisfy the cone-bounded hallways strictly flare condition if there exists $\lambda > 1$, $m \geq 1$ such that any cone-bounded hallway of length $2m$ is λ-hyperbolic.

In our case, this condition is easily checked by using Lemma 3.12. To see why, recall the construction of maximal cone subtrees in our case and see the discussion preceding the statement of Lemma 3.12.

Hence, we have the following theorem by applying [MR08, Theorem 4.6].

PROPOSITION 3.13. *Let $\phi \in \mathsf{Out}(\mathbb{F})$ be rotationless and exponentially growing outer automorphism equipped with a dual lamination pair $\Lambda_\phi^\pm$, which are topmost. Let $\Phi \in \mathrm{Aut}(\mathbb{F})$ be a lift of ϕ. In addition, let $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm) = \{[F^1], [F^2], \ldots, [F^k]\}$ denote the nonattracting subgroup system for $\Lambda_\phi^\pm$. If F^i denote representatives of $[F^i]$ such that $\Phi_i(F^i) = F^i$ for some lift Φ_i, then the extension group Γ in the short exact sequence*

$$1 \to \mathbb{F} \to \Gamma \to \langle \Phi \rangle \to 1$$

is strongly hyperbolic relative to the collection of subgroups $\{F^i \rtimes_{\Phi_i} \mathbb{Z}\}$.

3.4 Atoroidal case

Observe that if ϕ is *atoroidal* (i.e. does not have any periodic conjugacy class), then ϕ is necessarily exponentially growing (see [Gho21, Lemma 3.1]). This can also be seen by using the Bestvina–Feighn–Handel theorem developed in [BFH05] which implies that any polynomially growing outer automorphism necessarily has a periodic conjugacy class. Thus, we can use Proposition 3.13 to set up a recursion on each mapping tori $\{F^i \rtimes_{\Phi_i} \mathbb{Z}\}$ in the following way.

 (i) If the mapping tori $\{F^i \rtimes_{\Phi_i} \mathbb{Z}\}$ is hyperbolic, we stop.
 (ii) If the restriction of ϕ to F^j is polynomially growing then using the Kolchin-type theorem of Bestvina, Feighn and Handel [BFH05] we know that ϕ must fix some conjugacy class in F^j and therefore ϕ cannot be atoroidal. Hence, this case is not possible.
 (iii) If $\mathsf{rank}(F^i) \leq 2$, then again this implies ϕ fixes some conjugacy class and therefore cannot be atoroidal. Hence, this case is not possible.
 (iv) If none of the previous cases happen, then we may continue recursively by considering the restriction $\phi \in \mathsf{Out}(F^i)$ (which is also rotationless) and applying Proposition 3.13. As each subsequent component, say $F_\Phi^{i,j}$, obtained from F_Φ^i in this process is a proper free factor of $\mathbb{F}$ by itself (due to Lemma 3.1(v)), the rank of such $F_\Phi^{i,j}$ drops at each step of recursion. Therefore, this process must stop when we have $\mathsf{rank}(F^{i,j,k,\ldots,s}) = 3$ and the mapping tori corresponding to this invariant subgroup must be hyperbolic (any outer automorphism of a free group of rank 2 always fixes some conjugacy class).

This shows that for an atoroidal ϕ, the collection of parabolic subgroups obtained in Proposition 3.13 can be made finer (in finitely many steps) until eventually we have that each subgroup in the collection is a hyperbolic group. Now we use two well-known facts in the theory of hyperbolic groups.

 (a) If Γ is (strongly) hyperbolic relative to the finite collection $\{H_i\}_i$ and each H_i is (strongly) hyperbolic relative to the finite collection $\{K_j^i\}_j$, then Γ is (strongly) hyperbolic relative to the finite collection $\{K_j^i\}_{i,j}$.
 (b) If Γ is (strongly) hyperbolic relative to the finite collection $\{H_i\}_i$ and each H_i is hyperbolic, then Γ is hyperbolic.

Then we can conclude that Γ itself must be hyperbolic when ϕ is atoroidal. Thus, as a corollary of Proposition 3.13, we have obtained a new proof of Brinkmann's theorem [Bri00].

COROLLARY 3.14. *Let $\phi \in \mathsf{Out}(\mathbb{F})$ and $\Phi \in \mathrm{Aut}(\mathbb{F})$ be any lift of ϕ. Then ϕ is atoroidal if and only if $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ is a hyperbolic group.*

Proof. Suppose ϕ is atoroidal. Pass to a rotationless power ϕ^k of ϕ and consider the mapping torus of ϕ^k. The discussion shows that $\mathbb{F} \rtimes_{\Phi^k} \mathbb{Z}$ is hyperbolic. As $\mathbb{F} \rtimes_{\Phi^k} \mathbb{Z}$ is a finite-index subgroup of $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ they are quasi-isometric. Therefore $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ is also hyperbolic.

For the converse direction, observe that if $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ is hyperbolic, then $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ cannot contain a copy (up to finite index) of $\mathbb{Z} \times \mathbb{Z}$. Hence, ϕ cannot have a periodic conjugacy class. □

The converse of Proposition 3.13 has already been proven, and is present implicitly in the work of Macura [Mac02] as was first pointed out in the work of Hagen and Wise [HW16]. In [Mac02], Macura studied the divergence function for the free-by-cyclic extensions induced by polynomially growing outer automorphisms and proved that the divergence function for an extension induced by an outer automorphism of growth order r, is approximately x^{r+1}. However, it is well known that relatively hyperbolic groups have exponential divergence function (see [Sis12]). The author thanks Mark Hagen for pointing this out.

Remark. In the work of Macura, it is implicit that for a polynomially growing outer automorphism ϕ, the induced mapping torus group $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ is a *thick metric space* of order r (in the sense of [BDM09]) for any lift Φ of ϕ and, hence, $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ cannot be relatively hyperbolic with respect to any finite collection of subgroups. Hagen has written down an explicit proof of this in [Hag19].

Thus, in light of the above discussion, we have the following theorem.

THEOREM 3.15. *If $\phi \in \mathrm{Out}(\mathbb{F})$ and $\Phi \in \mathrm{Aut}(\mathbb{F})$ is any lift of ϕ, then the group $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ is strongly relatively hyperbolic if and only if ϕ is exponentially growing.*

Proof. Let $\Gamma = \mathbb{F} \rtimes_\Phi \mathbb{Z}$ denote the mapping torus. Pass to rotationless power ϕ' of ϕ and consider the mapping torus Γ' of ϕ'. Proposition 3.13 shows that Γ' is (strongly) relatively hyperbolic group. As Γ' is quasi-isometric to Γ, by using the fact that relative hyperbolicity is quasi-isometry invariant [Dru09], we can conclude that Γ is also (strongly) relatively hyperbolic.

For the converse direction, suppose that Γ is relatively hyperbolic. If ϕ is not exponentially growing, then it is either polynomially growing or of finite order. Hagen's work in [Hag19, Theorem 1.2] shows that Γ cannot be relatively hyperbolic in the polynomially growing case or finite order case. Hence, ϕ must be exponentially growing. □

Before we move on to the polynomially growing case we would like to remark that a somewhat similar looking result as in Proposition 3.13 was claimed in an yet unpublished paper of Gautero and Lustig [GL07]. What they show is that the mapping torus of any outer automorphism is (strongly) hyperbolic relative to the collection of 'canonical' subgroups which contain all the polynomially growing conjugacy classes. A result similar to the Gautero–Lustig theorem can be easily deduced by a repeated application of Proposition 3.13 on the peripheral subgroups given in the conclusion of that proposition (similar to the argument we gave for the atoroidal case, Corollary 3.14). The two facts that allow us to provide a inductive argument in this case are as follows.

(a) For a geometric lamination Λ_ϕ^+, the nonattracting subgroup system can be written as $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+) = \mathcal{F} \cup \{[\langle c_1 \rangle], [\langle c_2 \rangle], \ldots, [\langle c_k \rangle]\}$, where $\mathcal{F}$ is a free-factor system and $\langle c_i \rangle$ are infinite cyclic subgroups where the conjugacy classes $[c_i]$ represents the boundary components of the surface which supports the lamination. This fact can be extracted from the subgroup decomposition work [HM20] where the geometric models are developed to study geometric strata (see Remark 1.3 in [HM20] or the proof of Proposition 5.11 in [HM09, Case 3b, pp. 49–50]). This allows us to perform induction on the rank of the components of the nonattracting subgroup system.

(b) For a nongeometric lamination, we know that $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^+)$ is a free-factor system (Lemma 3.1(vi)).

We record this result as a corollary:

Corollary 3.16. *The mapping torus of every rotationless, exponentially growing $\phi \in \mathrm{Out}(\mathbb{F})$ is (strongly) hyperbolic relative to a finite collection of peripheral subgroups of the form $F^i \rtimes_{\Phi_i} \mathbb{Z}$, where Φ_i is a lift of ϕ that preserves F^i and the outer automorphism class of Φ_i restricted to F^i is polynomially growing.*

4. Polynomial growth case

For any $\phi \in \mathrm{Out}(\mathbb{F})$, we have the following result due to Genevois and Horbez. In an earlier work, Button and Kropholler (see [BK16, Corollary 4.3]) had proven a similar result but with 'virtually acylindrically hyperbolic' conclusion instead of 'acylindrically hyperbolic'.

Lemma 4.1. *[GH21, Corollary 1.5] Let $\phi \in \mathrm{Out}(\mathbb{F})$ and $\Phi \in \mathrm{Aut}(\mathbb{F})$ be any lift of ϕ. Then the mapping torus $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ is acylindrically hyperbolic if and only if ϕ has infinite order.*

Combining this lemma with Theorem 3.15 we conclude as follows.

Corollary 4.2. *Let $\phi \in \mathrm{Out}(\mathbb{F})$ be of infinite order. Then the extension group $\mathbb{F} \rtimes_\Phi \mathbb{Z}$ is acylindrically hyperbolic but not relatively hyperbolic if and only if ϕ is polynomially growing.*

To conclude of this section we would like to point out that this answers a problem posed by Minasyan and Osin [MO15, Problem 8.2].

5. Applications

5.1 Free-by-free hyperbolic extensions

Consider the short exact sequence

$$1 \to \mathbb{F} \to \Gamma \to Q \to 1,$$

where Q is a free subgroup of $\mathrm{Out}(\mathbb{F})$. In this section, we give the construction of a free-by-free hyperbolic extension Γ where the elements of quotient group Q are not necessarily fully irreducible. Thus, far in the study of $\mathrm{Out}(\mathbb{F})$, the only examples of free-by-free hyperbolic extensions which are known; necessarily assume that every element of Q is fully irreducible. Thus, Theorem 5.6, in a way, gives a new class of examples of free-by-free hyperbolic extensions.

Standing assumptions. We let ϕ, ψ be atoroidal outer automorphisms with dual lamination pairs $\Lambda_\phi^\pm$ and $\Lambda_\psi^\pm$ and assume that the following hold:

(i) $\{\Lambda_\phi^+, \Lambda_\phi^-\} \cap \{\Lambda_\psi^+, \Lambda_\psi^-\} = \emptyset$;
(ii) $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$ does not carry generic leaves of $\Lambda_\psi^\pm$ and $\mathcal{A}_{\mathrm{na}}(\Lambda_\psi^\pm)$ does not carry generic leaves of $\Lambda_\phi^\pm$;
(iii) $\phi^\pm$ and $\psi^\pm$ are both rotationless.

We now make the following observations.

(1) As the outer automorphism ϕ is atoroidal, the restriction of ϕ as an element of $\mathrm{Out}(F^i)$ (where $[F^i] \in \mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$) is also atoroidal. Similarly for ψ.
(2) As $\mathbb{F} \rtimes \langle \Phi \rangle$ is a hyperbolic group, we can look at it as a tree of hyperbolic metric spaces using the short exact sequence $1 \to \mathbb{F} \to \mathbb{F} \rtimes \langle \Phi \rangle \to \langle \Phi \rangle \to 1$. Bowditch [Bow07, pp. 85–86] shows that in this situation, hallways flare (see Theorem 3.7 in [MR08]).
(3) As hallways flare, we conclude that conjugacy flaring also holds.

We record these observations in the following lemma.

LEMMA 5.1. *Suppose $\phi \in \mathrm{Out}(\mathbb{F})$ is rotationless and atroidal and let $\Phi \in \mathrm{Aut}(\mathbb{F})$ be some lift of ϕ. There exists numbers $N_\phi, M_\phi > 0$ such that:*

(i) *(hallway flaring) for every $n \geq N_\phi$ and every nontrivial word $w \in \mathbb{F}$*

$$3|w| \leq \max\{|\Phi_\#^n(w)|, |\Phi_\#^{-n}(w)|\};$$

(ii) *(conjugacy flaring) for every $m \geq M_\phi$ and every nontrivial conjugacy class in $\mathbb{F}$*

$$3\|c\| \leq \max\{\|\phi_\#^m(c)\|, \|\phi_\#^{-m}(c)\|\}.$$

The following result shows that under our assumptions there is a mutual attraction of the laminations for ϕ, ψ.

LEMMA 5.2. *If ϕ, ψ are outer automorphisms which satisfy the standing assumptions above, then we have the following result.*

Generic leaves of $\Lambda_\phi^+, \Lambda_\phi^-$ are attracted to Λ_ψ^+ (respectively, Λ_ψ^-) under action of ψ (respectively, ψ^{-1}). Similarly, with roles of ϕ, ψ reversed.

Proof. Let γ_ϕ^+ denote a generic leaf of Λ_ϕ^+. We claim that γ_ϕ^+ cannot be a leaf of Λ_ψ^-. It is clear that γ_ϕ^+ cannot be a generic leaf of Λ_ψ^-, because otherwise the weak closure would be equal to both Λ_ϕ^+ and Λ_ψ^-, which would violate item (i) in the standing assumption. If H_t^- is the EG strata associated to Λ_ψ^- for some relative train-track map $f_\psi : G_\psi^- \to G_\psi^-$ then γ_ϕ^+ must have height at least t, because all strata below H_t^- are carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\psi^\pm)$. In addition, if γ_ϕ^+ has height greater than t, it cannot be a leaf of Λ_ψ^-.

If γ_ϕ^+ is a nongeneric leaf of Λ_ψ^- of height t, then [BFH00, Lemma 3.1.15] closure of γ_ϕ^+ is all of Λ_ψ^- which again contradicts item (i) of the standing assumption. Thus, γ_ϕ^+ cannot be a nongeneric leaf of Λ_ψ^- and, consequently, γ_ϕ^+ cannot be a leaf of Λ_ψ^-.

Choose attracting neighborhoods V_ψ^+ and V_ψ^- of Λ_ψ^+ and Λ_ψ^-, respectively, defined by long generic leaf segments of the respective laminations such that $\gamma_\phi^+ \notin V_\psi^-$.

By using the weak attraction Lemma 3.2 together with the standing assumption (ii), we know that either $\gamma_\phi^+ \in V_\psi^-$ or γ_ϕ^+ is weakly attracted to Λ_ψ^+ under iteration by ψ. However, because we have ruled out the first possibility, γ_ϕ^+ is necessarily attracted to Λ_ψ^+. This implies that generic leaves of Λ_ϕ^+ are attracted to Λ_ψ^+.

The proof of other conclusions in item (ii) follows from symmetric arguments. $\square$

Let $\{\alpha_i\}$ be a sequence of conjugacy classes and also denote their realization in G by α_i. Decompose each α_i as a concatenation of subpaths each of which are either in the complement of G_r or are paths of height r. Further decompose each height-r component into segments which are either generic leaf segments of Λ_ϕ^+ (denote such a segment by $\epsilon_{i,k}$ for some k) or a path which is not a generic leaf segment.

DEFINITION 5.3. We say that α_i is *approximate* Λ_ϕ^+ if for any $L > 0$, the ratio

$$\frac{\{\sum |\epsilon_{i,k}|_G \mid \text{summing over all such } k \text{ where } |\epsilon_{i,k}|_G \geq L\}}{|\alpha_i|_G}$$

converges to 1 as $i \to \infty$.

LEMMA 5.4. *Let ϕ, ψ be outer automorphisms of $\mathbb{F}$ which satisfy the standing assumptions. Then for any sequence of conjugacy classes $\{\alpha_i\}$, the sequence cannot approximate both Λ_ϕ^- and Λ_ψ^-.*

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Proof. Suppose that $\{\alpha_i\}$ approximates Λ_ψ^-. As in the proof of Lemma 5.2 we choose an attracting neighborhood V_ϕ^- for Λ_ϕ^- such that $\Lambda_\psi^- \notin V_\phi^-$.

In this setup, $\Lambda_\psi^- \notin V_\phi^-$ and generic leaves of Λ_ψ^- are not carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_\phi^\pm)$. Hence, by applying the uniformity part of the weak attraction theorem, we get an $M \geq 1$ such that $\phi_\#^m(\gamma_\psi^-) \in V_\phi^+$ for all $m \geq M$ for every generic leaf $\gamma_\psi^- \in \Lambda_\psi^-$. As V_ϕ^+ is an open set we can find an $I \geq 1$ such that $\phi_\#^m(\alpha_i) \in V_\phi^+$ for all $m \geq M, i \geq I$. This implies that $\{\alpha_i\}$ cannot approximate Λ_ϕ^-, because otherwise generic leaves of Λ_ϕ^- would get attracted to Λ_ϕ^+. □

Now we are ready to prove a version of Mosher's 3-of-4 stretch lemma [Mos97] and give a new example of a hyperbolic-by-hyperbolic hyperbolic group.

PROPOSITION 5.5 (3-of-4 stretch). *Let ϕ, ψ be outer automorphisms which satisfy the standing assumptions. In addition, we suppose that $\mathcal{L}(\phi^\pm) \cap \mathcal{L}(\psi^\pm) = \emptyset$. Then we have the following.*

(i) *There exists some $M \geq 0$ such that for any conjugacy class α, at least three of the four numbers*

$$\|\phi_\#^{n_i}(\alpha)\|, \quad \|\phi_\#^{-n_i}(\alpha)\|, \quad \|\psi_\#^{n_i}(\alpha)\|, \quad \|\psi_\#^{-n_i}(\alpha)\|$$

are greater an or equal to $3\|\alpha\|$, for all $n_i \geq M$.

(ii) *There exists some $N \geq 0$ such that for any word $w \in \mathbb{F}$, at least three of the four numbers*

$$|\Phi_\#^{n_i}(w)|, \quad |\Phi_\#^{-n_i}(w)|, \quad |\Psi_\#^{n_i}(w)|, \quad |\Psi_\#^{-n_i}(w)|$$

are greater than $3|w|$, for all $n_i \geq N$.

Proof. Proof of (i). Suppose there does not exist any such M_0. We argue to a contradiction by using the weak attraction theorem. By our supposition, we get a sequence of conjugacy classes α_i such that at least two of the four numbers $\|\phi_\#^{n_i}(\alpha_i)\|, \|\phi_\#^{-n_i}(\alpha_i)\|, \|\psi_\#^{n_i}(\alpha_i)\|, \|\psi_\#^{-n_i}(\alpha_i)\|$ are less than $3\|\alpha_i\|$ and $n_i > i$. Lemma 5.1 tells us that at least one of $\{\|\phi_\#^{n_i}(\alpha_i)\|, \|\phi_\#^{-n_i}(\alpha_i)\|\}$ is at least $3\|\alpha_i\|$ and at least one of

$\{\|\psi_\#^{n_i}(\alpha_i)\|, \|\psi_\#^{-n_i}(\alpha_i)\|\}$ is at least $3\|\alpha_i\|$ for all sufficiently large i.

For sake of concreteness suppose that

$$\|\phi_\#^{n_i}(\alpha_i)\| \leq 3\|\alpha_i\|, \quad \|\psi_\#^{n_i}(\alpha_i)\| \leq 3\|\alpha_i\| \quad \forall \, n_i. \tag{4}$$

The above assumptions show that α_i grows exponentially under iterates of both ϕ^{-1} and ψ^{-1}. Hence, after passing to a subsequence of α_i, we may assume that all α_i are weakly attracted to some attracting lamination Λ_1^- of ϕ^{-1} and Λ_2^- of ψ^{-1} (because $\mathcal{L}(\phi^\pm), \mathcal{L}(\psi^\pm)$ are finite sets). If α_i is also attracted to some lamination which contains Λ_1^- we replace Λ_1^- with the highest lamination such a inclusion chain and call it Λ_1^-. Similarly for Λ_2^-. Thus, after passing to a subsequence, we may also assume that none of the α_i are attracted to any lamination which properly contain Λ_1^- and Λ_2^-. Let Λ_i^+ be dual to Λ_i^- for $i = 1, 2$.

If Λ_1^- is carried by the nonattracting subgroup system of Λ_2^-, then α_i cannot approximate both Λ_1^- and Λ_2^- because the realization of any generic leaf of Λ_1^- in any CT representing ψ^{-1} will be entirely contained in the nonattracting subgraph for Λ_2^-. Otherwise when generic leaves of Λ_1^- are weakly attracted to Λ_2^- and vice versa, we use the additional hypothesis $\mathcal{L}(\phi^\pm) \cap \mathcal{L}(\psi^\pm) = \emptyset$ and follow the proof of Lemma 5.4 to conclude that α_i does not approximate both Λ_1^- and Λ_2^-.

For concreteness, suppose that α_i do not approximate Λ_1^-. Then we can choose some attracting neighborhood V_1^- of Λ_1^- which is defined by some long generic leaf segment and after passing to a subsequence if necessary we may assume that $\alpha_i \notin V_1^-$ for all i. Also note that α_i is not

carried by $\mathcal{A}_{\mathrm{na}}(\Lambda_1^{\pm})$ for all sufficiently large i. By using the uniformity part of the weak attraction Lemma 3.2, there exists some M^+ such that $\phi_{\#}^m(\alpha_i) \in V_1^+$ for all $m \geq M$.

Choose a CT $f : G \to G$ representing ϕ and let H_r be the EG stratum associated to Λ_1^+. Our assumption that Λ_1^- is the highest lamination to which α_i are attracted ensures that α_i are not attracted to any lamination which properly contain Λ_1^+ (due to duality). We can choose i to be sufficiently large we may assume that $n_i \geq M^+$ and so by using Lemma 3.5 we have that for some $\epsilon > 0$, $LEG_{H_r^\sigma}(\phi_{\#}^{n_i}(\alpha_i)) \geq \epsilon$. By using Lemma 3.6 (see the discussion preceding the statement) we obtain that for any $A > 0$ there exists some M_1^+ such that

$$|f_{\#}^m(\alpha_i)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \geq A|\alpha_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$$

for every $m > M_1^+$. This implies that for all sufficiently large i, $|f_{\#}^{n_i}(\alpha_i)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \geq A|\alpha_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$. Choosing a sequence $A_i \to \infty$ so that we have $A_i > i.|\alpha_i|/|\alpha_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}$ and after passing to a subsequence of $\{n_i\}$ we may assume that $|f_{\#}^{n_i}(\alpha_i)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \geq A_i|\alpha_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle} \geq i.|\alpha_i|$. Finally, using an $\mathbb{F}$−equivariant quasi-isometry between $\widetilde{G}$ and universal cover of standard rose for $\mathbb{F}$, we obtain some constant K'' such that $K''|\alpha_i| \geq \|\alpha_i\|$ for every i. We therefore get

$$3 \geq \frac{\|\phi_{\#}^{n_i}(\alpha_i)\|}{\|\alpha_i\|} \geq \frac{\|\phi_{\#}^{n_i}(\alpha_i)\|_{\mathrm{el}}}{\|\alpha_i\|} \geq \frac{|\phi_{\#}^{n_i}(\alpha_i)|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}}{K\|\alpha_i\|} \geq \frac{A_i \cdot |\alpha_i|_{\langle \mathcal{Z}, \hat{\sigma} \rangle}}{KK''|\alpha_i|} \geq \frac{i}{KK''}.$$

The first inequality is inequality (4) above. The second inequality follows from the fact the absolute length is at least as much as length in electrocuted metric. The third inequality is obtained by using Lemma 3.7. The fourth inequality uses Lemma 3.6 and the choice of K'' explained previously. The last inequality follows from the choice of A_i made previously. Thus, we get a contradiction to the existence of the sequence of conjugacy classes α_i.

The proof of (ii) is similar to proof of Proposition 3.10 (see Lemma 5.1). $\square$

THEOREM 5.6. *Let ϕ, ψ be outer automorphisms which satisfy the standing assumptions above and $\mathcal{L}(\phi^{\pm}) \cap \mathcal{L}(\psi^{\pm}) = \emptyset$. Then there exists some $M > 0$ such that for every $m, n \geq M$ the group $Q := \langle \phi^m, \psi^n \rangle$ is a free group of rank 2 and the extension group $\mathbb{F} \rtimes \widetilde{Q}$ is hyperbolic for any lift $\widetilde{Q}$ of Q.*

Proof. The conclusion about free groups follows directly from 3-of-4 stretch result in Proposition 5.5. The hyperbolicity of the extension group follows by using the Bestvina–Feighn Combination theorem [BF92] because Proposition 5.5(ii) implies that the annuli flare condition is satisfied. $\square$

It is worth pointing out that it is very easy to construct examples of ϕ, ψ such that no element of Q will be fully irreducible. To see this consider a free group of rank at least 6. Say $\mathbb{F} = \langle a, b, c, d, e, f \rangle$. Now take two fully irreducible elements without common nonzero powers, say ψ_1, ϕ_1, defined on $\langle a, b, c \rangle$. Now extend ϕ_1 to an outer automorphism of $\mathbb{F}$ so that every conjugacy class in $\langle d, e, f \rangle$ is attracted to the lamination of ϕ_1 under iterates of ϕ. Do the same for ψ. For any given relative train-track map for ϕ there is no restriction on the strata above the base stratum associated to the attracting lamination of ϕ_1, except that they must be all attracted to the attracting lamination of ϕ_1. Then ϕ, ψ satisfy all our standing assumptions for this section and we get a free-by-free hyperbolic extension. Observe that both ϕ and ψ have a common invariant conjugacy class of free factor $\langle a, b, c \rangle$ and, hence, none of the elements of the free group generated by powers of ϕ, ψ are fully irreducible.

COROLLARY 5.7 [BHF97, Theorem 5.2]. *Suppose ϕ, ψ are fully irreducible and atoroidal which do not have common powers. Then there exists some $M > 0$ such that for every $m, n \geq M$ the*

group $Q := \langle \phi^m, \psi^n \rangle$ is a free group of rank 2 and the extension group $\mathbb{F} \rtimes \widetilde{Q}$ is hyperbolic for any lift $\widetilde{Q}$ of Q.

Proof. As ϕ, ψ are fully irreducible, atoroidal, our standing assumptions are automatically satisfied. Now apply Theorem 5.6. $\qquad\square$

Remark. Caglar Uyanik has pointed out to the author that Theorem 5.6 can also be deduced from his work [Uya19], because our standing assumptions imply that the hypothesis of [Uya19, Proposition 4.2] (the version of Mosher's 3-of-4 stretch lemma in his work) is satisfied. The attracting laminations in our hypothesis act as the attracting and repelling simplices in the space of currents which is used for a ping-pong argument in that paper.

5.2 Quadratic isoperimetric inequality

Bridson and Groves [BG10] proved that the mapping torus of any outer automorphism of a free group satisfies the quadratic isoperimetric inequality. We can deduce the same theorem from our work here, and it is perhaps a simpler proof of the Bridson–Groves theorem. In another related work Macura [Mac00] has some interesting results on the quadratic isoperimetric inequality problem.

THEOREM 5.8. *The mapping torus of any $\phi \in \mathrm{Out}(\mathbb{F})$ satisfies the quadratic isoperimetric inequality.*

Proof. If ϕ is polynomially growing, then the result follows from the Bridson–Groves theorem for the special case of polynomially growing outer automorphisms. Otherwise, for exponentially growing ϕ, denote its mapping torus by Γ. Pass to a rotationless power, call it ϕ', and use Corollary 3.16 to conclude that the mapping torus of ϕ', Γ' say, is (strongly) hyperbolic relative to a collection of peripheral subgroups each of which satisfy the quadratic isoperimetric inequality (by applying polynomially growth case of the Bridson–Groves theorem on the peripheral subgroups). Farb's work in [Far98] shows that if the peripheral subgroups of a relatively hyperbolic group G, satisfies a quadratic isoperimetric inequality, then G satisfies the quadratic isoperimetric inequality. Hence, we conclude that Γ' satisfies a quadratic isoperimetric inequality.

As Γ' is a finite index subgroup of Γ and the property of satisfying a quadratic isoperimetric inequality is quasi-isometry invariant, Γ also satisfies a quadratic isoperimetric inequality. $\qquad\square$

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