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WEAK AMENABILITY OF GROUP ALGEBRAS OF LOCALLY COMPACT GROUPS

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ABSTRACT. A new proof is given for the weak amenability of the group algebras of locally compact groups.

Let $L^1(G)$ be the group algebra of a locally compact group G. In a recent paper [7] B. E. Johnson proved that $L^1(G)$ is weakly amenable (see also [5], [6] and [3] for earlier partial results). In this paper we give a different proof of Johnson's result, which simplifies the technicalities in [7] but utilizes the lattice structure of $L^\infty_{\mathbb{R}}(G)$.

Recall that if A is a Banach algebra, then its dual A^* can be made into a Banach A-module, with module actions defined by

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle,$$
$$\langle a \cdot f, b\rangle = \langle f, ba\rangle, \quad (f \in A^*, a, b \in A).$$

A linear map $D\colon A \longrightarrow A^*$ is a *derivation* if $D(ab) = D(a) \cdot b + a \cdot D(b)$ $(a, b \in A)$. For example, if $\varphi \in A^*$, then the map $\Delta_\varphi\colon a \longmapsto a \cdot \varphi - \varphi \cdot a$ is a derivation. Derivations Δ_φ are called *inner*. A Banach algebra A is *weakly amenable* if every continuous derivation from A into A^* is inner [1].

In our proof of the weak amenability of $L^1(G)$ we make use of the fact that $L^\infty(G)$ is also an $M(G)$-module, where the module actions are defined by

$$\langle f \cdot \mu, a\rangle = \langle f, \mu * a\rangle,$$
$$\langle \mu \cdot f, a\rangle = \langle f, a * \mu\rangle,$$
$$(f \in L^\infty(G), \mu \in M(G), a \in L^1(G)).$$

We say that a net $(\mu_i) \subset M(G)$ converges to $\mu \in M(G)$ in the *strong operator* (so) topology if for every $f \in L^1(G)$,

$$\mu_i * f \longrightarrow \mu * f$$

and

$$f * \mu_i \longrightarrow f * \mu$$

in the norm topology of $L^1(G)$.

The following lemma is standard [*cf.* 4, Proposition 1.1]. We include a proof for completeness.

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LEMMA 1. *Let $D\colon L^1(G) \longrightarrow L^\infty(G)$ be a continuous derivation. Then:*
(a) D has an extension to a continuous derivation $\bar{D}\colon M(G) \longrightarrow L^\infty(G)$.
(b) $\bar{D}$ is continuous when $M(G)$ is equipped with the so*-topology and $L^\infty(G)$ is equipped with the weak*-topology.*

PROOF. Let $\mu \in M(G)$, $f \in L^1(G)$, and let (e_α) be a bounded approximate identity for $L^1(G)$. By Cohen's factorization theorem, there exist $f_1, f_2 \in L^1(G)$ such that $f = f_1 * f_2$. Now

$$\begin{aligned} \langle D(\mu * e_\alpha), f\rangle &= \langle D(\mu * e_\alpha)\cdot f_1, f_2\rangle \\ &= \langle D(\mu * e_\alpha * f_1), f_2\rangle - \langle D(f_1), f_2 * \mu * e_\alpha\rangle \\ &\longrightarrow \langle D(\mu * f_1), f_2\rangle - \langle D(f_1), f_2 * \mu\rangle, \end{aligned} \tag{1}$$

so that the weak*-$\lim_\alpha D(\mu * e_\alpha)$ exists in $L^\infty(G)$.

Define

$$\bar{D}(\mu) = \text{weak}^* \text{-} \lim_\alpha D(\mu * e_\alpha).$$

It follows from (1) that

$$\bar{D}(\mu * f_1) = \mu \cdot \bar{D}(f_1) + D(\mu)\cdot f_1,$$

and similar calculations then show that $\bar{D}$ is a derivation. Finally, the so-weak* continuity of $\bar{D}$ follows from:

$$\langle \bar{D}(\mu), f\rangle = \langle D(\mu * f_1), f_2\rangle - \langle D(f_1), f_2 * \mu\rangle.$$

THEOREM 1. *The group algebra $L^1(G)$ is weakly amenable.*

PROOF. By Lemma 1, it suffices to show that a continuous derivation D from $M(G)$ into $L^\infty(G)$ is inner. For $t \in G$, let δ_t be the point mass at t. Then for any $x \in G$.

$$\begin{aligned} \delta_{t^{-1}} \cdot D(\delta_t) &= \delta_{t^{-1}} \cdot D(\delta_{tx^{-1}} * \delta_x) \\ &= \delta_{x^{-1}} \cdot [\delta_{(tx^{-1})^{-1}} \cdot D(\delta_{tx^{-1}})] \cdot \delta_x + \delta_{x^{-1}} \cdot D(\delta_x). \end{aligned} \tag{2}$$

For $\psi \in L^\infty(G)$, let $\mathrm{Re}(\psi)$ denote the real part of ψ and let

$$S = \Big\{\mathrm{Re}\big(\delta_{t^{-1}} \cdot D(\delta_t)\big) : t \in G\Big\}.$$

Then S is a subset of $L^\infty_{\mathbb{R}}(G)$, the vector lattice of real-valued functions in $L^\infty(G)$, and is bounded above by the constant function $\|D\|$ in $L^\infty_{\mathbb{R}}(G)$. Since $L^\infty_{\mathbb{R}}(G)$ is a complete vector lattice, $\varphi_1 = \sup(S)$ exists in $L^\infty_{\mathbb{R}}(G)$. Furthermore, it is easily verified that

$$\begin{aligned} \sup(\delta_{x^{-1}} \cdot S \cdot \delta_x) &= \delta_{x^{-1}} \cdot \sup(S) \cdot \delta_x, \quad \text{and} \\ \sup(\psi + S) &= \psi + \sup(S), \quad \big(x \in G, \psi \in L^\infty_{\mathbb{R}}(G)\big). \end{aligned} \tag{3}$$

Taking $\sup_{t\in G}$ of the real parts in (2), and using (3), we obtain:

$$\varphi_1 = \delta_{x^{-1}} \cdot \varphi_1 \cdot \delta_x + \delta_{x^{-1}} \cdot \mathrm{Re}\big(D(\delta_x)\big),$$

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or equivalently

$$\operatorname{Re}\big(D(\delta_x)\big) = \delta_x \cdot \varphi_1 - \varphi_1 \cdot \delta_x,$$

for all $x \in G$. Similarly, by considering imaginary parts and taking $\sup_{t\in G}$ in (2), we obtain a $\varphi_2 \in L^\infty_{\mathbb{R}}(G)$ such that

$$\operatorname{Im}\big(D(\delta_x)\big) = \delta_x \cdot \varphi_2 - \varphi_2 \cdot \delta_x$$

for all $x \in G$. Thus

$$D(\delta_x) = \delta_x \cdot \varphi - \varphi \cdot \delta_x \quad (x \in G),$$

where $\varphi = \varphi_1 + i\varphi_2$. Since every measure μ in $M(G)$ is the so-limit of a net (μ_i) with each μ_i a linear combination of point masses, Lemma 1(b) gives

$$D(\mu) = \mu \cdot \varphi - \varphi \cdot \mu \quad \big(\mu \in M(G)\big),$$

as required.

NON-AMENABILITY OF $M(G)$. Using arguments of the previous section, it is possible to show that for every continuous derivation D: $M(G) \longrightarrow M(G)^*$, there exists $\varphi \in M(G)^*$ such that

$$D(\delta_x) = \delta_x \cdot \varphi - \varphi \cdot \delta_x \quad (x \in G).$$

However, it is no longer possible to deduce that $D = \Delta_\varphi$. At least, this is not the case when G is a non-discrete, abelian group (see [2]).

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REFERENCES

1. W. G. Bade, P. C. Curtis, Jr. and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. (3) **55**(1987), 359–377.

2. P. C. Curtis, Jr. and R. J. Loy, *The structure of amenable Banach algebras*, J. London Math. Soc. (2) **40**(1989), 89–104.

3. N. Grønbaek, *Constructions preserving weak amenability*, Proc. Centre for Math. Anal., Australia Nat. Univ. **21**(1989), 186–202.

4. B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127**, 1972.

5. ———, *Derivations from $L^1(G)$ into $L^1(G)$ and $L^\infty(G)$*, Lecture Notes in Math. **1359**, Springer, Berlin, 1988, 191–198.

6. ———, *Weak amenability of group algebras of connected complex semisimple Lie groups*, Proc. Amer. Math. Soc (2) **111**(1991), 177–185.

7. ———, *Weak amenability of group algebras*, Bull. London Math. Soc. **23**(1991), 281–284.

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